Chapter 10 - Taylor Series

Name: Solutions

For each of the following problems, write up a complete and readable solution on your own paper. Make sure your name is clearly visible and all of the pages are stapled. Do not use a completely dull pencil, turn in your stream-of-consciousness scratch work, or write your final answers on the back of a postcard.

1. Find the Taylor series for each of the following functions from scratch. Find the radius of convergence of each series (you do not need to check the endpoints for this problem, though you need to know how to check them). Show all of your work.

   (a) $2\sin(x)$ around $x = 0$.
   We compute the first couple derivates to look for a pattern,

$$\begin{aligned} f(x) &= 2\sin(x) \\ f'(x) &= 2\cos(x) \\ f''(x) &= -2\sin(x) \\ f^{(3)}(x) &= -2\cos(x) \\ f^{(4)}(x) &= 2\sin(x) \end{aligned}$$

   So the values of the derivatives at $x = 0$ are

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 2 \\ f''(0) &= 0 \\ f^{(3)}(0) &= -2 \\ f^{(4)}(0) &= 0 \end{aligned}$$

So the Taylor series is

$$2\sin(x) = 2x - \frac{2}{3!}x^3 + \frac{2}{5!}x^5 - \frac{2}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n+1)!}x^{2n+1}$$

(b) $\cos(x)$ around $x = \pi$.
Again we find the first couple derivates to look for a pattern $\cos(x), -\sin(x), -\cos(x), \sin(x), \cos(x), \ldots$ so the values at $x = \pi$ are $-1, 0, 1, 0, -1, \ldots$. The Taylor series is

$$\cos(x) = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}$$

(c) $\frac{1}{1-3x}$ around $x = 0$.

As always, we write down derivates of $\frac{1}{1-3x}$: $\frac{1}{1-3x}$, $3(1-3x)^{-2}$, $3^3(2)(1-3x)^{-3}$, $3^4(3)(2)(1-3x)^{-4}$, $3^5(4)(3)(2)(1-3x)^{-5}$, etc. The values of the derivates at $x = 0$ are 1, 3, $3^2(2)$, $3^3(3!)$, $3^4(4!)$, $3^5(5!)$, etc. The Taylor series is

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + (3x)^4 + \cdots = \sum_{n=0}^{\infty}(3x)^n$$

Obviously we could also do this by just substituting $3x$ for x in the geometric series, but the problem asked us to do it from scratch.

(d) $\ln(x)$ around $x = 1$.

Start out with a bunch of derivatives,

$$\begin{aligned} f(x) &= \ln(x) \\ f'(x) &= x^{-1} \\ f''(x) &= -x^{-2} \\ f^{(3)}(x) &= 2x^{-3} \\ f^{(4)}(x) &= -(3!)x^{-4} \\ f^{(5)}(x) &= (4!)x^{-5} \end{aligned}$$

So the values at $x = 1$ are:

$$\begin{aligned} f(1) &= 0 \\ f'(1) &= 1 \\ f''(1) &= -1 \\ f^{(3)}(1) &= 2 \\ f^{(4)}(1) &= -(3!) \\ f^{(5)}(1) &= (4!) \end{aligned}$$

The Taylor series is

$$\begin{aligned} \ln(x) &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \end{aligned}$$

2. Write each Taylor series with $\sum$ notation, then check the radius of convergence (you can skip checking the endpoints).

   (a) $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$.
   The nth term is $\frac{1}{2n+1}x^{2n+1}$, so the series is $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$. Find the radius of convergence with the ratio test for power series,

$$\lim_{n\to\infty} \left| \frac{\frac{1}{2(n+1)+1}x^{2(n+1)+1}}{\frac{1}{2n+1}x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{2n+1}{2n+3}x^2 \right| = 1|x^2|$$

The radius of convergence is $R = \sqrt{\frac{1}{1}} = 1$.

