

# Singular Homology and Cohomology

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Recall that the standard  $n$ -simplex is given by

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

While simplicial homology relies on “nice” embeddings of  $\Delta^n \rightarrow X$  (e.g., if  $\sigma : \Delta^n \rightarrow X$ , then  $\sigma$  restricted to the interior of  $\Delta^n$  must be injective!), singular homology is a bit looser, only requiring the map  $\sigma : \Delta^n \rightarrow X$  to be continuous. The word “singular” is used to express the idea that  $\sigma$  need no longer be a “nice” embedding, but can have “singularities” where its image does not look at all like a simplex (e.g., faces/edges may nontrivially intersect).

## 1 Singular Homology

A **singular  $n$ -simplex** in a space  $X$  is a continuous map  $\sigma : \Delta^n \rightarrow X$ . Let  $C_n(X)$  denote the free abelian group with basis the set of singular  $n$ -simplices in  $X$ . Then, similar to simplicial chains, we have that the elements of  $C_n(X)$ , called  **$n$ -chains** (or **singular  $n$ -chains**), are finite formal sums  $\sum_i n_i \sigma_i$  for  $n \in \mathbb{Z}$  and  $\sigma_i : \Delta^n \rightarrow X$ , where  $\sigma_i$  is a singular  $n$ -simplex.

As in the simplicial case we have a boundary map  $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$  given by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]},$$

where  $\hat{v}_i$  indicates that  $v_i$  has been removed from the list, and  $[v_0, \dots, \hat{v}_i, \dots, v_n]$  is the canonical identification with  $\Delta^{n-1}$  (hence making  $\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} : \Delta^{n-1} \rightarrow X$ ).

This is (roughly) the same boundary map as in the simplicial case, so the proof that  $\partial_n \partial_{n+1} = 0$  readily follows, allowing us to define a chain complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Since we have  $\partial_n \partial_{n+1} = 0$ , it follows that  $\text{Im } \partial_{n+1} \subseteq \ker \partial_n$ , and hence we may define the **singular homology group**,

$$H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}.$$

*Example:* Let  $X = \{\text{pt.}\}$  be a one-point space, then for each  $n$  there is a unique singular  $n$ -simplex,  $\sigma_n$  (namely,  $\sigma_n \equiv \text{pt.}$ ). Thus,  $\partial(\sigma_n) = \sum_i (-1)^i \sigma_{n-1}$  is a sum of  $n+1$  terms, which will be 0 if  $n$  is odd, and  $\sigma_{n-1}$  if  $n$  is even (and  $n \neq 0$ ). Since  $C_n$  is the free abelian group generated by the singular  $n$ -simplices, and we only have one singular  $n$ -simplex, it follows that  $C_n \cong \mathbb{Z}$  for each  $n$ . Hence, we have the chain complex

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & C_4 & \xrightarrow{\partial_4} & C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\ & & & & & & \Downarrow & & & & & & \\ \dots & \longrightarrow & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

from which we can clearly see  $H_n(X) = 0$  for  $n > 0$  and  $H_0(X) \cong \mathbb{Z}$ .

In our definition of  $n$ -chains given above, we restricted the coefficients in the formal sum to be integer-valued, however, we can generalize the above homology theory by considering coefficients from any fixed abelian group,  $G$ , not just  $\mathbb{Z}$ .

Let  $G$  be a fixed abelian group, then we may construct chains of the form  $\sum_i n_i \sigma_i$  where each  $\sigma_i$  is a singular  $n$ -simplex and  $n_i \in G$ . Such  $n$ -chains form an abelian group, denoted  $C_n(X; G)$ . The boundary maps defined above can still be used for an arbitrary group, so it follows that the  $C_n(X; G)$  form a chain complex, allowing us to define the **homology groups with coefficients in  $G$** ,  $H_n(X, G)$ .

