Chapter 9 - Review Problems

Name: Solutions

For each of the following problems, write up a complete and readable solution on your own paper. Make sure your name is clearly visible and all of the pages are stapled. Do not use a completely dull pencil, turn in your stream-of-consciousness scratch work, or write your final answers on the back of a postcard.

1. Classify each of the following series as divergent, conditionally convergent, or absolutely convergent. Remember that absolute convergence implies convergence, this fact can often save you work whether the series converges or diverges. Show work that proves your answer is correct.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$

   Apply the alternating series test. We have $0 < \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$ for all n since $\ln(n)$ is an increasing function. Also have

   $$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$$

   So by the original series converges. Testing for absolute convergence we see that

   $$\frac{1}{n} < \frac{1}{\ln(n)} = \left|\frac{(-1)^{n+1}}{\ln(n)}\right|$$

   since $\ln(n) < n$. But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so $\sum_{n=2}^{\infty} \left|\frac{(-1)^{n+1}}{\ln(n)}\right|$ diverges. So the series is conditionally convergent.

   (b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\sqrt{\frac{n+2}{n}}\right)$

Start out by observing that $\ln\left(\sqrt{\frac{n+2}{n}}\right) = \frac{1}{2}(\ln(n+2) - \ln(n)) > 0$ so the series is either absolutely convergent or divergent. Use the integral comparison test,

$$\int_1^\infty \frac{1}{2}(\ln(x+2) - \ln(x))dx = \frac{1}{2} \lim_{R\to\infty} (x+2)\ln(x+2) - (x+2) - (x\ln(x) - x)\Big]_1^R$$
$$= \frac{1}{2} \lim_{R\to\infty} (x+2)\ln(x+2) - 2 - x\ln(x)\Big]_1^R$$

The integral will converge if $\lim_{R\to\infty}(R+2)\ln(R+2) - R\ln(R)$ exists.

$$\lim_{R\to\infty} (R+2)\ln(R+2) - R\ln(R) = \lim_{R\to\infty} R\ln(R+2) + 2\ln(R+2) - R\ln(R)$$
$$= \lim_{R\to\infty} R\ln\left(\frac{R+2}{R}\right) + 2\ln(R+2)$$

Since $\frac{R+2}{R} > 1$ we have $\ln\left(\frac{R+2}{R}\right) > 0$. Therefore $R\ln\left(\frac{R+2}{R}\right) > 0$ which implies that

$$\lim_{R\to\infty} R\ln\left(\frac{R+2}{R}\right) + 2\ln(R+2)$$

does not exist since $2\ln(R+2) \to +\infty$ as $R \to \infty$. (If the limit did exist then $R\ln\left(\frac{R+2}{R}\right)$ would have to be negative to cancel out $2\ln(R+2)$ as $R\to\infty$). Since the integral doesn't converge neither does the original series.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\pi n)}{n^2}$

Check absolute convergence first, $\sum_{n=1}^\infty \left|\frac{\sin(\pi n)}{n^2}\right| \le \sum_{n=1}^\infty \frac{1}{n^2}$ which converges. So the sum is absolutely convergent.

2. Decide if each of the series converges. You can use any method unless the problem specifies otherwise.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{(n+1)\ln(n+2)}$[1]

Writing out the terms of the series we have

$$\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{(n+1)\ln(n+2)} = \frac{1}{\ln(2)} - \frac{1}{3\ln(4)} + \frac{1}{4\ln(5)} - \frac{1}{5\ln(6)} + \dots$$

Applying the alternating series test we see that $0 < \frac{1}{(n+2)\ln(n+3)} < \frac{1}{(n+1)\ln(n+2)}$ and $\lim_{n\to\infty} \frac{1}{(n+1)\ln(n+2)|} = 0$, so the series converges.

(b) $\displaystyle\sum_{k=30}^{\infty} \frac{k}{\ln(k)^2}$

According the rule of relative rates of growth, we expect k to grow after than $\ln(k)^2$, so the limit of the terms should not be 0 (which will automatically imply divergence). By repeated application of L'Hopitals rule,

$$\lim_{k\to\infty} \frac{k}{\ln(k)^2} = \lim_{k\to\infty} \frac{1}{2\ln(k)\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{2\ln(k)} = \lim_{k\to\infty} \frac{1}{2\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{2}$$

The limit DNE so it is is not 0, so the series must diverge (remember, if the limit were 0 it wouldn't mean that it does converge, we'd need other tests for that!).

