

# Hida Theory

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# Outline

- 1 Ordinary Parts
- 2  $\Lambda$ -adic Forms
  - $\Lambda$ -adic Eisenstein Series
  - Ordinary  $\Lambda$ -adic Forms
  - Universal Ordinary Hecke Algebra
- 3 Ordinary  $\Lambda$ -adic Galois Representations
  - Pseudo-Representations

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# Ordinary Parts

- Let  $U$  be an indeterminate.
- $M$  a  $\mathbb{Z}_p[U]$ -module finitely generated as a  $\mathbb{Z}_p$ -module.

Consider the  $\mathbb{Z}_p$ -module morphism

$$\mathbb{Z}_p[U] \longrightarrow \text{End}_{\mathbb{Z}_p}(M).$$

The image of  $\mathbb{Z}_p[U]$  in  $\text{End}_{\mathbb{Z}_p}(M)$  is a finite  $\mathbb{Z}_p$ -algebra; call it  $A$ .

- $A = \prod A_i$  where each  $A_i$  is local.
- Project  $U$  onto these local factors of  $A$ .
- Let  $A^{\text{ord}}$  denote the product of local factors of  $A$  in which the image of  $U$  is a unit.

## Definition

The **ordinary part** of  $M$  is defined to be

$$M^{\text{ord}} = M \otimes_A A^{\text{ord}}.$$

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# Ordinary Parts

## Ordinary Projector

The **ordinary projector** on  $\mathcal{H}_k(qp^r, \chi; \mathcal{O})$  is  $e = \lim_{n \rightarrow \infty} T_p^n$ .

- If  $f \in \mathcal{M}_k(qp^r, \chi; \mathcal{O})$  is an eigenform of  $T_p$  with eigenvalue  $c_p$  then

$$f|_e = \begin{cases} f & \text{if } |c_p|_p = 1, \\ 0 & \text{if } |c_p|_p < 1. \end{cases}$$

- $f \in \mathcal{M}_k(qp^r, \chi; \mathcal{O})$  is **ordinary** if  $f|_e = f$ .
- $\mathcal{M}_k^{\text{ord}}(qp^r, \chi; \mathcal{O}) = \{f|_e : f \in \mathcal{M}_k(qp^r, \chi; \mathcal{O})\}$
- $\mathcal{H}_k^{\text{ord}}(qp^r, \chi; \mathcal{O}) = \{eH : H \in \mathcal{H}_k(qp^r, \chi; \mathcal{O})\}$

The spaces  $\mathcal{S}_k^{\text{ord}}$  and  $h_k^{\text{ord}}$  are defined similarly.

# Ordinary Parts

- Let  $A$  be a subring of  $\mathbb{C}$  with  $\mathbb{Z}[\chi] \subset A$ .
- $K = \text{Frac}(A)$ .
- $\mathcal{M}_{k,0}(N, \chi; A) = \{f \in \mathcal{M}_k(N, \chi; K) : a_n(f) \in A \forall n \geq 1\}$ .

## Proposition

For every  $k \geq 0$  we have

$$\begin{aligned}\text{Hom}_{\mathcal{O}}(\mathcal{H}_k^{\text{ord}}(qp^r, \chi; \mathcal{O}), \mathcal{O}) &\cong \mathcal{M}_{k,0}^{\text{ord}}(qp^r, \chi; \mathcal{O}), \\ \text{Hom}_{\mathcal{O}}(\mathcal{M}_{k,0}^{\text{ord}}(qp^r, \chi; \mathcal{O}), \mathcal{O}) &\cong \mathcal{H}_k^{\text{ord}}(qp^r, \chi; \mathcal{O}).\end{aligned}$$

And an equivalent result holds for cusp forms and their corresponding Hecke algebras.

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And an equivalent result holds for cusp forms and their corresponding Hecke algebras.