(b) $x + 2x^2 + 3x^3 + 4x^4 + \dots$.
Formula for the nth term is $a_n = nx^n$, the sum is $\sum_{n=0}^{\infty} nx^n$. Again the ratio of convergence is easy,

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = \lim_{n\to\infty} 1|x| = 1|x|$$

The radius of convergence is $R = \frac{1}{1} = 1$.

(c) $-\frac{1}{2} + \frac{2}{6}(x-1) - \frac{4}{24}(x-1)^2 + \frac{8}{120}(x-1)^3 - \frac{16}{720}(x-1)^4 + \dots$
Clearly the numerator is growing with a factor of 2^n, with a $(-1)^n$ to cause the sign to switch. The denominator contains a factorial, so $a_n = \frac{(-1)^{n+1}2^n}{(n+2)!}(x-1)^n$ and the sum is $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^n}{(n+2)!}(x-1)^n$. Find the radius of convergence,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}2^{n+1}}{(n+3)!}(x-1)^{n+1}}{\frac{(-1)^{n+1}2^n}{(n+2)!}(x-1)^n} \right| = \lim_{n\to\infty} \frac{2}{n+3}|x-1| = 0$$

The radius of convergence is "$R = 1/0 = \infty$".

3. For this problem you'll be asked to find Taylor series for new functions by using known results. Each problem will ask for two or more Taylor series; you will have to use the previous Taylor series somehow to find the next Taylor series.

   (a) Consider the following two Taylor series:

      i. Find the Taylor series of e^x around $x = 0$.[1]
      For every n we have $f^{(n)}(x) = e^x$, so $f^{(n)}(4) = e^4$ for every n. The Taylor series for e^x around $x = 0$ is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

---

[1] There was an error in the original problem that made the substitution really mess.

ii. Find the Taylor series of e^{x^3} around $x = 0$.
Just substitute x^3 in for x,

$$e^{x^3} = 1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^{3n}$$

(b) Consider the following two Taylor series:

i. Find the Taylor series for $\frac{1}{(1-x)^2}$ around $x = 0$ (use the Taylor series for $\frac{1}{1-x}$).

This is just the derivative of $\frac{1}{1-x}$, so we take the derivative of the Taylor series,

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \\ \frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \dots\right) \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \dots \end{aligned}$$

ii. Find the Taylor series for $\frac{x}{(1-x)^2}$ around $x = 0$.
Just multiply both sides by x,

$$\frac{x}{(1-x)^2} = x(1+2x+3x^2+4x^3+\dots) = x+2x^2+3x^3+4x^4+\dots$$

(c) Consider the following trio of Taylor series:

i. Find the Taylor series for $\frac{1}{1+x}$ around $x = 0$ (use the Taylor series for $\frac{1}{1-x}$).
Substitute $-x$ for x in the Taylor series for the geometric

formula,

$$\begin{aligned}\frac{1}{1+x} = \frac{1}{1-(-x)} &= 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \dots \\ &= 1 - x + x^2 - x^3 + x^4 - \dots\end{aligned}$$

ii. Find the Taylor series for $\frac{1}{1+x^2}$ around $x = 0$.
Substitute x^2 for x in the previous result,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

iii. Find the Taylor series for $\arctan(x)$ around $x = 0$.
We know that $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$, so we can get the series for $\arctan(x)$ by integrating the previous result,

$$\begin{aligned}\arctan(x) = \int \frac{1}{1+x^2} dx &= \int \left(1 - x^2 + x^4 - x^6 + x^8 - \dots\right) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots\end{aligned}$$

Note: we have technically ignored the constant of integration, since $\arctan(x)$ is just one antiderivative of $\frac{1}{1+x^2}$, but the choice is justified since $\arctan(0) = 0$ and $0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \dots = 0$. Also note that this gives us a formula for π, since $\tan\left(\frac{\pi}{4}\right) = 1$ implies that $\arctan(1) = \frac{\pi}{4}$. So

