Math 129 - Section 12
Exam #2

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Students Name (please print): Solutions

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ______________________________

Date: ______________________________

You should work alone and show all of your work on all the problems.

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Points					

1. *(12 pts)* Evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt[3]{\cos(x)}}dx$. Make sure to show all your work and don't take any shortcuts.

Note, the integral is improper, so we need to use a limit at the left endpoint.

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt[3]{\cos(x)}}dx = \lim_{b\to\pi/2^-} \int_0^b \frac{\sin(x)}{\sqrt[3]{\cos(x)}}dx$$

Make the substitution $u = \cos(x)$, $du = -\sin(x)dx$

$$\begin{aligned} &= \lim_{b\to\pi/2^-} \int -u^{-\frac{1}{3}}du = \lim_{b\to\pi/2^-} \left[-\frac{3}{2}(\cos(x))^{\frac{2}{3}}\right]_0^b \\ &= \lim_{b\to\pi/2^-} -\frac{3}{2}(\cos(b))^{\frac{2}{3}} + \frac{3}{2}(\cos(0))^{\frac{2}{3}} \\ &= \frac{3}{2} \end{aligned}$$

2. *(8 pts)* Which of the following integrals are improper integrals?

(a) $\int_1^{\infty} \frac{1}{1+x^2}dx$ **Improper**

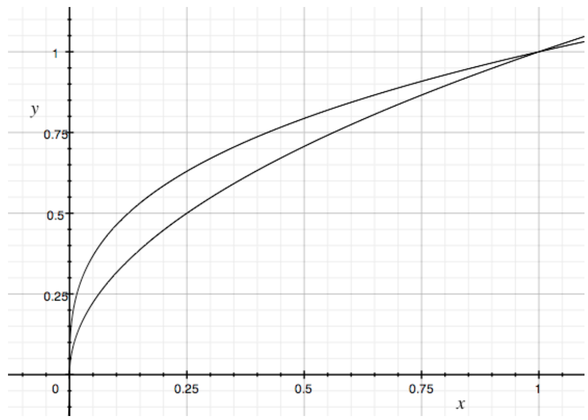
(b) $\int_1^{\infty} \frac{4}{x}dx$ **Improper**

(c) $\int_0^{5} \frac{1}{x^2+2x+1}dx$

(d) $\int_0^{\frac{1}{2}} \frac{1}{x\ln(x)}dx$ **Improper**

3. Consider the region bounded by the functions $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{2}}$.

(a) *(4 pts)* Sketch a graph of the region, indicating which function is which.



The lower graph is $y = x^{1/2}$, the upper is $y = x^{1/3}$

(b) *(4 pts)* What are the coordinates of the endpoints of the region?
The endpoints $(0, 0)$ and $(1, 1)$.

(c) *(10 pts)* Write the integral for the volume of the solid of revolution obtained by rotating the region around the y-axis. You do not have to solve the integral.

Note that for this we need to take horizontal slices, so the radius is given by the inverse functions $f^{-1}(y)$.

$$\int_0^1 \pi \left((y^2)^2 - (y^3)^2\right) dy$$

(d) *(10 pts)* Write the integral for the volume of the solid of revolution obtained by rotating the region around the line $y = 4$. You do not have to solve the integral.

Note that the distance from $y = x^{1/2}$ to $y = 4$ is $4 - x^{1/2}$ in this case, and the volume of the region is the volume obtained by subtracting the inner shape from the outer shape (note that $x^{1/3}$ is the inner shape here since $y = 4$ is above both of them).

$$\int_0^1 \pi \left((4 - x^{\frac{1}{2}})^2 - (4 - x^{\frac{1}{3}})^2\right) dx$$

4. *(12 pts)* A group of students at the University of Arizona who had never seen ice before attempted to make an icicle during a lab period. By dripping water in a freezer they made a cone of height 4 cm and radius 1 cm at the base. The density of the icicle is $\delta(h) = 1 + \sqrt{h}\ g/cm^3$, where h is the number of cm above the base of the cone. Find the total mass of the icicle.

   At a height of h above the base the radius of the cone is $r = 1 - \frac{1}{4}h$[1]. The mass of each horizontal slice is $(\Delta h)\delta(h)\pi r^2 = (\Delta h)(1+\sqrt{h})\pi(1-\frac{1}{4}h)^2$. The corresponding integral is

$$\int_0^4 (1+\sqrt{h})\pi(1-\frac{1}{4}h)^2 dh = \frac{268}{105}\pi \approx 8.0185 \text{ g}$$

5. *(12 pts)* Let $f(x)$ and $g(x)$ be continuous functions such that $\int_2^\infty f(x)dx$ and $\int_2^\infty g(x)dx$ converge, and let $a > 0$ be a fixed constant. For each of the following integrals, indicate whether the integral will **always converge**, or **always diverge**, or **depends on** $f(x)$ **and** $g(x)$.