# Ordinary Parts

## Theorem

Let  $k \geq 2$  and  $\omega$  be the Teichmüller character, then

$$\begin{aligned} \operatorname{rank}_{\mathcal{O}} \mathcal{H}_k^{\text{ord}}(qp^r, \chi\omega^{-k}; \mathcal{O}) &= \operatorname{rank}_{\mathcal{O}} \mathcal{M}_k^{\text{ord}}(qp^r, \chi\omega^{-k}; \mathcal{O}) \\ &= \operatorname{rank}_{\mathcal{O}} \mathcal{M}_2^{\text{ord}}(qp^r, \chi\omega^{-2}; \mathcal{O}). \end{aligned}$$

The equivalent result also holds for ordinary cusp forms.



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# $\Lambda$ -adic Forms

Let us fix some notation:

- Dirichlet Characters
  - $\chi$  – level  $Np$
  - $\chi_\zeta$  – conductor  $p^r$  (If  $|\zeta| = p^{r-1}$ ,  $\chi_\zeta$  maps the image of  $u \in U_1 = 1 + p\mathbb{Z}_p$  in  $(\mathbb{Z}/p^r\mathbb{Z})^\times$  to  $\zeta$ .)
- $\nu_p$  –  $p$ -adic cyclotomic character ( $g(\zeta) = \zeta^{\nu_p(g)}$   
 $\forall g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and for all  $p$ -power roots of unity,  $\zeta$ .)
- $\Lambda = \mathbb{Z}_p[[X]]$  – Iwasawa algebra.
- $K$  – finite field extension of  $\text{Frac}(\Lambda)$ .
- $I$  – integral closure of  $\Lambda$  in  $K$ .
- $\kappa$  – character mapping  $U_1 \rightarrow \Lambda^\times$  with  $u \mapsto 1 + X$ , i.e.,

$$\kappa(u^s) = (1 + X)^s = \sum_{m=0}^s \binom{s}{m} X^m \in \Lambda^\times.$$

# $\Lambda$ -adic Forms

## Definition

A  **$\Lambda$ -adic form**  $F$  of level  $N$  and character  $\chi$  is a formal  $q$ -expansion

$$F = \sum_{n=0}^{\infty} a_n(F)q^n \in I[[q]],$$

such that for all specializations  $\nu : I \rightarrow \overline{\mathbb{Q}_p}$  extending

$$\nu_{k,\zeta} : \Lambda \rightarrow \overline{\mathbb{Q}_p} \quad x \mapsto \zeta u^k - 1,$$

where  $k > 1$  and  $\zeta \in \mu_p^{r-1}$  with  $r \geq 1$ , the specialized  $q$ -expansion

$$f_\nu = \nu(F) = \sum_{n=0}^{\infty} \nu(a_n(F))q^n \in \overline{\mathbb{Q}_p}[[q]]$$

is the image under a fixed embedding  $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}_p}$  of the  $q$ -expansion of a classical modular form of weight  $k$ , level  $Np^r$  and character  $\chi_\nu = \chi\omega^{-k}\chi_\zeta$ .

# $\Lambda$ -adic Forms

Let  $F$  be a  $\Lambda$ -adic form.

- If the  $q$ -expansion of  $F$  is in  $I[[q]]$ ,  $F$  is called an  **$I$ -adic form**.
- If all the specializations  $f_\nu = F(\zeta u^k - 1)$  are cusp forms,  $F$  is **cuspidal**.

Let

$$\mathcal{M}(N, I) = \bigoplus_{\chi} \mathcal{M}(N, \chi, I) \quad \mathcal{S}(N, I) = \bigoplus_{\chi} \mathcal{S}(N, \chi, I)$$

denote the spaces of all  $I$ -adic forms and cusp forms, respectively.



# $\Lambda$ -adic Hecke Operator

We would like to describe a Hecke operator which acts on  $\mathcal{M}(N, I)$  and  $\mathcal{S}(N, I)$ .

For notation, let  $\langle n \rangle := \omega(n)^{-1}n$ .