## 2 Equivalence of Simplicial and Singular Homology

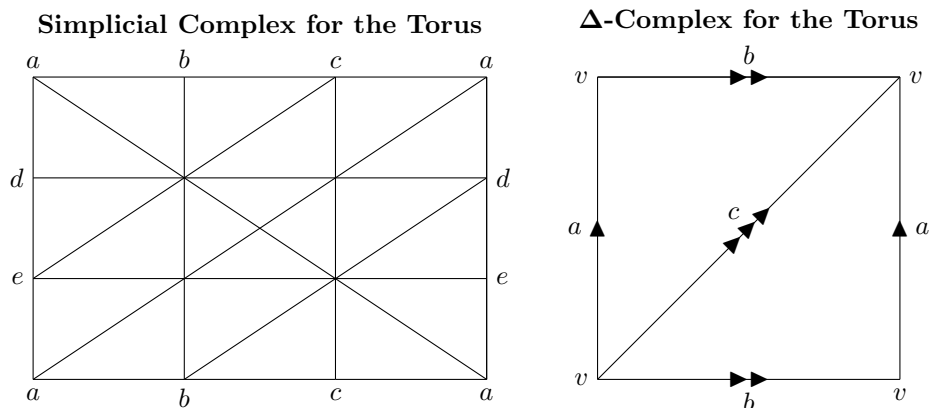
Before establishing the equivalence between simplicial and singular homology, we need to first make a few preliminary statements.

Recall that a  $\Delta$ -complex structure on a space  $X$  is a collection of maps  $\sigma_\alpha : \Delta^n \rightarrow X$ , with  $n$  depending on  $\alpha$  such that

1. The restriction of  $\sigma_\alpha$  to the interior of  $\Delta^n$  is injective, and each point of  $X$  is in the image of exactly one such restriction.
2. Each restriction of  $\sigma_\alpha$  to a face of  $\Delta^n$  is one of the maps  $\sigma_\beta : \Delta^{n-1} \rightarrow X$ .
3. A set  $A \subset X$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\Delta^n$  for each  $\sigma_\alpha$ .

### 2.1 Comparison of Simplicial and $\Delta$ - Complexes

Traditionally, simplicial homology is defined by simplicial complexes, which are simply  $\Delta$ -complexes whose simplices are uniquely determined by their vertices. It can be shown (Hatcher, Problem 2.1.23) that every  $\Delta$ -complex can be subdivided to be a simplicial complex. Hence, every  $\Delta$ -complex is homeomorphic to a simplicial complex.



**Theorem** – Homeomorphic spaces have isomorphic homology groups.

Since each  $\Delta$ -complex is homeomorphic to a simplicial complex, the above theorem then tells us that we may use  $\Delta$ -complexes to calculate the simplicial homology of a space, rather than the (more complicated) simplicial complexes.

### 2.2 Relative Homology Groups

It sometimes happens that by ignoring a certain amount of data or structure one obtains a simpler, more flexible theory, which can occasionally give results not readily obtainable in the original setting (e.g., working in  $\mathbb{Z}/n\mathbb{Z}$ ). Relative homology is an example for which this occurs; in this case, one ignores all singular chains in a subspace of the given space.

More concretely, given a space  $X$  and a subspace  $A \subset X$ , we let  $C_n(X, A)$  denote the quotient group  $C_n(X)/C_n(A)$ , i.e., chains in  $A$  are trivial in  $C_n(X, A)$ . The usual boundary map induces a boundary map on this quotient group giving rise to the relative homology groups,  $H_n(X, A)$ .

### 2.3 Equivalence Theorem

Let  $X$  be a space with  $A \subset X$ . For notation, we will let  $C_n(X, A)$  denote the singular relative  $n$ -chains, and  $\Delta_n(X, A)$  denote the simplicial relative  $n$ -chains. There is then a canonical homomorphism  $H_n^\Delta(X, A) \rightarrow H_n(X, A)$  induced by the chain map  $\Delta_n(X, A) \rightarrow C_n(X, A)$  sending each  $n$ -simplex of  $X$  to its characteristic map  $\sigma : \Delta^n \rightarrow X$ .