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^4}{n^2 - 6n + 2 - n!}$

Whoever wrote this problem tried to sneak the most important part in the back where we wouldn't notice it: the $n!$. Since factorials grow much faster than any polynomial, our expectation (not our final answer!) should be that it will converge. The easiest way to do this is to "reduce" to $\sum \frac{n^4}{-n!}$ since thats the

[1] There was a typo in the original problem, sorry.

important part of series. We can do this by a comparison test,

$$\lim_{n\to\infty}\left|\frac{\frac{n^4}{-n!}}{\frac{n^4}{n^2-6n+2-n!}}\right| = \lim_{n\to\infty}\left|\frac{n^2-6n+2-n!}{-n!}\right| = \lim_{n\to\infty}\left|\frac{-n^2}{n!}+\frac{6n}{n!}+1\right| = 1$$

So $\sum_{n=0}^{\infty}\frac{n^4}{n^2-6n+2-n!}$ converges if $\sum_{n=1}^{\infty}\frac{n^4}{n!}$ converges. We finish the problem with the ratio test for $\sum_{n=1}^{\infty}\frac{n^4}{n!}$,

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}}\right| = \lim_{n\to\infty}\frac{(n+1)^4}{(n+1)n^4} = \lim_{n\to\infty}\frac{(n+1)^3}{n^4} = 0$$

So the series converges, which means the original series converges.

(d) $\displaystyle\sum_{n=3}^{\infty}\frac{(-1)^n}{\ln(\ln(n))}$

Remember that $\ln(n)$ is a (very slowly) increasing function, so $\ln(\ln(n))$ is an even more slowly increasing function, and $\frac{1}{\ln(\ln(n+1))} < \frac{1}{\ln(\ln(n))}$. Since $\lim_{n\to\infty}\ln(\ln(n)) = \infty$, we know that $\lim_{n\to\infty}\frac{1}{\ln(\ln(n))} = 0$. By the alternating series test the series converges.

(e) $\displaystyle\sum_{m=0}^{\infty}\frac{1}{\sqrt{m+1}(\sqrt{m+2})}$

Note that this series is going to behave like $\sum\frac{1}{\sqrt{m^2}} = \sum\frac{1}{m}$ so we can compare it to the harmonic series. Taking the limit of the ratio of the coefficients,

$$\begin{aligned}\lim_{m\to\infty}\frac{\frac{1}{m}}{\frac{1}{\sqrt{m^2+3m+2}}} &= \lim_{m\to\infty}\frac{\sqrt{m^2+3m+2}}{m} = \lim_{m\to\infty}\sqrt{\frac{m^2+3m+2}{m^2}}\\ &= \lim_{m\to\infty}\sqrt{1+\frac{3}{m}+\frac{2}{m^2}} = \sqrt{\lim_{m\to\infty}1+\frac{3}{m}+\frac{2}{m^2}}\\ &= 1\end{aligned}$$

So both series converge or diverge, but the harmonic series diverges.[2]

(f) *Tricky:* $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)! + n^2}{(2n+1)! - n}$

For our guess, we should think that this will behave like $\frac{(2n)!}{(2n+1)!} = \frac{1}{2n+1}$ which will diverge. Apply the comparison test to $\frac{1}{2n+1}$,

$$\lim_{n\to\infty} \left| \frac{(2n)! + n^2}{(2n+1)! - n} \frac{2n+1}{1} \right| = \lim_{n\to\infty} \left| \frac{(2n)! + n^2}{(2n+1)(2n)! - n} \frac{2n+1}{1} \right|$$
$$= \lim_{n\to\infty} \frac{(2n)! + n^2}{(2n)! - \frac{n}{2n+1}} = 1$$

And we know that $\sum \frac{1}{2n+1}$ diverges, so the original series diverges.

(g) *Tricky:* $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n)}{\cos(2n) - 3}$

There is a complicated way to do this and there is an extremely simple way to do this. The very simple way (again, always check for obvious ways to cut a problem short first) is to just see that $\displaystyle\lim_{n\to\infty} \frac{\sin(2n)}{\cos(2n) - 3} \neq 0$ (in fact the limit does not exist). This proves that the series cannot converges.

(h) *Very Tricky:*[3] $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

This is mostly just a matter of writing out what $n!$ and n^n mean,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} = \sum_{n=1}^{\infty} \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \frac{n-3}{n} \cdots \frac{3}{n} \frac{2}{n} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{2}{n} \frac{1}{n}$$

since $\frac{n}{n}, \frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \ldots, \frac{3}{n}$ are all < 1. We know that $\sum_{n=1}^{\infty} \frac{2}{n}\frac{1}{n} = \sum_{n=1}^{\infty} \frac{2}{n^2}$ converges.