If  $F \in \mathcal{M}(N, I)$ , then the coefficients of the formal  $q$ -expansion for  $T_n F$  are given by

$$a_m(T_n F)(X) = \sum_{b|(m,n)} \kappa(\langle b \rangle)(X) \chi(b) b^{-1} a_{mn/b^2}(F)(X).$$



# $\Lambda$ -adic Eisenstein Series

Recall, the classical Eisenstein series is given by:

$$E_{k,\chi}(z) = \frac{L(1-k, \chi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n \in \mathcal{M}_k(N, \chi)$$

where  $\sigma_{k-1,\chi}(n) = \sum_{d|n} \chi(d) d^{k-1}$ .

Note that  $E_{k,\chi}$  has level  $N$ , but  $\Lambda$ -adic forms interpolate modular forms of weight divisible by  $Np$ .

# $\Lambda$ -adic Eisenstein Series

## Definition

The  **$p$ -stabilization** of  $E_{k,\chi}$  is given by

$$E_{k,\chi}^{(p)}(z) = E_{k,\chi}(z) - \chi(p)p^{k-1}E_{k,\chi}(pz),$$

which has  $q$ -expansion

$$E_{k,\chi}^{(p)}(z) = \frac{L^{(p)}(1-k, \chi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n)q^n,$$

where

$$\sigma_{k-1,\chi}^{(p)}(n) = \sum_{\substack{d|n \\ (d,p)=1}} \chi(d)d^{k-1}$$

and

$$L^{(p)}(s, \chi) = (1 - \chi(p)p^{-s})L(s, \chi),$$

and is of level divisible by  $Np$ .

# $p$ -adic $L$ -Functions

- 1964 – Kubota & Leopoldt
  - Analytic  $p$ -adic  $L$ -functions
  - $p$ -adic interpolation of special values of Dirichlet  $L$ -functions.
- 1972 – Iwasawa
  - Arithmetic  $p$ -adic  $L$ -functions
  - Stems from the arithmetic of cyclotomic fields.

## Definition

The **Kubota-Leopoldt  $p$ -adic  $L$ -function**,  $L_p(s, \chi)$  is a continuous function for  $s \in \mathbb{Z}_p \setminus \{1\}$  satisfying

$$L_p(1 - k, \chi) = (1 - \chi\omega^{-k}(p)p^{k-1})L(1 - k, \chi\omega^{-k})$$

for  $k \geq 1$ .

# $p$ -adic $L$ -Functions

For notation, set

$$H_\chi(X) = \begin{cases} \chi(u)(1-X) - 1 & \text{if } \chi \text{ is of type } U_1, \\ 1 & \text{otherwise.} \end{cases}$$

Iwasawa showed the existence of a unique power series  $G_\chi(X) \in I$  such that

$$L_p(1-s, \chi) = \frac{G_\chi(u^s - 1)}{H_\chi(u^s - 1)}.$$

Also, if  $\rho$  is a character of type  $U_1$  then

$$G_{\chi\rho}(X) = G_\chi(\rho(u)(1+X) - 1).$$

# $\Lambda$ -adic Eisenstein Series

## Proposition

If  $\chi \neq 1$ , then there is a  $\Lambda$ -adic form

$$\mathcal{E}_\chi = \sum_{n=0}^{\infty} A_{n,\chi}(X)q^n \in I[[q]]$$

which specializes to  $E_{k,\psi}^{(p)}$ , with  $\psi = \chi\omega^{-k}\chi_\zeta$  under the homomorphism of  $I$  to  $\overline{\mathbb{Q}_p}$  induced by  $\nu_{k,\zeta}$  for  $k > 1$  and  $\zeta \in \mu_{p^r-1}$  for  $r \geq 1$ .

If  $\chi = 1$  then  $\mathcal{E}_\chi$  still exists, but is, strictly speaking, not a  $\Lambda$ -adic form since its constant term will have denominator  $X$ .

# $\Lambda$ -adic Eisenstein Series

Proof.

