Domain of a Composition

1 Definition

Given the function f and g, the **composition of f with g** is a function defined as

$$(f \circ g)(x) = f(g(x)).$$

The **domain** of $f \circ g$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f.

2 Explanation

We have seen that determining the domain of a composition can be a bit difficult, so let's try to break down the above definition to get a better idea of what we are actually looking for.

First, it is clear we need to find the domain of g:

"The domain of $f \circ g$ is the set of all real numbers x in the domain of g...".

This step shouldn't be too difficult, just remember that 99% of the time you will be looking for which x-values give you a zero in a denominator, or make you take the square-root (or any even-root, i.e. 4th-root, 6th-root, etc.) of a negative number (these x-values will **not** be in the domain).

The latter portion of the definition is where the difficulty generally occurs:

"...$g(x)$ is in the domain of f".

Let us begin by calculating the domain of f (again this should not be too difficult), but instead of writing the domain in interval notation (as we usually do) let's use inequalities, i.e. $<, \leq, >, \geq, \neq$.

So now we have conditions on which **x-values** are in the domain of f, but when we look back at the definition see that we are not interested in the **x-values** in the domain of f, but rather the **$g(x)$-values** in the domain of f. Thus, substitute $g(x)$ in for x inside of the inequalities you found for the domain of f. Once you have made this substitution, solve for x. These will now be the x-values for which $g(x)$ is in the domain of f (which is what we want).

After doing this we know the x-values in the domain of g, and we know which x-values keep $g(x)$ in the domain of f (it may be easier at this point to write these x-values in interval notation). Hence, to find the domain of $f \circ g$, we simply need to intersect these two intervals (i.e. find the x-values which are in **both** of our intervals).

3 Examples

As always it's easier to understand a new concept if you see a few examples, so we now will work through a few problems using the above explanation.

Example 1

Let $f(x) = \frac{x}{x+2}$ and $g(x) = \frac{-4}{x}$. Find $(f \circ g)(x)$ and determine its domain.

Solution.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{-4}{x}\right) = \frac{\frac{-4}{x}}{\frac{-4}{x}+2} = \frac{\frac{-4}{x}}{\frac{-4+2x}{x}} = \frac{-4}{x} \cdot \frac{x}{-4+2x} = \frac{-2}{x-2}.$$

First we will determine the domain of g. Since $g(x) = \frac{-x}{4}$ we see that if $x = 0$, then we will have a 0 in the denominator, and hence 0 is not in the domain of g. Therefore,

$$\text{Domain of } g = (-\infty, 0) \cup (0, \infty) \qquad (\text{i.e. } x \neq 0).$$

Now we want to find the domain of f. Since $f(x) = \frac{x}{x+2}$ we see that if $x = -2$ we will have a 0 in the denominator, so -2 is not in the domain of f. Thus,

$$\text{Domain of } f = (-\infty, -2) \cup (-2, \infty) \qquad (\text{i.e. } x \neq -2).$$

Remember that we don't want to know which x's are in the domain of f, but rather which $g(x)$'s are in the domain of f. Hence, we substitute $g(x)$ in for x to find,

$$\frac{-4}{x} = g(x) \neq 2.$$

Solving this for x we see

$$\begin{aligned} -4 &\neq -2x \\ 2 &\neq x. \end{aligned}$$

This means that for $x \neq 2$ (in interval notation: $(-\infty, 2) \cup (2, \infty)$) we will have that $g(x)$ is in the domain of f.

Finally, we need to find the intersection of our two intervals (i.e. find all numbers that are in **both** intervals):

$$\begin{aligned} \text{Domain of } f \circ g &= (\text{Domain of } g) \cap (x\text{-values where } g(x) \text{ is in the domain of } f) \\ &= ((-\infty, 0) \cup (0, \infty)) \cap ((-\infty, 2) \cup (2, \infty)) \\ &= (-\infty, 0) \cup (0, 2) \cup (2, \infty) \\ &= \{x | x \neq 0, x \neq 2\}. \end{aligned}$$

Example 2

Let $f(x) = x^2 + 8$ and $g(x) = \sqrt{x-3}$. Find $(f \circ g)(x)$ and determine its domain.

