MODIFIED MANIN SYMBOLS PROPERTIES, EXAMPLE & IDEAS

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1. Basic Properties

Suppose we are in the quotient of $H_1(X_1(p), C_1^0, \mathbb{Z}_p)^+$ on which the diamond operators $\langle d\rangle^{-1}$ act via a fixed character χ. Then we have, for $u, v \neq 0$,

$$[u : v] = \chi(v)[uv^{-1} : 1],$$

since

$$[u : v] = \langle v\rangle^{-1}[uv^{-1} : 1] = \chi(v)[uv^{-1} : 1].$$

Any Manin symbol with nonzero entries can now be written in the form $\chi(b)[a : 1]$ for some a, b, so we define a modified Manin symbol to be $[a]$ where $[a] = [a : 1]$.

We have the "standard" relations on Manin symbols,

$$\begin{aligned} [u : v] &= [-u : -v] && (1.1)\\ [u : v] &= -[-v : u] && (1.2)\\ [u : v] &= [u : u + v] + [u + v : v], && (1.3) \end{aligned}$$

and since we are in the plus-part,

$$[u : v] = [-u : v]. \qquad (1.4)$$

Applying these relations to our modified symbols we see

$[u] = [u : 1] = -[1 : u] = -\chi(u)[u^{-1} : 1] = -\chi(u)[u^{-1}]$ (by (1.3) & (1.4))

$[u] = [u : 1] = [-u : 1] = [-u]$ (by (1.4))

$[u] = [u : 1] = [-u : -1] = \chi(-1)[-u(-1)^{-1} : 1] = \chi(-1)[u : 1] = \chi(-1)[u]$ (by (1.2))

and

$$\begin{aligned} [u] = [u : 1] = [u : u + 1] + [u + 1 : 1] &= -[u + 1 : u] + [u + 1]\\ &= -\chi(u)[(u + 1)u^{-1} : 1] + [u + 1]\\ &= -\chi(u)[u^{-1} + 1] + [u + 1]. \qquad \text{(by (1.3), (1.3) \& (1.4))} \end{aligned}$$

Simplifying the above we have

$$\chi(-1) = 1 \ (\chi \text{ is even}) \tag{1.5}$$
$$[u] = [-u] \tag{1.6}$$
$$[u] = -\chi(u)[u^{-1}] \tag{1.7}$$
$$[u] = -\chi(u)[u^{-1}+1] + [u+1]. \tag{1.8}$$

Now we consider the T_2-action on these modified symbols,

$$\begin{aligned} T_2[u] = T_2[u:1] &= [u:2] + [2u:1] + [2u:u+1] + [u+1:2] \\ &= \chi(2)\left[\frac{u}{2}\right] - \chi(2u)\left[\frac{u^{-1}}{2}\right] - \chi(2u)\left[\frac{u^{-1}+1}{2}\right] + \chi(2)\left[\frac{u+1}{2}\right] \\ &= \chi(2)\left(\left[\frac{u}{2}\right] + \left[\frac{u+1}{2}\right]\right) - \chi(2u)\left(\left[\frac{u^{-1}}{2}\right] + \left[\frac{u^{-1}+1}{2}\right]\right). \end{aligned}$$

Finally, we are interested in relations obtained from the equality $(2 + \langle 2\rangle^{-1})[u] = T_2[u]$, i.e.,

$$(2+\chi(2))[u] = \chi(2)\left(\left[\frac{u}{2}\right] + \left[\frac{u+1}{2}\right]\right) - \chi(2u)\left(\left[\frac{u^{-1}}{2}\right] + \left[\frac{u^{-1}+1}{2}\right]\right),$$

which can also be written as

$$(1+2\chi(2)^{-1})[u] = \left[\frac{u}{2}\right] + \left[\frac{u+1}{2}\right] - \chi(u)\left(\left[\frac{u^{-1}}{2}\right] + \left[\frac{u^{-1}+1}{2}\right]\right).$$

We also note that

$$(1+2\chi(2)^{-1})[1] = \left[\frac{1}{2}\right] + [1] - \chi(1)\left(\left[\frac{1}{2}\right] + [1]\right).$$

Since $\chi(1) = 1$ for any character, it follows that $[1] = 0$. Using this, along with relation (1.6), we may restrict our attention to symbols $[u]$ for which $2 \le u \le \frac{p-1}{2}$.