Nonconstant Terms ( $A_{n,\chi}(X)$ ,  $n > 0$ )

- $d \in \mathbb{Z}_p^\times$ ,  $(d, p) = 1$ ,  $\Rightarrow \langle d \rangle \in U_1$
- $A_d(X) := \frac{1}{d}(1 + X)^{s(\langle d \rangle)}$
- $A_d(\zeta u^k - 1) = \frac{\zeta^{s(\langle d \rangle) u^{ks(\langle d \rangle)}}}{d} = \frac{\chi_\zeta(\langle d \rangle) \langle d \rangle^k}{d} = \omega^{-k}(d) d^{k-1} \chi_\zeta(d)$
- $A_{n,\chi}(X) := \sum_{\substack{d|n \\ (d,p)=1}} \chi(d) A_d(X)$
- $A_{n,\chi}(\zeta u^k - 1) = \sum_{\substack{d|n \\ (d,p)=1}} \psi(d) d^{k-1} = \sigma_{k-1,\psi}^{(p)}(n).$

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which has  $q$ -expansion

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where

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and

$$L^{(p)}(s, \chi) = (1 - \chi(p)p^{-s})L(s, \chi),$$

and is of level divisible by  $Np$ .

# $\Lambda$ -adic Eisenstein Series

Proof. (cont.)

Constant Terms ( $A_{0,\chi}(X)$ )

- $A_{0,\chi}(X) := \frac{G_\chi(X)}{2H_\chi(X)}$ .
- Suppose  $\chi \neq 1$  ( $\chi = 1$  is similar).
- $H_\chi(X) = 1$

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Proof. (cont.)

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$$\begin{aligned}
 A_{0,\chi}(\zeta u^k - 1) &= \frac{G_\chi(\zeta u^k - 1)}{2H_\chi(\zeta u^k - 1)} = \frac{G_{\chi\chi\zeta}(u^k - 1)}{2H_{\chi\chi\zeta}(u^k - 1)} \\
 &= \frac{L_p(1 - k, \chi\chi\zeta)}{2} \\
 &= \frac{L^{(p)}(1 - k, \psi)}{2}. \quad \square
 \end{aligned}$$

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# Ordinary $\Lambda$ -adic Forms

Using the same ideas as we discussed for modular forms we may define the spaces of ordinary  $\Lambda$ -adic modular forms and ordinary  $\Lambda$ -adic cusp forms,

$$\mathcal{M}^{\text{ord}}(N, I) \quad \text{and} \quad \mathcal{S}^{\text{ord}}(N, I),$$

each of which have a decomposition

$$\mathcal{M}^{\text{ord}}(N, I) = \bigoplus_{\chi} \mathcal{M}^{\text{ord}}(N, \chi, I),$$

and

$$\mathcal{S}^{\text{ord}}(N, I) = \bigoplus_{\chi} \mathcal{S}^{\text{ord}}(N, \chi, I).$$

# Ordinary $\Lambda$ -adic Forms

## Definitions

$F \in \mathcal{M}^{\text{ord}}(N, I)$  is:

- a **newform** if each specialization  $f_\nu$  is  $N$ -new.
- an **eigenform** if each specialization  $f_\nu$  is a classical eigenform.
- **primitive** if it is an eigenform, a newform, and normalized such that  $a_1(F) = 1$ .
- **ordinary** if  $a_p(f_\nu)$  is a  $p$ -adic unit for each specialization.



# Ordinary $\Lambda$ -adic Forms

## Theorem (Hida, Wiles)

Let  $p$  be an odd prime.

- 1 There are finitely many primitive, ordinary  $\Lambda$ -adic forms  $F$  of tame level  $N$ .
- 2 Each classical,  $p$ -stabilized, primitive, ordinary form is a member of some primitive ordinary  $\Lambda$ -adic form  $F$ .
- 3 The form  $F$  from 2 is unique up to Galois conjugacy.
- 4 Given a normalized, ordinary,  $\Lambda$ -adic eigenform  $F$ , one may associate a Galois representation  $\rho_F$  to it, with several natural properties.