**Theorem** – The homomorphisms  $H_n^\Delta(X, A) \rightarrow H_n(X, A)$  are isomorphisms for all  $n$  and all  $\Delta$ -complex pairs  $(X, A)$ .

*Sketch of Proof:* The following lemma provides us with some long-exact sequences to work with:

**Lemma** – Let  $X$  be a space with  $A \subset X$ . There is a natural long exact sequence

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots,$$

where the map  $H_n(A) \rightarrow H_n(X)$  is induced by the inclusion map and  $H_n(X) \rightarrow H_n(X, A)$  is induced by the quotient map. The boundary map  $\partial H_n(X, A) \rightarrow H_{n-1}(A)$  has the following description: If a class  $[\alpha] \in H_n(X, A)$  is represented by a relative cycle  $\alpha$ , then  $\partial[\alpha]$  is the class of the cycle  $\partial\alpha$  in  $H_{n-1}(A)$ .

Let  $X^k$  denote the  $k$ -skeleton of  $X$ , consisting of all simplices of dimension  $k$  or less. Then using this lemma, along with the homomorphisms  $H_n^\Delta(X, A) \rightarrow H_n(X, A)$  we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^k, X^{k-1}) & \rightarrow & H_n^\Delta(X^{k-1}) & \rightarrow & H_n^\Delta(X^k) & \rightarrow & H_n^\Delta(X^k, X^{k-1}) & \rightarrow & H_{n-1}^\Delta(X^{k-1}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^k, X^{k-1}) & \rightarrow & H_n(X^{k-1}) & \rightarrow & H_n(X^k) & \rightarrow & H_n(X^k, X^{k-1}) & \rightarrow & H_{n-1}(X^{k-1}). \end{array}$$

Inductively, we may assume that the second and fifth (vertical) maps are isomorphisms (taking  $A = \emptyset$ ). The first and fourth (vertical) maps then are shown to be isomorphisms (which requires a bit of work). Since we know that all but the middle map are isomorphisms, it follows by the “Five-Lemma” that the middle map must also be an isomorphism (by the Five-Lemma), completing the proof.

(Justification that the first and fourth vertical maps are isomorphisms: The simplicial chain group  $\Delta_n(X^k, X^{k-1})$  is zero for  $n \neq k$ , and is free abelian with basis the  $k$ -simplices of  $X$  when  $n = k$ . Hence  $H_n^\Delta(X^k, X^{k-1})$  has exactly the same description. The corresponding singular homology groups  $H_n(X^k, X^{k-1})$  can be computed by considering the map  $\Phi : \coprod_\alpha (\Delta_\alpha^k, \partial\Delta_\alpha^k) \rightarrow (X^k, X^{k-1})$  formed by the characteristic maps  $\Delta^k \rightarrow X$  for all the  $k$ -simplices of  $X$ . Since  $\Phi$  induces a homeomorphism of quotient spaces  $\coprod_\alpha \Delta_\alpha^k / \coprod_\alpha \partial\Delta_\alpha^k \cong X^k / X^{k-1}$ , it induces isomorphisms on all singular homology groups. Thus  $H_n(X^k, X^{k-1})$  is zero for  $n \neq k$ , while for  $n = k$  this group is free abelian with basis represented by the relative cycles given by the characteristic maps of all the  $k$ -simplices of  $X$ , in view of the fact that  $H_k(\Delta^k, \partial\Delta^k)$  is generated by the identity map  $\Delta^k \rightarrow \Delta^k$ . Therefore, the map  $\Delta_k^\Delta(X^k, X^{k-1}) \rightarrow H_k(X^k, X^{k-1})$  is an isomorphism.)

