MATH 129 - SECTION 12
Exam #3

Instructors Name: Ryan Smith
Students Name (please print): **Solutions**____

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ____

Date:____

You should work alone and show all of your work on all the problems.

1. *(10 pts)* Mark each statement as true or false. You do now have to show any work.

   (a) Every power series must converge at least one point. **True**

   (b) If $\sum_{n=0}^{\infty} |a_n + b_n|$ converges then $\sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=0}^{\infty} |b_n|$ must converge. **False**

   (c) If $a_n = f(n)$ for a continuous, decreasing function $f(x)$, then $\sum_{n=0}^{\infty} a_n$ converges if $\int_{100}^{\infty} f(x)dx$ converges. **True**

   (d) If $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. **True**

   (e) A finite geometric series $a + ar + ar^2 + \cdots + ar^n$ only converges if $|r| < 1$. **False**

2. *(20 pts)* Find the interval of convergence for each of the following power series. Make sure to pay attention to the endpoints.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-1)^n}{n^3}$

   Ratio test for the radius of convergence.

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}(x-1)^{n+1}}{(n+1)^3}}{\frac{3^n(x-1)^n}{n^3}} \right| = \lim_{n\to\infty} \left| \frac{3n^3(x-1)}{(n+1)^3} \right| = 3|x-1| \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 3|x-1|$$

   Therefore the radius of convergence is $\frac{1}{3}$. We check the endpoints $\frac{4}{3}$ and $\frac{2}{3}$ by plugging in the values,

$$\sum_{n=1}^{\infty} \frac{3^n(\frac{4}{3}-1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ Which converges by the integral comparison test.}$$

$$\sum_{n=1}^{\infty} \frac{3^n(\frac{2}{3}-1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \text{ Which converges by the alternating series test.}$$

   Therefore the interval of convergence is $\frac{2}{3} \le x \le \frac{4}{3}$.

   (b) $\displaystyle \frac{2!}{\ln(2)}(x+2)^2 + \frac{3!}{\ln(3)}(x+2)^3 + \frac{4!}{\ln(4)}(x+2)^4 + \frac{5!}{\ln(5)}(x+2)^5 + \dots$

   Ratio test for the radius of convergence. The formula for the nth term[1] is $\frac{n!}{\ln(n)}(x+2)^n$.

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{\ln(n+1)}(x+2)^{n+1}}{\frac{n!}{\ln(n)}(x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\ln(n)}{(n!)\ln(n+1)}(x+2) \right| = \lim_{n\to\infty} \left| \frac{(n+1)\ln(n)}{\ln(n+1)}(x+2) \right|$$
$$= \lim_{n\to\infty} (n+1)\left(\ln(n) + \frac{n+1}{n}\right)|x+2| = \text{ DNE}$$

   Therefore the radius of convergence is 0 and the interval of convergence is $x = -2$.

[1] Written this way the sum starts from $n = 2$.

3. The March Madness tournament started with 64 teams in the first round[2]. Each round the teams are paired up, and only the winners advance to the next round (there are no ties). The tournament stops when there is only one team left.

   (a) *(2 pts)* How many games are played in the first round?
   $64\left(\frac{1}{2}\right) = 32.$

   (b) *(2 pts)* How many games will be played in the nth round?
   $64\left(\frac{1}{2}\right)^n.$

   (c) *(2 pts)* How many rounds will the tournament have?
   $2^6 = 64$, so a total of 6 rounds.

   (d) *(4 pts)* Use a series to determine how many games will be played in the entire tournament.

$$\sum_{n=1}^{6} 64\left(\frac{1}{2}\right)^n = 64\left(\frac{1}{2}\right)\sum_{n=0}^{5}\left(\frac{1}{2}\right)^n = 32\frac{1-\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = 64\left(\frac{63}{64}\right) = 63$$

   (e) *(6 pts)* Suppose instead that there were 4096 teams in the tournament. How many matches would they play then? Simplify your answer.
   First note that $4096 = 2^{12}$, so there will be a total of 12 rounds and the nth round will have $2^{12}\left(\frac{1}{2}\right)^n$ games.

$$\sum_{n=1}^{12} 2^{12}\left(\frac{1}{2}\right)^n = 2^{11}\sum_{n=0}^{11}\left(\frac{1}{2}\right)^n = 2^{11}\frac{1-\left(\frac{1}{2}\right)^{12}}{1-\frac{1}{2}} = 2^{12}\left(1-\frac{1}{2^{12}}\right) = 2^{12}-1 = 4095$$

[2] We are ignoring the 4 play-in games.

4. *(30 pts)* Classify each of the following series as absolutely convergent, conditionally convergent, or divergent. You must show work to justify your conclusion.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$

Applying the alternating series test, we see that $0 < \frac{1}{3(n+1)} < \frac{1}{3n}$ for all n and $\displaystyle\lim_{n\to\infty} \frac{1}{3n} = 0$. Next we check for absolute convergence by looking at $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{3n}\right| = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. So the series converges, but not absolutely, which is the definition of conditional convergence.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{\sqrt{n+1}}$

This fails the alternating series test since

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1 \neq 0$$

Therefore the series diverges.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}n^2}{(2n)!}$

To save work we can start with checking for absolute convergence since we expect $(2n)!$ to grow much faster than n^2. Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}(n+1)^2}{(2(n+1))!}}{\frac{(-1)^{n+1}n^2}{(2n)!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2(2n)!}{n^2(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2(2n+2)(2n+1)} = 0$$

This means that the series converges, so the series is absolutely convergent.

5. *(12 pts)* Does $\sum_{n=0}^{\infty} \frac{n}{e^{n^2}}$ converge? Justify your answer.
There are many ways to show this, we'll use the integral comparison test with the substitution $u = x^2$ (which you can check doesn't change the bounds),

$$\int_1^{\infty} \frac{x}{e^{x^2}} dx = \int_1^{\infty} \frac{1}{2} e^{-u} du = \lim_{R\to\infty} \frac{-1}{2} e^{-u} \Big|_1^R = \lim_{R\to\infty} \frac{-R}{2} + \frac{1}{2} = \frac{1}{2}$$

The integral converges so the sum also converges.

6. *(12 pts)* Consider the Taylor series for $f(x)$ around $x = 1$ given by

$$f(x) = -\frac{1}{2} + \frac{2}{(3)2}(x-1) - \frac{4}{(4)(3)(2)}(x-1)^2 + \frac{8}{(5)(4)(3)(2)}(x-1)^3 - \frac{16}{(6)(5)(4)(3)(2)}(x-1)^4 + \dots$$

(a) What is the coefficient of $(x-1)^n$?

$$a_n = \frac{(-1)^{n+1} 2^n}{(n+2)!}$$

(b) What is $f^{(10)}(1)$?
For the 10th derivative at $x = 1$ we look at the term $a_{10}(x-1)^{10}$. Taking the derivative 10 times yields

$$a_{10}(10)!(x-1)^0 = \frac{(-1)^{10+1} 2^{10}}{(10+2)!}(10)! = -\frac{2^{10}}{12(11)} = -\frac{256}{33}$$

So $f^{(10)}(1) = -\frac{256}{33}$.