# Ordinary $\Lambda$ -adic Forms

## Theorem (Wiles)

The space of ordinary  $\Lambda$ -adic modular forms (resp. ordinary  $\Lambda$ -adic cusp forms) of character  $\chi$  is free of finite rank over  $\Lambda$ ; i.e.,  $\mathcal{M}^{\text{ord}}(N, \chi, I)$  and  $\mathcal{S}^{\text{ord}}(N, \chi, I)$  are free of finite rank over  $\Lambda$ .



# Universal Ordinary Hecke Algebra

## Definition

The **universal ordinary Hecke algebra**  $\mathcal{H}^{\text{ord}}(N; \Lambda)$  is the subalgebra of  $\text{End}_{\Lambda}(\mathcal{M}^{\text{ord}}(N, \Lambda))$  generated by all of the Hecke operators  $T_n$  over  $\Lambda$ .

We define a pairing

$$\begin{aligned}\langle \cdot, \cdot \rangle : \mathcal{H}^{\text{ord}}(N; A) \times \mathcal{M}^{\text{ord}}(N, A) &\rightarrow A \\ \langle H, F \rangle &= a_1(HF) \in A.\end{aligned}$$

We also define the space

$$m^{\text{ord}}(N, A) = \{f \in \mathcal{M}^{\text{ord}}(N, K) : a_n(F) \in A \text{ for } n \geq 1\},$$

where  $K$  is the quotient field of  $A$ .

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# Universal Ordinary Hecke Algebra

## Theorem

For any extension  $A$  of  $\Lambda$ , the above pairing induces the isomorphisms

$$\mathrm{Hom}_A(\mathcal{H}^{\mathrm{ord}}(N; A), A) \cong m^{\mathrm{ord}}(N, A),$$

$$\mathrm{Hom}_A(m^{\mathrm{ord}}(N, A), A) \cong \mathcal{H}^{\mathrm{ord}}(N; A),$$

$$\mathrm{Hom}_A(h^{\mathrm{ord}}(N; A), A) \cong \mathcal{S}^{\mathrm{ord}}(N, A),$$

$$\mathrm{Hom}_A(\mathcal{S}^{\mathrm{ord}}(N, A), A) \cong h^{\mathrm{ord}}(N; A).$$

In particular,  $\mathcal{H}^{\mathrm{ord}}(N; A)$  and  $h^{\mathrm{ord}}(N; A)$  are free of finite rank over  $A$ .

# Outline

- 1 Ordinary Parts
- 2  $\Lambda$ -adic Forms
  - $\Lambda$ -adic Eisenstein Series
  - Ordinary  $\Lambda$ -adic Forms
  - Universal Ordinary Hecke Algebra
- 3 Ordinary  $\Lambda$ -adic Galois Representations
  - Pseudo-Representations

# Ordinary $\Lambda$ -adic Galois Representations

## Theorem 1 (Hida, Wiles)

Let  $F$  be a normalized  $l$ -adic eigenform of level  $N$  in  $\mathcal{S}^{\text{ord}}(N, \chi, l)$ , and let  $\lambda$  denote the corresponding  $l$ -algebra homomorphism  $\lambda : h^{\text{ord}}(N, \chi, l) \rightarrow l$ . Then there exists a unique Galois representation  $\rho_F : G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(K)$  such that

- 1  $\rho_F$  is continuous and absolutely irreducible,
- 2  $\rho_F$  is unramified outside  $Np$ ,
- 3 for each prime  $q \nmid Np$ , we have

$$\det(1 - \rho_F(\text{Frob}_q)T) = 1 - \lambda(T_q)T + (\chi \kappa \nu_p^{-1})(q)T^2,$$

where  $\text{Frob}_q$  is the Frobenius element at  $q$ .

# Ordinary $\Lambda$ -adic Galois Representations

Let us briefly explain what we mean by the reduction of  $\rho_F$  modulo  $P$ .

For any prime ideal  $P$  of  $I$ , let  $Q(I/P)$  denote the field of fractions of  $I/P$ .

