

25 Ordered Rings and Fields

Let's consider the group $\mathbb{R} \setminus \{0\}$ (with operation \times); what would the easiest way to classify elements in this group into two equivalence classes?

Given an element, x , in $\mathbb{R} \setminus \{0\}$, it's trivial to determine whether x is positive or negative, i.e., it's trivial to determine if $0 < x$. It is, however, not always so easy to determine which of two arbitrary elements of this group is greater. For instance, consider $\frac{15}{23}$ and $\frac{94}{143}$, which is greater? (Converting to decimals, $\frac{15}{23} \approx 0.6521739130$ and $\frac{94}{143} \approx 0.6573426573$, so we can see this way.) One can compute $\frac{15}{23} - \frac{94}{143} \approx -0.005168744299$, which is clearly negative, so we can conclude that $\frac{94}{143}$ is larger.

Thus, given any two elements $x, y \in \mathbb{R} \setminus \{0\}$, we can determine which element is greater simply by determining whether $x - y$ is positive or negative. Hence, the entire problem of ordering elements boils down to knowing which elements are positive; and this is something we would like to carry over to more general rings and fields.

****NOTE**** We assume throughout this section that all rings have nonzero unity 1.

Definition

An **ordered ring** is a ring R together with a nonempty subset P of R satisfying the following properties:

1. **Closure** – For all $a, b \in P$, both $a + b$ and ab are in P .
2. **Trichotomy** – For each $a \in R$, one and only one of the following holds:

$$a \in P, \quad a = 0, \quad -a \in P.$$

Elements of P are called **positive**.

Obviously if R is an ordered ring with positive set P and S is a subring of R , then $P \cap S$ will be a positive set for S and hence gives an ordering of S ; known as the **induced ordering**.

The usual orderings on \mathbb{Z} , \mathbb{Q} and \mathbb{R} are usually referred to as the **natural orderings**.

We now give an unfamiliar example of an ordering:

Example

Let R be an ordered ring with positive set P . There are two natural ways to define an ordering of the polynomial ring $R[x]$. Let

$$f(x) = a_r x^r + a_{r+1} x^{r+1} + \cdots + a_n x^n,$$

where $a_r, a_n \neq 0$ and $r \leq n$ be a nonzero polynomial in $R[x]$; so clearly $a_r x^r$ is the lowest degree term and $a_n x^n$ is the highest degree term.

For our two orderings, define the following positive sets:

$$P_{\text{low}} = \{f(x) \in R[x] : a_r \in P\},$$

and

$$P_{\text{high}} = \{f(x) \in R[x] : a_n \in P\}.$$

The requisite properties for P_{low} and P_{high} to be positive sets follow directly from the definition of addition and multiplication in $R[x]$ and the fact that P is a positive set.

For instance, $f(x) = -2x + 3x^4 \in \mathbb{Z}[x]$ would be positive under the ordering given by P_{high} but negative under the ordering P_{low} where P is the natural ordering on \mathbb{Z} .

Theorem 25.3

Let R be an ordered ring. All squares of nonzero elements of R are positive, R has characteristic 0, and there are no zero divisors.

Proof. Let P be the positive set corresponding to the ordering on R . If $a \in R \setminus \{0\}$, then either a or $-a$ is in P , and so by closure, $a^2 = (-a)^2$ is also in P ; thus all squares are positive.

Using this, $1 = 1^2$ is always positive, so again by closure, $1 + 1 + \cdots + 1$ is always positive and hence never zero. Thus the characteristic of R is zero.

Finally, if $a, b \in R \setminus \{0\}$ then either a or $-a$ and either b or $-b$ is in P . Once again by closure, either ab or $-ab$ is in P . By trichotomy, ab cannot be zero, and so an ordered ring can have no zero divisors. □

Corollary 25.4

We can consider \mathbb{Z} to be embedded in any ordered ring R , and the induced ordering of \mathbb{Z} from R is the natural ordering of \mathbb{Z} . The only possible ordering of \mathbb{R} is the natural ordering.