2. Example

Let $p = 11$, fix an even character χ, and for notation set $\beta = 1 + 2\chi(2)^{-1}$. For reference, we include a table of the possible even characters modulo 11:

	1	2	3	4	5	6	7	8	9	10
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	$e^{2i\pi/5}$	$e^{-4i\pi/5}$	$e^{4i\pi/5}$	$e^{-2i\pi/5}$	$e^{-2i\pi/5}$	$e^{4i\pi/5}$	$e^{-4i\pi/5}$	$e^{2i\pi/5}$	1
χ_3	1	$e^{4i\pi/5}$	$e^{2i\pi/5}$	$e^{-2i\pi/5}$	$e^{-4i\pi/5}$	$e^{-4i\pi/5}$	$e^{-2i\pi/5}$	$e^{2i\pi/5}$	$e^{4i\pi/5}$	1
χ_4	1	$e^{-4i\pi/5}$	$e^{-2i\pi/5}$	$e^{2i\pi/5}$	$e^{4i\pi/5}$	$e^{4i\pi/5}$	$e^{2i\pi/5}$	$e^{-2i\pi/5}$	$e^{-4i\pi/5}$	1
χ_5	1	$e^{-2i\pi/5}$	$e^{4i\pi/5}$	$e^{-4i\pi/5}$	$e^{2i\pi/5}$	$e^{2i\pi/5}$	$e^{-4i\pi/5}$	$e^{4i\pi/5}$	$e^{-2i\pi/5}$	1

From Section 1 we may focus on the T_2-action on symbols $[u]$ for $2 \le u \le \frac{p-1}{2}$. Upon computing the action (and using the relation $[u] = [-u]$ where appropriate) we see

$$\beta[2] = [4] - \chi(2)([3] + [2]), \tag{2.1}$$
$$\beta[3] = [4] + [2] - \chi(3)([2] + [3]), \tag{2.2}$$
$$\beta[4] = [2] + [3] - \chi(4)([4] + [2]), \tag{2.3}$$
$$\beta[5] = [3] + [3] - \chi(5)([5]). \tag{2.4}$$

Rewriting (2.1) we see

$$(\beta + \chi(2))[2] = [4] - \chi(2)[3].$$

We note here that $\beta + \chi(u) = \frac{\chi(2)(\chi(u)+1)+2}{\chi(2)}$ equals zero if and only if $\chi(2) = -1$ and $\chi(u) = 1$, which we see from our table never happens for an even character.

Thus, we may write the symbol [2] as

$$[2] = \frac{1}{\beta + \chi(2)}([4] - \chi(2)[3]). \tag{2.5}$$

Moving on to the symbol [3], from (2.2) we have

$$(\beta + \chi(3))[3] = [4] + (1 - \chi(3))[2], \tag{2.6}$$

which brings us to our first cases.

Case 1 ($\chi(3) = 1$)**.** *If $\chi(3) = 1$ (which would mean χ is identically 1 in this example), then it follows*

$$[3] = \frac{1}{\beta + \chi(3)}[4] = \frac{1}{4}[4] \tag{2.7}$$

Case 2 ($\chi(3) \neq 1$)**.** *We replace the symbol [2] in (2.6) according to our equality from (2.5) to see*

$$(\beta + \chi(3))[3] = [4] + \frac{1 - \chi(3)}{\beta + \chi(2)}([4] - \chi(2)[3]).$$

Which can be rewritten as

$$\left(\beta + \chi(3) + \frac{\chi(2)(1 - \chi(3))}{\beta + \chi(2)}\right)[3] = \left(1 + \frac{1 - \chi(3)}{\beta + \chi(2)}\right)[4].$$

One may now check (Mathematica works nicely) that these coefficients on [3] and [4] are always nonzero for the characters in question. Hence,

$$[3] = \left(1 + \frac{1 - \chi(3)}{\beta + \chi(2)}\right)\left(\frac{1}{\beta + \chi(3) + \frac{\chi(2)(1-\chi(3))}{\beta+\chi(2)}}\right)[4]. \tag{2.8}$$

Looking at the symbol [4], from (2.3) we see

$$(\beta + \chi(4))[4] = (1 - \chi(4))[2] + [3]. \tag{2.9}$$

Similar to what we saw for the symbol [3], we again have two cases.

Case 3 ($\chi(4) = 1$)**.** *If $\chi(4) = 1$ (which again would mean that χ is identically 1 in this example), then*

$$4[4] = [3].$$

Using this, plus the equality in (2.7) from Case 1 (where we also have assumed χ is identically 1), we see

$$4[4] = [3] = \frac{1}{4}[4],$$

which implies that $[4] = 0$. From this if then follows (from (2.7)) that $[3] = 0$, and hence (from (2.5)) $[2] = 0$. Thus, $[4] = [3] = [2] = [1] = 0$.