## Definition

A Galois representation  $\overline{\rho}_{F,P}$  into  $GL_2(Q(I/P))$  is called a **residual representation** of  $\rho_F$  at  $P$  if  $\overline{\rho}_{F,P}$  is continuous under the  $\mathfrak{m}$ -adic topology of  $Q(I/P)$ , it is semi-simple, and it satisfies the following properties:

- 1  $\overline{\rho}_{F,P}$  is unramified outside  $Np$ ,
- 2 For any prime  $q \nmid Np$ ,

$$\det(1 - \overline{\rho}_{F,P}(\text{Frob}_q)T) = 1 - \lambda(T_q)(P)T + ((\chi_{K\nu_p^{-1}})(q))(P)T^2.$$

# Ordinary $\Lambda$ -adic Galois Representations

- $f$  – normalized eigenform in  $S_k(\Gamma_0(Np^r), \psi; K_f)$
- $K_f$  – the number field generated by  $\{a_n(f)\}_{n=1}^{\infty}$
- $\lambda_f$  – the corresponding algebra homomorphism  $h_k(\Gamma_0(Np^r), \psi; K_f) \rightarrow K_f$

We have the following theorem due to Eichler-Shimura for  $k = 2$ , Deligne for  $k \geq 2$ , and finally Deligne-Serre for  $k = 1$ .



# Ordinary $\Lambda$ -adic Galois Representations

## Theorem

For each maximal ideal  $\wp$  of  $\mathcal{O}_{K_f}$  lying over  $P$ , there exists a unique 2-dimensional Galois representation  $\rho_{f,\wp} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(K_{f,\wp})$  such that

- 1  $\rho_{f,\wp}$  is continuous and absolutely irreducible,
- 2  $\rho_{f,\wp}$  is unramified outside  $Np$ ,
- 3 for each prime  $q \nmid Np$ , we have

$$\det(1 - \rho_{f,\wp}(\mathrm{Frob}_q)T) = 1 - \lambda_f(T_q)T + \psi(q)q^{k-1}T^2.$$

## Corollary 2

There exists a unique, odd Galois representation  $\rho_{F(P)} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(Q(I/P))$ , and hence into  $\mathrm{GL}_2(\widetilde{I/P})$ , with the same properties as in the above Theorem.

# Ordinary $\Lambda$ -adic Galois Representations

We will use the following lemma to show that Deligne's representation is the same as the reduction of the representation from Theorem 1.

## Lemma

Let  $G$  be a group,  $K$  be a field of characteristic 0 and let  $\rho_1$  and  $\rho_2$  be two finite dimensional linear representations of  $G$  over  $K$ . If  $\rho_1$  and  $\rho_2$  are semi-simple and  $\text{Tr}(\rho_1(g)) = \text{Tr}(\rho_2(g))$  for all  $g \in G$ , then  $\rho_1$  and  $\rho_2$  are isomorphic over  $K$ .



# Ordinary $\Lambda$ -adic Galois Representations

We can now show that the residual representation  $\overline{\rho}_{F,P}$  attached to  $\rho_F$  is the same as Deligne's representation,  $\rho_{F(P)}$ .

Note that we have  $\langle n \rangle = \omega(n)^{-1} n = u^{s(\langle n \rangle)}$  where  $s(n) = \frac{\log(\langle n \rangle)}{\log(u)}$ .

Then we see

$$\begin{aligned}
 \kappa(\langle n \rangle)(\zeta u^k - 1) &= \kappa(u^{s(\langle n \rangle)})(\zeta u^k - 1) &= (\zeta u^k)^{s(\langle n \rangle)} \\
 &= \zeta^{s(\langle n \rangle)} (u^{s(\langle n \rangle)})^k \\
 &= \zeta^{s(\langle n \rangle)} \omega^{-k}(n) n^k \\
 &= \chi_\zeta(\langle n \rangle) \omega^{-k}(n) n^k \\
 &= \chi_\zeta(n) \omega^{-k}(n) n^k.
 \end{aligned}$$



# Ordinary $\Lambda$ -adic Galois Representations

Recall the trace polynomials from of these representations are given by

$$\det(1 - \overline{\rho}_{F_P}(\text{Frob}_q)T) = 1 - \lambda(T_q)(P)T + ((\chi_{\kappa\nu_p^{-1}})(q))(P)T^2,$$

and

$$\det(1 - \rho_{f,\wp}(\text{Frob}_q)T) = 1 - \lambda_f(T_q)T + \psi(q)q^{k-1}T^2,$$

respectively.