[2]The step with interchanging the limit and the square root is justified because if $f(x)$ and $g(x)$ are continuous functions for all x then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$. Continuity is a really important assumption for this, you can get into a lot of trouble using this fact at home if you don't check that $\lim_{x\to a} g(x)$ exists before applying this.

[3]This problem is optional, you do not need to know how to solve this for our course.

3. Find the interval of convergence of the following power series. Make sure to check the endpoints.

(a) $\displaystyle\sum_{n=0}^{\infty} 3^{-n}(x-1)^n$

All of these will be checked with the ratio test,

$$\lim_{n\to\infty}\left|\frac{3^{-n-1}(x-1)^{n+1}}{3^{-n}(x-1)^n}\right| = \lim_{n\to\infty}\frac{1}{3}|x-1| = \frac{1}{3}|x-1|$$

The radius of convergence is $\frac{1}{\frac{1}{3}} = 3$, the endpoints are $-2, 4$. Check each endpoint, $\sum_{n=0}^{\infty} 3^{-n}(-2-1)^n = \sum_{n=0}^{\infty}(-1)^n$ which diverges since it fails the alternating series test. $\sum_{n=0}^{\infty} 3^{-n}(4-1)^n = \sum_{n=0}^{\infty} 1$ which definitely diverges. So the interval of convergence is $-2 < x < 4$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{n}$

This basically the same as the last problem[4].

$$\lim_{n\to\infty}\left|\frac{\frac{(x+1)^{n+1}}{n+1}}{\frac{(x+1)^n}{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x+1| = 1|x+1|$$

Radius of convergence 1, check the endpoints $-2, 0$. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ converges by the alternating series test, but $\sum_{n=0}^{\infty}\frac{1}{n}$ diverges. So the interval of convergence is $-2 \le x < 0$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{(n!)}x^n}{2^n}$

This is a more fun problem[5].

$$\lim_{n\to\infty}\left|\frac{\frac{\sqrt{(n+1)!}x^{n+1}}{2^{n+1}}}{\frac{\sqrt{(n!)}x^n}{2^n}}\right| = \lim_{n\to\infty}\frac{1}{2}\sqrt{\frac{(n+1)!}{n!}}|x| = \lim_{n\to\infty}\frac{1}{2}\sqrt{n+1}|x|$$

[4] Ok, I admit this is a fairly boring problem, but there's supposed to be a mix of routine stuff and more interesting problems.

[5] The kind people hate to see on tests.

The limit does not exist, which means that the radius of convergence is 0 and the interval of convergence is $x = 0$.

(d) $\sum_{n=0}^{\infty} 10^n x^{2n+1}$

$$\lim_{n\to\infty} \left| \frac{10^{n+1} x^{2(n+1)+1}}{10^n x^{2n+1}} \right| = \lim_{n\to\infty} 10|x|^2 = 10|x|^2$$

The tests in the book aren't so clear on what to do next - which is why we should understand the ideas behind the tests instead of memorizing a bunch of formulas. The ratio test for the radius of convergence works because it's comparing our series to a geometric series. In particular, if $10|x|^2 < 1$ then the series is bounded by a geometric series whose ratio is < 1 and thus converges. Ignoring the endpoints for a brief moment, we have $10|x|^2 = 1$ (after all, this is a test for the biggest the ratio can be and still converge), so $|x| = \frac{1}{\sqrt{10}}$. Therefore the radius of convergence is $\frac{1}{\sqrt{10}}$. Now we check the endpoints, $-\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}$.

$$\sum_{n=0}^{\infty} 10^n \left(-\frac{1}{\sqrt{10}}\right)^{2n+1} = \sum_{n=0}^{\infty} 10^n \left(\frac{1}{10}\right)^n \left(\frac{-1}{\sqrt{10}}\right) = \left(\frac{-1}{\sqrt{10}}\right) \sum_{n=0}^{\infty} 1$$

Similarly,

$$\sum_{n=0}^{\infty} 10^n \left(\frac{1}{\sqrt{10}}\right)^{2n+1} = \sum_{n=0}^{\infty} 10^n \left(\frac{1}{10}\right)^n \left(\frac{1}{\sqrt{10}}\right) = \left(\frac{1}{\sqrt{10}}\right) \sum_{n=0}^{\infty} 1$$

So the radius of convergence is $-\frac{1}{\sqrt{10}} < x < \frac{1}{\sqrt{10}}$.