Solution.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-3}) = (\sqrt{x-3})^2 + 8 = x - 3 + 8 = x + 5.$$

First we will determine the domain of g. Since $g(x) = \sqrt{x-3}$ we see that if $x < 3$ we will take the square root of a negative number, so these values are **not** in the domain. Therefore,

$$\text{Domain of } g = [3, \infty) \qquad (\text{i.e. } x \geq 3).$$

Now we want to find the domain of f. Since $f(x) = x^2 + 8$, we see that there are no square roots or denominators, so the domain of f is the set of all real numbers,

$$\text{Domain of } f = (-\infty, \infty) \qquad (\text{i.e. } -\infty < x < \infty).$$

Again, we are only interested in when $g(x)$ is in the domain of f, but this time things are very easy. We know that every real number is in the domain of f, so as long as $g(x)$ is a real number it will be in the domain of f. We know the only times that $g(x)$ fails to be a real number are when x is not in the domain of g. Hence, $g(x)$ will be in the domain of f when x is in the domain of g, i.e. in the interval $[3, \infty)$.

Thus, we look at the intersection of our intervals to determine the domain of $f \circ g$:

$$\begin{aligned}\text{Domain of } f \circ g &= (\text{Domain of } g) \cap (x\text{-values where } g(x) \text{ is in the domain of } f) \\ &= ([3, \infty)) \cap ([3, \infty)) \\ &= [3, \infty) \\ &= \{x | x \geq 3\}.\end{aligned}$$

Now we will work though two examples which are more difficult.

Example 3

Let $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$. Find $(f \circ g)(x)$ and determine its domain.

Solution.

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{1 - x^2}.$$

First we will determine the domain of g. Clearly the domain of g will be all real numbers (there are no denominators or square roots), so

$$\text{Domain of } g = (-\infty, \infty) \qquad (\text{i.e. } -\infty < x < \infty).$$

Now we will determine the domain of f. Since $f(x) = \sqrt{1 - x}$ we know that if $x > 1$ we will have a square-root of a negative number, so these values will not be in the domain. Hence,

$$\text{Domain of } f = (-\infty, 1] \qquad (\text{i.e. } x \leq 1).$$

Again, we want to know when for which x-values $g(x)$ will be in the domain of f, so we make a substitution:

$$x^2 = g(x) \leq 1.$$

This is where things get tricky. If we were to 'solve for x' as we usually do, we would most likely take the square-root of both sides to find $x \leq \pm 1$. If we wrote this in interval notation, we would have $(-\infty, -1]$ (since we would require both $x \leq 1$ and $x \leq -1$ at the same time), but there are obviously numbers inside that interval which violate the condition $x^2 \leq 1$. For instance, -2 is inside the interval $(-\infty, -1]$, but $(-2)^2 = 4$ which is definitely not less than or equal to 1! So we need to try a different approach.

Instead of just jumping in and solving for x, let us take a moment and think about what the inequality $x^2 \leq 1$ is telling us. The inequality is saying that if we multiply a number by itself, the resulting product is less than or equal to 1. If we pick a number larger than 1 and multiply it by itself, we will alway get a number that is larger that 1. For instance,

$$1.000000001 > 1 \quad \text{and} \quad (1.000000001)^2 = 1.000000002000000001 > 1.$$

Also, we should note that x^2 is an even function, which means $(-x)^2 = x^2$. Hence, if we pick any negative number and square it, it would be the same as squaring the 'positive version' of that number. Hence, if we take any number less than -1 and square it, the result will again be greater than 1. Thus, we may rule our all x values with $x > 1$ and $x < -1$.

It is clear that both 1 and -1 satisfy our condition $x^2 \leq 1$. You should also note that the square of any number between 0 and 1 is actually *smaller* than your original number! For instance, $(0.25)^2 = 0.0625 \leq 1$. So any number between 0 and 1 will satisfy our condition. Again using the fact that x^2 is an even function, it follows that any number between -1 and 0 will also satisfy our condition.

Therefore, putting all of this together, the only x-values which satisfy $x^2 \leq 1$ are the x-values such that $-1 \leq x \leq 1$ (i.e. inside the interval $[-1.1]$). Hence, the x-values where $g(x)$ is in the domain of f are in the interval $[-1,1]$. So, we find

$$\begin{aligned}
\text{Domain of } f \circ g &= (\text{Domain of } g) \cap (x\text{-values where } g(x) \text{ is in the domain of } f) \\
&= ((-\infty,\infty)) \cap ([-1,1]) \\
&= [-1,1] \\
&= \{x| -1 \leq x \leq 1\}.
\end{aligned}$$

Example 4

Let $f(x) = \sqrt{x+3}$ and $g(x) = \frac{1}{x-2}$. Find $(f \circ g)(x)$ and determine its domain.