Hence, if  $P|_{\Lambda} = \ker(\nu_{k,\zeta})$ , then

$$\begin{aligned} \lambda(T_q)(P) &= a_q(F)(P) = a_q(F(P)) = a_q(f_\nu) = \lambda_{f_\nu}(T_q) \\ &= \lambda_f(T_q), \end{aligned} \quad (1)$$

and

$$((\chi_{\kappa\nu_p^{-1}})(q))(P) = (\chi\omega^{-1}\chi_\zeta)(q)q^{k-1} = \chi_\nu(q)q^{k-1}, \quad (2)$$

as desired.

# Ordinary $\Lambda$ -adic Galois Representations

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Hence, if  $P|_{\Lambda} = \ker(\nu_{k,\zeta})$ , then

$$\begin{aligned} \lambda(T_q)(P) &= a_q(F)(P) = a_q(F(P)) = a_q(f_{\nu}) = \lambda_{f_{\nu}}(T_q) \\ &= \lambda_f(T_q), \end{aligned} \quad (1)$$

and

$$((\chi_{\kappa\nu_p^{-1}})(q))(P) = (\chi_{\omega^{-1}}\chi_{\zeta})(q)q^{k-1} = \chi_{\nu}(q)q^{k-1}, \quad (2)$$

as desired.

# Pseudo-Representations

- $B$  is a commutative topological ring with unity.
- 2 is invertible in  $B$ .
- If  $B$  is an integral domain, let  $Q(B)$  denote its field of fractions.

## Definition

Let  $G$  be a profinite group with an identity  $e$ , and a special element  $c$  of order 2. A **pseudo-representation**  $\pi : G \rightarrow B$  is a triple  $\pi = \{A, D, X\}$  of continuous maps

$$A : G \rightarrow B,$$

$$D : G \rightarrow B, \text{ and,}$$

$$X : G \times G \rightarrow B,$$

# Pseudo-Representations

## Definition (cont.)

satisfying the following axioms:

- 1  $A(\sigma\tau) = A(\sigma)A(\tau) + X(\sigma, \tau)$ ,
- 2  $D(\sigma\tau) = D(\sigma)D(\tau) + X(\tau, \sigma)$ ,
- 3  $X(\sigma\tau, \gamma) = A(\sigma)X(\tau, \gamma) + D(\tau)X(\sigma, \gamma)$ ,
- 4  $X(\sigma, \tau\gamma) = A(\gamma)X(\sigma, \tau) + D(\tau)X(\sigma, \gamma)$ ,
- 5  $A(e) = D(e) = A(c) = 1, D(c) = -1$ ,
- 6  $X(\sigma, e) = X(e, \sigma) = X(\sigma, c) = X(c, \sigma) = 0$ ,
- 7  $X(\sigma, \tau)X(\gamma, \eta) = X(\sigma, \eta)X(\gamma, \tau)$ .

When we need to specify  $\pi$  we denote  $A, D, X$  by  $A_\pi, D_\pi, X_\pi$ .

# Pseudo-Representations

## Definition

The **trace** and **determinant** of a pseudo-representation  $\pi$  are defined to be

- $\text{Tr}(\pi)(\sigma) := A(\sigma) + D(\sigma)$ ,
- $\text{Det}(\pi)(\sigma) := A(\sigma)D(\sigma) - X(\sigma, \sigma)$ .

