MATH 129 - SECTION 12
Exam #4

Instructors Name: Ryan Smith
Students Name (please print): Solutions______________________________

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ____

Date:____

You should work alone and show all of your work on all the problems.

1. *(10 pts)* Match each differential equation with the corresponding slope field.

(a) C______ $\frac{dy}{dx} = xy$

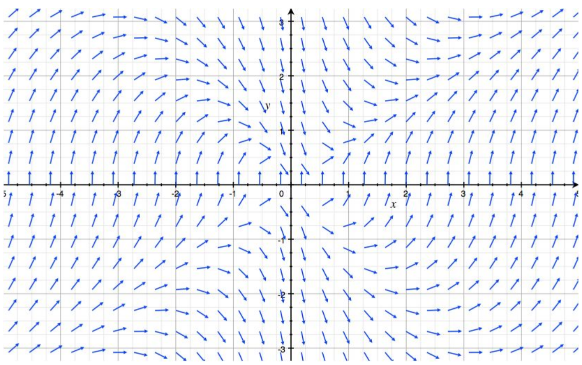
(b) B______ $\frac{dy}{dx} = \frac{1}{x+y}$

(c) E______ $\frac{dy}{dx} = \frac{1}{x(y-1)}$

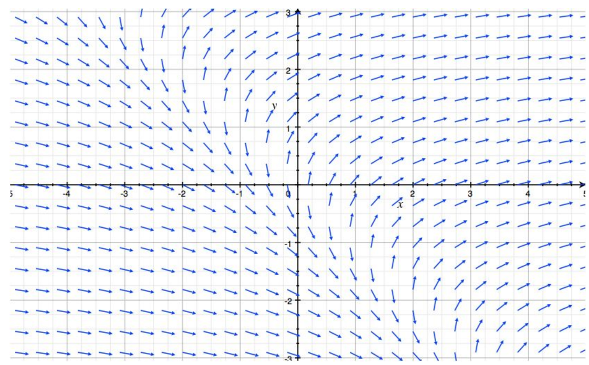
(d) A______ $\frac{dy}{dx} = \ln\left(\frac{x^2}{y^2}\right)$

(e) D______ $\frac{dy}{dx} = e^y$

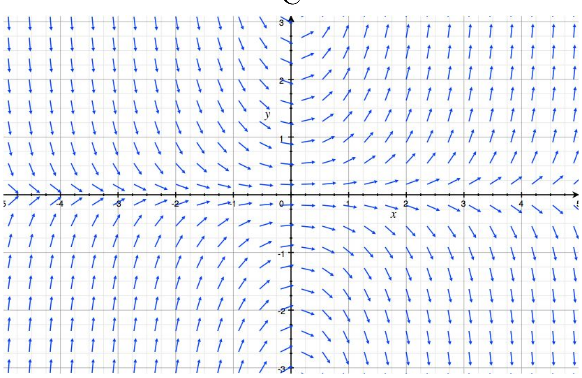
A



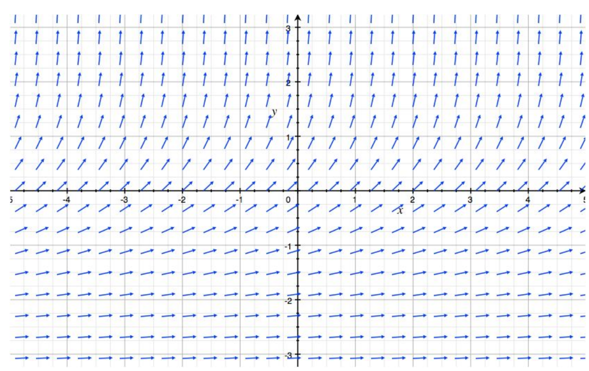
B



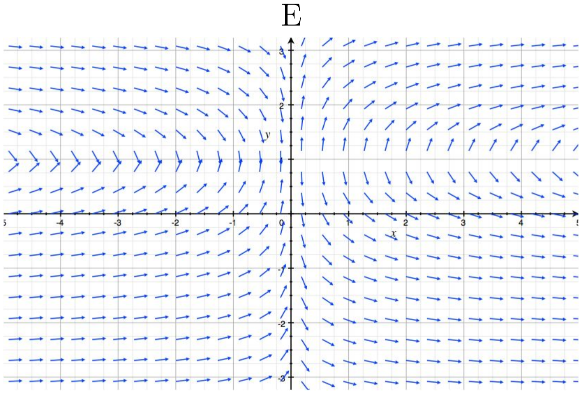
C



D



E



F

None of the above

2. Consider the function $f(z) = \frac{z^2}{e^{z^2}}$.

(a) *(16 pts)* Find the Taylor series for $f(z)$ around $z = 0$.
FIrst note that $\frac{1}{e^{z^2}} = e^{-z^2}$, so substituting $-z^2$ for z in the Taylor series for e^z gives us

$$e^{-z^2} = 1 + (-z^2) + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!} + \frac{(-z^2)^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!}$$

For the original problem we just need to multiply the Taylor series by z^2,

$$\begin{aligned}\frac{z^2}{e^{z^2}} &= z^2 + z^2(-z^2) + z^2\frac{(-z^2)^2}{2!} + z^2\frac{(-z^2)^3}{3!} + z^2\frac{(-z^2)^4}{4!} + \ldots \\ &= z^2 - z^4 + \frac{z^6}{2!} - \frac{z^8}{3!} + \frac{z^{10}}{4!} - \ldots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2}}{n!}\end{aligned}$$

(b) *(10 pts)* Use your results from the previous part to compute $\int_0^2 \frac{z^2}{e^{z^2}} dz$. Write your final answer with $\sum$ notation.