Case 4 ($\chi(4) \neq 1$)**.** *We begin by replacing the symbols* [2] *and* [3] *in* (2.9) *according to* (2.5) *and* (2.8)*, respectively, to see*

$$(\beta+\chi(4))[4] = (1-\chi(4))\left(\frac{1}{\beta+\chi(2)}\right)([4]-\chi(2)[3])+\left(1+\frac{1-\chi(3)}{\beta+\chi(2)}\right)\left(\frac{1}{\beta+\chi(3)+\frac{\chi(2)(1-\chi(3))}{\beta+\chi(2)}}\right)[4].$$

This can be rewritten as

$$\left(\beta+\chi(4)-\frac{1-\chi(4)}{\beta+\chi(2)}-\left(1+\frac{1-\chi(3)}{\beta+\chi(2)}\right)\left(\frac{1}{\beta+\chi(3)+\frac{\chi(2)(1-\chi(3))}{\beta+\chi(2)}}\right)\right)[4] = \frac{-\chi(2)(1-\chi(4))}{\beta+\chi(2)}[3] \tag{2.10}$$

Once again using (2.8) *to substitute for* [3] *in* (2.10) *and rewriting we find*

$$\left(\beta+\chi(4)-\frac{1-\chi(4)}{\beta+\chi(2)}-\left(1+\frac{1-\chi(3)}{\beta+\chi(2)}\right)\left(\frac{1}{\beta+\chi(3)+\frac{\chi(2)(1-\chi(3))}{\beta+\chi(2)}}\right)\left(-1+\frac{\chi(2)(1-\chi(4))}{\beta+\chi(2)}\right)\right)[4] = 0. \tag{2.11}$$

One may now check (Mathematica was used again here) that the coefficient on the symbol [4] *in* (2.11) *is nonzero for any of our even characters. Thus,* $[4] = 0$*, and we see from Case 2, equation* (2.8)*, that* $[3] = 0$*. Finally, (similar to Case 3) equation* (2.5) *then tells us that* $[2] = 0$*, so once again we have* $[4] = [3] = [2] = [1] = 0$*.*

Lastly, we want to examine the symbol [5], keeping in mind that we have shown above (regardless of the choice of even character) that $[3] = 0$. From (2.4) we have

$$\beta[5] = -\chi(5)[5],$$

or equivalently,

$$(\beta+\chi(5))[5] = 0. \tag{2.12}$$

Since we know $\beta + \chi(5) \neq 0$, it then follows that $[5] = 0$.

Therefore, we have shown that, for any even character χ, the T_2-action will give us the relation $[u] = 0$ for all $1 \leq u \leq 5$. This implies that the χ-eigenspace of $H_1(X_1(11), C_1^0, \mathbb{Z}_p)/T_2H_1(X_1(11), C_1^0, \mathbb{Z}_p)$ has dimension zero for every even character, and hence the entire quotient has dimension zero.

3. Ideas

With Propositions 4.0.1, 4.0.2 and 4.0.3 below, we should be able to predict when and where a particular symbol appears throughout all possible T_2-relations for a fixed prime. Using this knowledge, one should then be able to determine a relationship between values of a fixed χ which will determine whether we can eliminate particular symbols as being necessary generators, as seen in the example above.

We know that for any p we only need to consider symbols with $2 \leq u \leq \frac{p-1}{2}$. Thus far (as seen in our example above), we have been assuming the best course of action is to begin with the symbol [2], write it in terms of other symbols, then move on to [3] doing the same thing (making any substitutions where necessary), then [4], etc. We are currently in the process of attempting to do this for a general p, which complicates matters a bit since we don't know what inverses will look like for an arbitrary prime. However, it is our hope that the propositions from Section 4 will still allow us to identify equivalent symbols and make substitutions where appropriate. However, we are unsure how to completely generalize the process, i.e., right now we are simply moving through the symbols by one starting at [2] which makes it difficult to predict what the relation for $[\frac{p-1}{2}]$ will look like, without recursively going through all smaller symbols.

With this in mind, we hope there may be another (smarter) way to traverse the symbols making substitutions that would appear more regular and predicable for general p, e.g., would it be smarter to begin with a general $[u]$, then possibly move to $[2u]$, then $[3u]$, etc.

4. Relations Among Symbols and Actions

We first fix a character χ and a u with $2 \leq u \leq \frac{p-1}{2}$ and consider the relation

$$(1+2\chi(2)^{-1})[u] = \left[\frac{u}{2}\right] + \left[\frac{u+1}{2}\right] - \chi(u)\left(\left[\frac{u^{-1}}{2}\right] + \left[\frac{u^{-1}+1}{2}\right]\right). \tag{4.1}$$

In this section we describe conditions for which either the symbol $[u]$, or any of the symbols on the right-hand-side of (4.1) appear in the relation for some other symbol $[u+n]$. In other words, the relation for $[u+n]$ is

$$(1+2\chi(2)^{-1})[u+n] = \left[\frac{u+n}{2}\right] + \left[\frac{u+n+1}{2}\right] - \chi(u+n)\left(\left[\frac{(u+n)^{-1}}{2}\right] + \left[\frac{(u+n)^{-1}+1}{2}\right]\right), \tag{4.2}$$

so we describe when $[u]$ appears on the right-hand-side of (4.2) and also when the terms on the right-hand-side of (4.1) appear anywhere in (4.2).