## 3 Singular Cohomology

Above we calculated singular homology groups by first constructing a chain complex  $\cdots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots$  of singular chains, and then by computing  $\ker \partial_n / \text{Im } \partial_{n+1}$ . To obtain singular cohomology groups we replace the chain groups  $C_n$  by their dual groups  $\text{Hom}(C_n, G)$  and the boundary maps  $\partial_n$  by their dual maps  $\delta_n$ , then we compute the singular cohomology groups  $\ker \delta_n / \text{Im } \delta_{n-1}$ . Next seek to make this argument more formal.

Given a space  $X$  and an abelian group  $G$ , we define the group  $C^n(X; G)$  of **singular  $n$ -cochains with coefficients in  $G$**  to be the dual group  $\text{Hom}(C_n(X), G)$  of the singular chain group  $C_n(X)$ . Thus an  $n$ -cochain  $\phi \in C^n(X; G)$  assigns to each singular  $n$ -simplex,  $\sigma : \Delta^n \rightarrow X$ , a value  $\phi(\sigma) \in G$ .

Since the singular  $n$ -simplices form a basis for  $C_n(X)$ , these values can be chosen arbitrarily, and hence the  $n$ -cochains are exactly equivalent to functions from singular  $n$ -simplices to  $G$ .

The **coboundary map**  $\delta : C^n(X; G) \rightarrow C^{n+1}(X; G)$  is the dual  $\partial^*$ , so for a cochain  $\phi \in C^n(X; G)$ , its coboundary,  $\delta\phi$  is the composition

$$C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\phi} G.$$

Hence, for a singular  $(n+1)$ -simplex,  $\sigma : \Delta^{n+1} \rightarrow X$ , we have

$$\delta\phi(\sigma) = \sum_i (-1)^i \phi(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{n+1}]})$$

Since  $\delta$  is the dual of  $\partial$ , and  $\partial^2 = 0$  it follows that we have  $\delta^2 = 0$  as well. Hence, we may construct a chain complex

$$\begin{array}{ccccccc} \dots & \longleftarrow & C^{n+1}(X; G) & \xleftarrow{\delta^n} & C^n(X; G) & \xleftarrow{\delta^{n-1}} & C^{n-1}(X; G) & \longleftarrow & \dots & \longleftarrow & C^0(X; G) & \longleftarrow & 0 \\ & & & & & \Downarrow & & & & & & & & \\ \dots & \longleftarrow & \text{Hom}(C_{n+1}, G) & \xleftarrow{\delta^n} & \text{Hom}(C_n, G) & \xleftarrow{\delta^{n-1}} & \text{Hom}(C_{n-1}, G) & \longleftarrow & \dots & \longleftarrow & \text{Hom}(C_0, G) & \longleftarrow & 0, \end{array}$$

which allows us to define the **singular cohomology groups with coefficients in  $G$**  to be  $H^n(X; G) = \ker \delta^n / \text{Im } \delta^{n-1}$ .

## 4 Relationship Between Homology and Cohomology

We now seek a way to relate the concepts of homology and cohomology (in a way other than simply dualizing chains).

**Universal Coefficient Theorem for Cohomology** – If a chain complex  $C$  of free abelian groups has homology groups  $H_n(C)$ , then the cohomology groups  $H^n(C; G)$  of the cochain complex  $\text{Hom}(C_n, G)$  are determined by the split exact sequences

$$0 \longrightarrow \text{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \longrightarrow \text{Hom}(H_n(C), G) \longrightarrow 0.$$

In particular,  $H^n(C; G) \cong \text{Hom}(H_n(C), G) \oplus \text{Ext}(H_{n-1}(C), G)$ .

Note that computing  $\text{Ext}(H, G)$  for finitely generated  $H$  is not difficult using the following three properties:

- $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$ .
- $\text{Ext}(H, G) = 0$  if  $H$  is free.
- $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$ .

### References:

1. Allen Hatcher, *Algebraic Topology*. (2002) Cambridge University Press.
2. James Munkres, *Elements of Algebraic Topology*. (1984) Perseus Publishing.