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

(d) Consider the following trio of Taylor series:

i. Find the Taylor series of $\ln(1+x)$ around $x = 0$.
The Taylor series is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

ii. Find the Taylor series of $\ln(1+x^3)$ around $x=0$.
Do the usual substitution, x goes to x^3,

$$\begin{aligned}\ln(1+x^3) &= x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \dots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{3n}\end{aligned}$$

iii. Find the Taylor series of $\frac{3x^2}{1+x^3}$ around $x=0$.
Take the derivative of the result from the previous part,

$$\begin{aligned}\frac{3x^2}{1+x^3} &= 3x^2 - \frac{6x^5}{2} + \frac{9x^8}{3} - \frac{12x^{11}}{4} + \dots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3n}{n} x^{3n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} 3x^{3n-1}\end{aligned}$$

(e) For this problem you can assume that $w > 0$. Consider the following Taylor series:

i. Find the Taylor series of $\cos(\sqrt{w})$.
Since the problem technically didn't specify where we have to expand around, we should choose $w = 0$. Start with the series for $\cos(w)$,

$$\cos(w) = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \dots$$

In general it would be hard to compose with $\sqrt{w}$ with Taylor series, but all of the powers in the series for $\cos(w)$ are even.

$$\begin{aligned}\cos(\sqrt{w}) &= 1 - \frac{(\sqrt{w})^2}{2!} + \frac{(\sqrt{w})^4}{4!} - \frac{(\sqrt{w})^6}{6!} + \frac{(\sqrt{w})^8}{8!} - \dots \\ &= 1 - \frac{w}{2!} + \frac{w^2}{4!} - \frac{w^3}{6!} + \frac{w^4}{8!} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^n\end{aligned}$$

ii. Find the Taylor series of $\frac{\sin(\sqrt{w})}{\sqrt{w}}$[2].
Our first thought might be to divide the previous result by $\sqrt{w}$, but then it wouldn't be in the right form to be a Taylor series. The next idea is to take the derivative of the previous part, $\frac{d}{dw}(\cos(\sqrt{w})) = -\frac{\sin(\sqrt{w})}{\sqrt{w}}\frac{-1}{2}$. So

$$\begin{aligned}\frac{\sin(\sqrt{w})}{\sqrt{w}} &= 2\sum_{n=0}^{\infty} \frac{d}{dw}\left(\frac{(-1)^n}{(2n)!} w^n\right) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(2n)!} w^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} w^{n-1}\end{aligned}$$

4. Use your outstanding ability to recognize Taylor series to solve the following equations.

   (a) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \dots = 2$
   This is the Taylor series for e^x around $x = 0$, so $e^x = 2$ and $x = \ln(2)$.

   (b) $(x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5} - \dots = 4$
   This is saying that $\ln(x) = 4$, so $x = e^4$.

   (c) *Tricky:* $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} = \frac{\sqrt{2}}{2}$
   The first tip off if the only even powers in the Taylor series, after

---

[2] Typo in the original problem.

which we notice that it looks an awful lot like the series for $\sin(x)$. It's the series for $\sin(x^2)$, so the equation is $\sin(x^2) = \frac{\sqrt{2}}{2}$. Taking arcsin of both sides, $x^2 = \frac{\pi}{2}$, $x = \sqrt{\frac{\pi}{2}}$.

(d) *Tricky:* $x^2 + x^3 + x^4 + x^5 + x^6 + \cdots = \frac{1}{2}$

Looking at this, its nearly a geometric series. We just need to factor out x^2,

$$\begin{aligned} x^2(1 + x + x^2 + x^3 + x^4 + \dots) &= \frac{1}{2} \\ x^2 \frac{1}{1-x} &= \frac{1}{2} \\ x^2 &= \frac{1}{2} - \frac{1}{2}x \\ x^2 + \frac{1}{2}x - \frac{1}{2} &= 0 \\ (x+1)\left(x - \frac{1}{2}\right) &= 0 \\ x &= -1, \frac{1}{2} \end{aligned}$$

Note, $x = -1$ isn't a valid solution since the geometric series won't converge for $x = -1$.