For notation, we will write $[u]_i$ to refer to the ith term on the right-hand-side of one of our relations, i.e., $[u]_1 = \left[\frac{u}{2}\right]$, $[u]_2 = \left[\frac{u+1}{2}\right]$, etc. Using this notation we can state not only when a symbol reappears, but exactly where it does in the relation.

Proposition 4.0.1. *The symbol $[u]$ appears in the relation (and location) for the following symbols:*

$$\begin{array}{cccc} [2u]_1 & [2u-1]_2 & \left[(2u)^{-1}\right]_3 & \left[(2u-1)^{-1}\right]_4 \\ [-2u]_1 & [-2u-1]_2 & \left[-(2u)^{-1}\right]_3 & \left[(-2u-1)^{-1}\right]_4 . \end{array}$$

Proof. Immediate after investigating the congruences between $[u]$ and $[u+n]_i$ for $i = 1, 2, 3, 4$. □

We also note that it is possible for $[u]$ to appear on the right-hand-side of its own relation.

Proposition 4.0.2. *The symbol $[u]$ appears as one of the $[u]_i$ under the following conditions:*

1. $[u] = [u]_2$ *if and only if* $3u \equiv -1 \pmod p$
2. $[u] = [u]_3$ *if and only if* $2u^2 \equiv \pm 1 \pmod p$
3. $[u] = [u]_4$ *if and only if* $2u \equiv -1 \pmod p$ *or* $2u^2 + u \equiv -1 \pmod p$

Proof. Immediate after investigating congruences once again. □

After detailing the instances for which $[u]$ may or may not appear in other relations, we now would like to describe when any of the $[u]_i$ appear as $[u+n]_j$ for some j, e.g., under what conditions does $\left[\frac{u}{2}\right] = [u]_1 = [u+n]_4 = \left[\frac{(u+n)^{-1}+1}{2}\right]$.

Proposition 4.0.3. *The symbol $[u]_i$ is equivalent to the symbol $[u+n]_j$ under the following conditions:*

(1) $n \equiv \frac{1-u^2}{u} \pmod p$ (*i.e.,* $[u+n] = [u^{-1}]$) $\Rightarrow [u]_1 = [u+n]_3, [u]_2 = [u+n]_4, [u]_3 = [u+n]_1, [u]_4 = [u+n]_2$.

(2) $n \equiv \frac{u^2-u-1}{1-u} \pmod p$ (*i.e.,* $[u+n] = [(u-1)^{-1}]$) $\Rightarrow [u]_1 = [u+n]_4$.

(3) $n \equiv \frac{1-u-u^2}{1+u} \pmod p$ (*i.e.,* $[u+n] = [(u+1)^{-1}]$) $\Rightarrow [u]_2 = [u+n]_3$.

(4) $n \equiv \frac{1-u-u^2}{u} \pmod p$ (*i.e.,* $[u+n] = [u^{-1}+1]$) $\Rightarrow [u]_3 = [u+n]_2$.

(5) $n \equiv \frac{u^2}{1-u} \pmod p$ (*i.e.,* $[u+n] = [(u^{-1}-1)^{-1}]$) $\Rightarrow [u]_3 = [u+n]_4$.

(6) $n \equiv \frac{1+u-u^2}{u} \pmod p$ (*i.e.,* $[u+n] = [u^{-1}+1]$) $\Rightarrow [u]_4 = [u+n]_1$.

(7) $n \equiv \frac{u^2}{-1-u} \pmod p$ (*i.e.,* $[u+n] = [(u^{-1}+1)^{-1}]$) $\rightarrow [u]_4 = [u+n]_3$.

(8) $n \equiv -1 \pmod p$ (*i.e.,* $[u+n] = [u-1]$) $\Rightarrow [u]_1 = [u+n]_2$.

(9) $n \equiv 1 \pmod p$ (*i.e.,* $[u+n] = [u+1]$) $\Rightarrow [u]_2 = [u+n]_1$.

Proof. Immediate. □

We also note here that for some n-values particular cases from Proposition 4.0.3 correspond. For example, if $n = u+1$, then cases (5) and (6) can be combined – both equivalences then become $2u^2 \equiv 1 \pmod p$. Similarly, if $n = u-1$, then cases (4) and (7) are equivalent to $2u^2 \equiv 1 \pmod p$.

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