The order of the integral and the sum are interchangeable for all the functions we are working with, so

$$\begin{aligned}\int_0^2 \frac{z^2}{e^{z^2}} dz &= \int_0^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2}}{n!} \right) dz = \sum_{n=0}^{\infty} \int_0^2 \frac{(-1)^n z^{2n+2}}{n!} dz = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^2 z^{2n+2} dz \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{z^{2n+3}}{2n+3} \right]_0^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{2^{2n+3}}{2n+3}\end{aligned}$$

3. Complex numbers.

(a) *(6 pts)* Write $\sqrt{2} + \sqrt{2}i$ in polar coordinates.
The radius is $r = \sqrt{2+2} = 2$, and the angle is $\theta = \arctan\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\pi}{4}$. So

$$\sqrt{2} + \sqrt{2}i = 2e^{\frac{\pi}{4}i}$$

(b) *(10 pts)* Let $z = 3e^{\frac{\pi}{3}i}$. Find a square root of z exactly, writing your answer as $a + bi$. Then approximate $\sqrt{z}$ to 3 decimals.

Taking the square root will give us a new r, θ which we then need to convert back to $a + bi$ form.

$$\left(3e^{\frac{\pi}{3}i}\right)^{\frac{1}{2}} = 3^{\frac{1}{2}} e^{\frac{\pi}{6}i} = \sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = \sqrt{3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{3}{2} + i\frac{\sqrt{3}}{2}$$

The decimal approximation is

$$\sqrt{z} \approx 1.500 + 0.866i$$

4. *(14 pts)* Solve $y' - xy^2 - x = 0$ subject to the initial value condition $y(1) = \sqrt{3}$.

$$\begin{aligned}
\frac{dy}{dx} &= x(y^2+1) \\
\int \frac{1}{y^2+1} dy &= \int x dx \\
\arctan(y) &= \frac{x^2}{2} + C \\
y &= \tan\left(\frac{x^2}{2} + C\right)
\end{aligned}$$

We can apply the initial value condition now,

$$\begin{aligned}
\sqrt{3} &= \tan\left(\frac{1}{2} + C\right) \\
\frac{\pi}{3} &= \frac{1}{2} + C
\end{aligned}$$

So the final solution is

$$y(x) = \tan\left(\frac{x^2}{2} + \frac{\pi}{3} - \frac{1}{2}\right)$$

5. *(14 pts)* Solve the equation

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \cdots = 3$$

The nth term is clearly $2^n x^n = (2x)^n$, so this is just the substitution of a geometric series.

$$\begin{aligned}
\frac{1}{1-2x} &= 3 \\
x &= \frac{1}{3}
\end{aligned}$$

The quantity $\frac{1}{3}$ is within the radius of convergence $-1 < x < 1$, so it is a real solution.

6. Recall that Newton's law of cooling states that the rate an object cools is proportional to the difference in temperature between the object and the surrounding environment. A blacksmith heats a steel rod to 2100 degree Fahrenheit. The blacksmith is forging in June when the temperature outside is 110 degrees. After sitting on the anvil for 5 minutes the rod has cooled to 1500 degree.

   (a) *(14 pts)* Write down the differential equation for the temperature of the rod and solve it.
   The basic differential equation is $\frac{dH}{dt} = -k(H - 110)$ since the rate of heat loss is proportional to the difference in temperature between the rod and the environment. It is separable, so we can solve it

$$\begin{aligned} \int \frac{1}{H-110} dH &= \int -k dt \\ \ln|H - 110| &= -kt + C \\ H - 110 &= e^{-kt+C} \\ H(t) &= 110 + e^{-kt+C} \end{aligned}$$

   According to the problem the rod is 2100 degree at time $t = 0$,

$$\begin{aligned} 2100 &= 110 + e^{C} \\ C &= \ln(1990) \end{aligned}$$

   Now we use the initial value condition that the temperature is 1500 after 5 minutes,

$$\begin{aligned} 1500 &= 110 + 1990e^{-5k} \\ \frac{1390}{1990} &= e^{-5k} \\ k &= \frac{-1}{5} \ln\left(\frac{1390}{1990}\right) \end{aligned}$$

   Final answer,

$$H(t) = 110 + 1990e^{\frac{1}{5}\ln\left(\frac{1390}{1990}\right)t}$$

(b) *(6 pts)* The blacksmith waits until the rod is 140 degrees before he handles it without gloves. Assuming he doesn't die of heatstroke first, how long will he have to wait before it it safe to handle the rod with his bare hands?

$$140 = 110 + 1990e^{\frac{1}{5}\ln\left(\frac{1390}{1990}\right)t}$$

$$\ln\left(\frac{130}{1990}\right) = \frac{1}{5}\ln\left(\frac{1390}{1990}\right)t$$

$$t = \frac{5\ln\left(\frac{130}{1990}\right)}{\ln\left(\frac{1390}{1990}\right)}$$

$$t \approx 58.45 \text{ minutes}$$