Solution.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = \sqrt{\frac{1}{x-2}+3} = \sqrt{\frac{1+3(x-2)}{x-2}} = \sqrt{\frac{3x-5}{x-2}}.$$

First we will determine the domain of g. Since $g(x) = \frac{1}{x-2}$ we know that if $x = 2$ we will have a zero in the denominator, so 2 is not in the domain of g. Thus,

$$\text{Domain of } g = (-\infty, 2) \cup (2, \infty) \qquad (\text{i.e. } x \neq 2).$$

Now we will determine the domain of f. Since $f(x) = \sqrt{x+3}$, we know that if $x < -3$ we will have the square-root of a negative number, and hence these values will not be in the domain. Hence,

$$\text{Domain of } f = [-3, \infty) \qquad (\text{i.e. } x \geq 3).$$

We are only interested in when $g(x)$ is in the domain of f, so we make our usual substitution:

$$\frac{1}{x-2} = g(x) \geq -3.$$

Again, we run into a slight problem here. We want to solve for x, but we need to remember that if you have an inequality and multiply both sides by a negative number, you **must** change the direction of the inequality. Note that if $x < 2$ then $x - 2$ will be negative! So if $x < 2$ and we multiply both sides by $x - 2$ we must remember to change the direction of our inequality!

Thus, we need to break our problem up into two cases: when $x > 2$ and when $x < 2$ (note, we do not need to consider $x = 2$, since this would give us a zero in the denominator). Let us first suppose that $x > 2$, then we know that $x - 2 \geq 0$, so we can multiply both sides of our inequality without needing to do anything special:

$$\begin{aligned}
\frac{1}{x-2} &\geq -3 \\
(x-2) \cdot \frac{1}{x-2} &\geq -3(x-2) \\
1 &\geq -3x + 6 \\
-5 &\geq -3x.
\end{aligned}$$

To keep going, we will need to multiply both sides by $-\frac{1}{3}$, and since this is negative, we must change the direction of our inequality!

$$\begin{aligned}
-5 &\geq -3x \\
\frac{5}{3} &\leq x.
\end{aligned}$$

At the start we required $x > 2$ and we just found that we need $x \geq \frac{5}{3}$ for $g(x)$ to be in the domain of f, so putting these inequalities together we find that we need $x > 2$ for $g(x)$ to be in the domain of f (i.e. we need x inside the interval $(2, \infty)$). (Note, we also could have done this case mentally. If $x > 2$ then we know $x - 2 > 0$ and so $\frac{1}{x-2} > 0 > -3$; hence so every $x > 2$ satisfies our inequality, which is what we found above!)

Now we need to consider our other case, $x < 2$. When $x < 2$ we stated above that we must change the direction of our inequality if we multiply by $x - 2$. So we see:

$$\begin{aligned} \frac{1}{x-2} &\geq -3 \\ (x-2) \cdot \frac{1}{x-2} &\leq -3(x-2) \\ 1 &\leq -3x + 6 \\ -5 &\leq -3x. \end{aligned}$$

Again, we will multiply by $-\frac{1}{3}$, so we need to change the direction of our inequality!

$$\begin{aligned} -5 &\leq -3x \\ \frac{5}{3} &\geq x. \end{aligned}$$

At the start we required $x < 2$, and we just found that we need $x \leq \frac{5}{3}$ for $g(x)$ to be in the domain of f. So, if we put these inequalities together, we find that we need $x \leq \frac{5}{3}$ for $g(x)$ to be in the domain of f (i.e. we need x inside the interval $(-\infty, \frac{5}{3}]$).

Putting our two cases together, we have found that if $x > 2$ or $x \leq \frac{5}{3}$, then $g(x)$ will be in the domain of f; which in interval notation can be written $(-\infty, \frac{5}{3}] \cup (2, \infty)$. So finally we can determine the domain of $f \circ g$:

$$\begin{aligned} \text{Domain of } f \circ g &= (\text{Domain of } g) \cap (x\text{-values where } g(x) \text{ is in the domain of } f) \\ &= ((-\infty, 2) \cup (2, \infty)) \cap \left(\left(-\infty, \frac{5}{3} \right] \cup (2, \infty) \right) \\ &= \left(-\infty, \frac{5}{3} \right] \cup (2, \infty) \\ &= \{x | x \leq \frac{5}{3}, x > 2\}. \end{aligned}$$