Theorem 25.5

Let R be an ordered ring with positive set P . Let $<$, read “is less than”, be the relation on R defined by

$$a < b \text{ if and only if } (b - a) \in P$$

for $a, b \in R$. The relation $<$ has the following properties for all $a, b, c \in R$:

1. **Trichotomy** – One and only one of the following holds:

$$a < b, \quad a = b, \quad b < a. \tag{1}$$

2. **Transitivity** – If $a < b$ and $b < c$, then $a < c$.

3. **Isotonicity** – If $b < c$, then $a + b < a + c$. If $b < c$ and $0 < a$, then $ab < ac$ and $ba < ca$.

Conversely, given a relation $<$ on a nonzero ring R satisfying these three conditions, the set $P = \{x \in R : 0 < x\}$ satisfies the two criteria for a set of positive elements, and the relation $<_P$ defined as in (1) with this P is the given relation $<$.

The proof of this theorem follows easily from the definitions.

Example

Let R be an ordered ring, and consider the orderings of $R[x]$ given by P_{low} and P_{high} .

Taking P_{low} , for every $a > 0$ in R , $a - x$ is positive and hence $x < a$. Also, since 1 is always positive, $x = x - 0$ is positive, and so $0 < x$. Thus, $0 < x < a$ for every $a \in R$.

Similarly, $x^i - x^j \in P_{\text{low}}$ whenever $i < j$, and hence $x^j < x^i$ if $i < j$. Thus, monomials have the ordering

$$0 < \cdots < x^6 < x^5 < x^4 < x^3 < x^2 < x < a$$

for any positive $a \in R$.

Taking $R = \mathbb{Z}$ in this example we see that in this ordering of $\mathbb{Z}[x]$ there are infinitely many positive elements that are less than any positive integer!

****NOTE**** There is a typo in the textbook. In Example 25.6 (p. 230) the text says to take $R = \mathbb{R}$; however, this leads to a trivial conclusion.

The preceding example is of interest because it is an example of an ordering which is **not Archimedean**.

Definition

An ordering of a ring R is an **Archimedean ordering** if for every positive $a, b \in R$ there exists a positive integer n such that $na > b$.

We now want to give two examples describing types of ordered rings and fields that can be thought of as generalizations of the ring $R[x]$.

Let R be a ring and F be a field. We know

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_j \in R \text{ with } a_j = 0 \text{ for all but finitely many } j \right\},$$

and we can generalize this by no longer putting a requirement on the coefficients:

$$R[[x]] = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_j \in R \right\}.$$

We can further generalize this idea by no longer requiring the powers of x to be positive:

$$F((x)) = \left\{ \sum_{i=N}^{\infty} a_i x^i : a_j \in F \right\},$$

where N can be any integer (positive or negative!).

We say that $R[[x]]$ is the **ring of formal power series**, and as the name would suggest, it is a ring. The set of positive elements P_{low} from $R[x]$ trivially extends to this ring; however, we cannot use P_{high} .

We call $F((x))$ the **field of formal Laurent series**, again, as the same suggests, it is a field (inverses of elements can be found via long division). And again P_{low} extends to this field, but P_{high} does not.

Theorem 25.10

Let R be an ordered ring with positive set P and let $\phi : R \rightarrow R'$ be a ring isomorphism. The subset $P' = \phi(P)$ satisfies the requirements for a positive set of elements of R' and, furthermore, in the ordering of R' given by P' we have $\phi(a) < \phi(b)$ in R' if and only if $a < b$ in R .

Definition

An isomorphism from a ring (or group/field) R onto itself is called an **automorphism** of R , and a homomorphism from R to itself is called an **endomorphism**.

Example

Consider the ring $\mathbb{Z}[\sqrt{2}]$, and define an automorphism ϕ by $\phi(m+n\sqrt{2}) = m-n\sqrt{2}$. Since $\sqrt{2}$ is positive, it follows that its image $\phi(\sqrt{2}) = -\sqrt{2}$ must also be positive in the ordering induced by ϕ (as in Theorem 25.10)!

Theorem 25.13

Let D be an ordered integral domain with positive set P , and let F be a field of quotients of D . The set

$$P' = \left\{ x \in F : x = \frac{a}{b} \text{ for } a, b \in D \text{ and } ab \in P \right\}$$

is well defined and gives an order on F that induces the given order on D . Furthermore, P' is the only subset of F with this property.