Chapter 10 - Taylor Series

Name: _____

For each of the following problems, write up a complete and readable solution on your own paper. Make sure your name is clearly visible and all of the pages are stapled. Do not use a completely dull pencil, turn in your stream-of-consciousness scratch work, or write your final answers on the back of a postcard.

1. Find the Taylor series for each of the following functions from scratch. Find the radius of convergence of each series (you do not need to check the endpoints for this problem, though you need to know how to check them). Show all of your work.
   (a) $2\sin(x)$ around $x = 0$.
   (b) $\cos(x)$ around $x = \pi$.
   (c) $\frac{1}{1-3x}$ around $x = 0$.
   (d) $\ln(x)$ around $x = 1$.
2. Write each Taylor series with $\sum$ notation, then check the radius of convergence (you can skip checking the endpoints).
   (a) $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$.
   (b) $x + 2x^2 + 3x^3 + 4x^4 + \ldots$.
   (c) $-\frac{1}{2} + \frac{2}{6}(x-1) - \frac{4}{24}(x-1)^2 + \frac{8}{120}(x-1)^3 - \frac{16}{720}(x-1)^4 + \ldots$
3. For this problem you'll be asked to find Taylor series for new functions by using known results. Each problem will ask for two or more Taylor series; you will have to use the previous Taylor series somehow to find the next Taylor series.
   (a) Consider the following two Taylor series:

i. Find the Taylor series of e^x around $x = 4$.

ii. Find the Taylor series of e^{x^3} around $x = 4$.

(b) Consider the following two Taylor series:

i. Find the Taylor series for $\frac{1}{(1-x)^2}$ around $x = 0$ (use the Taylor series for $\frac{1}{1-x}$).

ii. Find the Taylor series for $\frac{x}{(1-x)^2}$ around $x = 0$.

(c) Consider the following trio of Taylor series:

i. Find the Taylor series for $\frac{1}{1+x}$ around $x = 0$ (use the Taylor series for $\frac{1}{1-x}$).

ii. Find the Taylor series for $\frac{1}{1+x^2}$ around $x = 0$.

iii. Find the Taylor series for $\arctan(x)$ around $x = 0$.

(d) Consider the following trio of Taylor series:

i. Find the Taylor series of $\ln(1 + x)$ around $x = 0$.

ii. Find the Taylor series of $\ln(1 + x^3)$ around $x = 0$.

iii. Find the Taylor series of $\frac{3x^2}{1+x^3}$ around $x = 0$.

(e) For this problem you can assume that $w > 0$. Consider the following Taylor series:

i. Find the Taylor series of $\cos(\sqrt{w})$.

ii. Find the Taylor series of $\frac{\cos(\sqrt{w})}{\sqrt{w}}$.

4. Use your outstanding ability to recognize Taylor series to solve the following equations.

(a) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots = 2$

(b) $(x + 1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5} - \cdots = 4$

(c) *Tricky:* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} = \frac{\sqrt{2}}{2}$

(d) *Tricky:* $x^2 + x^3 + x^4 + x^5 + x^6 + \cdots = \frac{1}{2}$