With these definitions we now have the following two identities.

$$A(\sigma) = \frac{\text{Tr}(\pi)(\sigma) + \text{Tr}(\pi)(c\sigma)}{2},$$

and

$$D(\sigma) = \frac{\text{Tr}(\pi)(\sigma) - \text{Tr}(\pi)(c\sigma)}{2}.$$

# Pseudo-Representations

## Theorem

Suppose  $\{P_n\}_{n=1}^{\infty}$  is a sequence of height 1 prime ideals of  $I$ . For each  $n \geq 1$ , suppose  $\pi_{P_n}$  is a pseudo-representation of  $G$  into  $I/P_n$ . Suppose that the  $\pi_{P_n}$  are compatible on a dense subset  $\Sigma$  of  $G$ , i.e., there exist two functions  $T$  and  $D$  on  $\Sigma$ , with values in  $I$ , such that for any  $n \geq 1$

$$\begin{aligned}\mathrm{Tr}(\pi_{P_n})(\sigma) &\equiv T(\sigma) \pmod{P_n}, \\ \mathrm{Det}(\pi_{P_n})(\sigma) &\equiv D(\sigma) \pmod{P_n},\end{aligned}$$

for all  $\sigma \in \Sigma$ .

# Pseudo-Representations

## Theorem (cont.)

Let  $\mathfrak{a}_n = \bigcap_{i=1}^n P_i$ . Then there exists a pseudo-representation  $\pi$  of  $G$  into  $I$ , such that for all  $\sigma \in \Sigma$  and  $n \geq 1$ ,

$$\mathrm{Tr}(\pi)(\sigma) \equiv T(\sigma) \pmod{\mathfrak{a}_n},$$

$$\mathrm{Det}(\pi)(\sigma) \equiv D(\sigma) \pmod{\mathfrak{a}_n},$$

and hence for every  $\sigma \in \Sigma$ ,

$$\mathrm{Tr}(\pi)(\sigma) = T(\sigma), \quad \mathrm{Det}(\pi)(\sigma) = D(\sigma).$$

# Pseudo-Representations

## Theorem 3 (Wiles)

Suppose that  $\{P_n\}_{n=1}^{\infty}$  is an infinite set of distinct prime ideals of  $I$  of height one. Let  $\widetilde{I/P_n}$  denote the integral closure of  $I/P_n$  in  $Q(I/P_n)$ . Suppose that, for each  $n$ , we are given a continuous, odd representation

$$\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\widetilde{I/P_n}),$$

which is unramified outside  $Np$ . Suppose that for each prime  $q \nmid Np$ , there exists elements  $a_q$  and  $\epsilon_q$ , in  $I$ , such that

$$\begin{aligned} \mathrm{Trace} \rho_n(\mathrm{Frob}_q) &= a_q \pmod{P_n}, & \text{and} \\ \mathrm{Det} \rho_n(\mathrm{Frob}_q) &= \epsilon_q \pmod{P_n}. \end{aligned}$$

# Pseudo-Representations

## Theorem 3 (cont.)

Then there exists a continuous, odd representation

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(K),$$

with

$$\mathrm{Trace} \rho(\mathrm{Frob}_q) = a_q,$$

$$\mathrm{Det} \rho(\mathrm{Frob}_q) = \epsilon_q,$$

for  $q \nmid Np$ . Also,  $\rho$  is irreducible if  $\rho_n$  is irreducible for some  $n$ .

# Pseudo-Representations

## Proof of Theorem 1

Consider the set of arithmetic primes  $\{P_n\}_{n=1}^{\infty}$  and let  $\rho_n$  denote the representation associated to  $F(P_n)$  as in Corollary 2. For every prime  $q \nmid Np$  we take

$$a_q = \lambda(T_q) \in I \text{ and } \epsilon_q = (\chi \kappa \nu_p^{-1})(q) \in I.$$

By the computations in Equations (1) and (2), for every  $q \nmid Np$ ,  $a_q$  and  $\epsilon_q$  satisfy the conditions of Theorem 3. Hence, there exists a representation  $\rho$ , such that for each prime  $q \nmid Np$

$$\det(1 - \rho(\text{Frob}_q)T) = 1 - \lambda(T_q)T + (\chi \kappa \nu_p^{-1})(q)T^2.$$

This  $\rho$  is our desired  $\rho_F$ . □

# Questions?



Questions?

Thank you!

