Group Actions

Introduction

We know that any group G has an operation associated to it (generally, $+$ or $\cdot$, or even composition) so that we may apply our operation to any two elements *within* the group. A natural question would be whether we can define an operation, call it $*$, so that we can compute $g * a$ where $g \in G$ but a can be an element of an arbitrary space, X (not even necessarily a group), and the notion of a *group action* provides the answer. We can discuss a group action on many different algebraic structures; for instance, if $a \in X$ where X is only an arbitrary *set* then X will be refered to as a *G-set* (as we further describe below). If we include some added structure, and say that the elements $a \in X$ with X a *group* then X will be a *G-module*; and including even more structure so that $a \in X$ with X a *ring*, then X will be a *G-algebra* – the latter two objects are more complex algebraic structures and will not be considered here.

Group Action on a Set

Let X be an arbitrary set and G a group. We should note that it's possible to let X be a group, we just need to ignore the fact that the group has an operation, i.e., we could let $X = \mathbb{Z}$, which is a group under $+$, but we ignore the operation $+$ and simply treat X as a set of numbers, $X = \{\dots, -1, 0, 1, \cdots\}$.

Definition. *An* ***action of*** G ***on*** X *is a map* $* : G \times X \to X$ *(usually written multiplicatively, i.e.,* $g * x = gx$*) such that*

1. $ex = x$ *for all* $x \in X$,
2. $(g_1 g_2)(x) = g_1(g_2 x)$ *for all* $x \in X$ *and all* $g_1, g_2 \in G$.

Under these conditions, X *is a* ***G-set****.*

Essentially, one should think of a G-set as an object that is closed under the operation $*$, i.e., any element in the set can be "multiplied" by any element in the group and still remain in the set.

Example. *Let* $X = \{1, 2, 3, \dots, n\}$ *and* $G = S_n$*, then* X *is trivially a* S_n*-set. Remember, one should think of elements of* S_n *as functions on* X*. For example, let* $g = (1\ 2) \in S_n$ *then* $g * 2 = g2$ *should be thought of as* $g * 2 = g(2) = 1$ *since* g *tells us that* $2 \mapsto 1$*.*

One can generalize the above example by taking X to be an arbitrary set and letting $G = S_X$, the group of all permutations of X. Any element of S_X tells us where to send every element of X (again by thinking of those group elements as functions on X), providing a group action on X and hence making X a S_X-set. In fact, if $H \leq G$ is any subgroup, then a similar reasoning implies that X is also an H-set.

Note, if X is a G-set then it follows that X is also an H-set for any subgroup $H \leq G$.

Recall in the proof of Cayley's Theorem, if G is a group with $g \in G$ we can define a map, $\lambda_g : G \to G$, called the *left-multiplication by g* map, defined by $\lambda_g(h) = gh$ for all $h \in G$. The map λ_g was shown to be injective and provide a permutation of the group G; inducing a map $\phi : G \to S_G$ given by $\phi(g) = \lambda_g$ for all $g \in G$.

The next theorem mimics this construction for G-sets.

Theorem. *Let* X *be a* G*-set. For each* $g \in G$*, then function* $\sigma_g : X \to X$ *defined by* $\sigma_g(x) = gx$ *for* $x \in X$ *is a permutation of* X*. This function induces a homomorphism* $\phi : G \to S_X$ *given by* $\phi(g) = \sigma_g$*, such that* $(\phi(g))(x) = gx$*.*

Suppose that X is a G-set and consider the set

$$N = \{g \in G : gx = x \text{ for all } x \in X\}.$$

We want to show that N is actually a normal subgroup of G.

Let $\phi : G \to S_X$ be the homomorphism from the theorem above. If we can show that $N = \ker \phi$ then since the kernel of a homomorphism is always a normal subgroup, we will have that $N \lhd G$. By definition, $\ker \phi = \{g \in G : \phi(g) = e \in S_X\}$, i.e., if $g \in \ker \phi$ then $\phi(g)$ represents the identity permutation (1) which fixes all elements of X, but this is exactly the definition of our set N (an argument showing that $N \subseteq \ker \phi$ and $\ker \phi \subseteq N$ could be given, but these containments should be clear from the definitions).

Thus, $N = \{g \in G : gx = x \text{ for all } x \in X\}$ is a normal subgroup of G which allows us to form the quotient group G/N. We may then regard X as a G/N-set, where $(gN)x = gx$. Note, the definition of $(gN)x = gx$ makes sense since X is a G-set; using the properties of a G-set, $(gN)x = g(Nx)$ but every element of N acts on x trivially (by the definition of N) so $Nx = x$, thus implying $(gN)x = g(Nx) = gx$.

Definition. *Using the notation above, if $N = \{e\}$, then the identity element of G is the only element that fixes each $x \in X$, and in this case we say that G* ***acts faithfully on*** *X.*

It's important to note that most actions that arise naturally (such as the action of S_n on $X = \{1, 2, \ldots, n\}$) are, in fact, faithful actions. Also, using our work above, if an action is not faithful (i.e., $N \neq \{e\}$) then we can consider the quotient G/N which will provide a faithful action (since we've essentially "killed" (equated to the identity) all of the non-identity elements of G that fixed all elements of our set).

Definition. *A group G is* ***transitive*** *on a G-set X if for each $x_1, x_2 \in X$ there exists a $g \in G$ such that $gx_1 = x_2$.*

Hence, a group is transitive on a G-set X if every element of X can be written as the product of g and some other (not necessarily distinct) element of X.