   (a) $\int_2^\infty \frac{f(x)}{x}dx$ **Always converges**

   (b) $\int_2^\infty \frac{f(x)+g(x)}{a}dx$ **Always converges**

   (c) $\int_1^\infty f(x)dx$ **Always converges / Depends on** $f(x)$[2]

   (d) $\int_2^\infty g(x+a)dx$ **Always converges**

   (e) $\int_2^\infty (a+f(x))dx$ **Always diverges**

   (f) $\int_2^\infty f(x)g(x)dx$ **Always converges / Depends on** $f(x)$ **and** $g(x)$[3]

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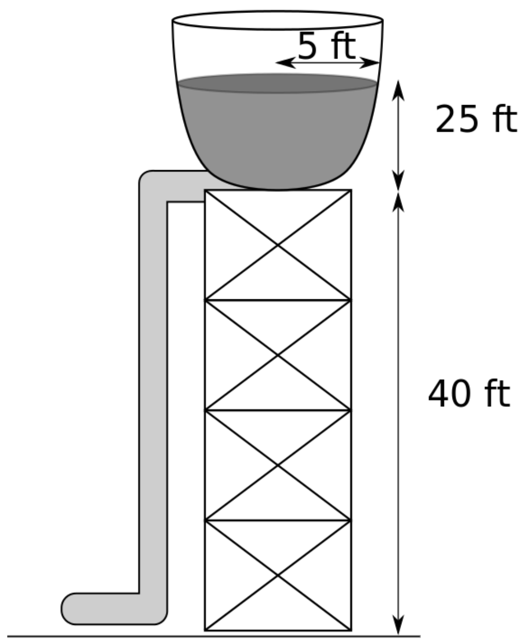
[1] Technically the problem didn't specify whether the base of the cone was at the top or bottom of the picture, so I gave full credit as long as the solution was consistent (which includes any diagrams agreeing with the mathematical setup).

[2] The answer depends on whether you assume that $f(x)$ must be continuous on $[0,1]$. If you read it to be saying that it is only assumed to be continuous on $[2,\infty)$ then it depends on $f(x)$.

[3] Since the function are continuous it will always converge, but if you miss that part of the question then it will depends on the choice of $f(x)$ and $g(x)$. For example, $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges but $\int_0^1 \frac{1}{(\sqrt{x})^2}dx$ obviously diverges.

6. *(14 pts)* A bottom of a new water tower gently slopes upward before getting very steep. The base of the tower is 40 ft above the ground. The tower is round, with a cross section given by the quadratic $f(x) = x^2$. If the municipal authorities want to fill the tower to a depth of 25 ft, and water weighs 62.2 lbs per cubic foot, how much work will it take to fill the tower from the ground?



We should cut the tower into *horizontal* slices since each slice will get lifted the same distance off the ground (and thus the amount of work for each slice will be easy to calculate). The radius of a slice at height h is $\sqrt{h}$. So the amount of mass in each slice is $\pi(\sqrt{h})^2\Delta h(62.2)$. That slice gets lifted $40 + h$ ft, so the work per slice is

$$\pi h \Delta h (62.2)(40 + h) g$$

The total work is

$$W = \int_0^{25} \pi h (62.2)(40 + h)(32.2)\, dh \approx 1.11 \times 10^8 \text{ ft lbs}$$

7. *(14 pts)* Prove that the integral $\displaystyle\int_1^\infty \frac{x+\cos(\ln(x))}{x^3+1}dx$ converges. You do not need to find the value.

First observe that $\cos(\ln(x)) \leq 1$ for all x, and the entire function is never negative, so

$$0 \leq \frac{x+\cos(\ln(x))}{x^3+1} \leq \frac{x+1}{x^3+1}$$

Next note that $\dfrac{x-1}{x^3+1} < \dfrac{x-1}{x^3}$, so we have

$$0 \leq \frac{x+\cos(\ln(x))}{x^3+1} \leq \frac{x+1}{x^3}$$

Integrating everything,

$$0 \leq \int_1^\infty \frac{x+\cos(\ln(x))}{x^3+1} \leq \int_1^\infty \frac{x+1}{x^3} = \int_1^\infty (x^{-2}+x^{-3})dx = \frac{1}{2}+\frac{1}{3}$$

So the original integral is bounded above and bellow by convergent integrals, so it converges.