Math 124 - Section 2
Exam #1

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Students Name (please print): Solutions

1. A landscapping company in Tucson has found that the demand for cactus drops at a constant rate as they raise their prices. They've found that when they sell cactus for \$15 per cactus they sell $11,000$ each month, and when they raise the price to \$25 per cactus they sell $7,000$ per month.[1]

   (a) Write a function for their sales S as a function of price p.
   `The problem says constant rate, so the function will be linear. First we find the slope`

   $$m = \frac{7000 - 11000}{25 - 15} = -400$$

   `Then we plug into` $y = -400x + b$ `to find` $7000 = -400(25) + b$, `so` $b = 17,000$. `The function is`

   $$S(p) = -400p + 17,000$$

   (b) Find the price at which no one will buy their cactus.
   `Solve for the` x `intercept:` $0 = -400p + 17,000$ `so` $p = \$42.50$

2. Which of the following statements is / are true? You do **not** have to justify your answer.

   I. $e^{\sin(x)}$ is periodic. `True`

   II. $25e^{2x}$ is proportional to $2e^{25x}$. `False`

   III. If $f(x)$ has an inverse, then the domain of $f^{-1}(x)$ is the range of $f(x)$. `True`

   IV. Every polynomial has at least one zero. `False`

[1] Parts a) and b) were originally out of order, and part c) was dropped because it was poorly worded.

3. Let $f(x) = \log x$

   (a) On what interval(s) is $f(x)$ positive and on what interval(s) is $f(x)$ negative?
   Positive on $(1, \infty)$ and negative on $(0, 1)$

   (b) On what interval(s) is $f(x)$ increasing and on what interval(s) is $f(x)$ decreasing?
   Increasing on $(0, \infty)$, nowhere decreasing

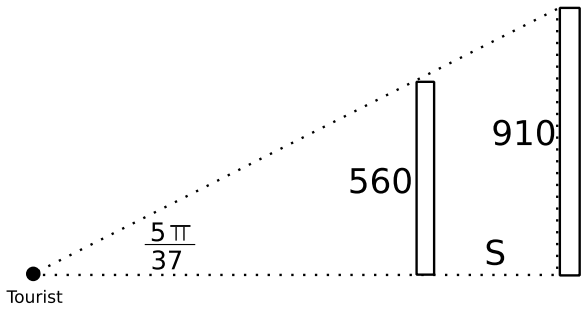
   (c) On what interval(s) is $f(x)$ concave up and on what interval(s) is $f(x)$ concave down?
   Concave down on $(0, \infty)$, nowhere concave up

4. Let $r(t) = t^2 + \frac{1}{t}$ and $s(t) = 1 + \frac{1}{t}$. Find and simplify $(r \circ s)(t)$

$$\begin{aligned} r\left(1 + \frac{1}{t}\right) &= \left(1 + \frac{1}{t}\right)^2 + \frac{1}{1 + \frac{1}{t}} = \left(\frac{t+1}{t}\right)^2 + \frac{1}{\frac{t+1}{t}} \\ &= \frac{(t+1)^2}{t^2} + \frac{t}{t+1} = \frac{(t+1)^2(t+1)}{t^2(t+1)} + \frac{t^3}{t^2(t+1)} \\ &= \frac{(t+1)^3 + t^3}{t^2(t+1)} = \frac{2t^3 + 3t^2 + 3t + 1}{t^2(t+1)} \end{aligned}$$

5. In 1976 it was estimated that there were 1200 giant Panda bears left in the wild, by 1987 there were only 800. Use a continuous growth / decay model to find when there would have only been 300 Pandas left.
   The basic model will be $A = Pe^{kt}$ where A is the amount left, P is the inital population, k a constant, and t the time. Measuring time from 1976, at $t = 0$ there were 1200 so $P = 1200$. We find the constant k by using the other data point $800 = 1200e^{11k}$, $k = \frac{\ln(\frac{2}{3})}{11} \approx -0.36860$. Now we find when there would have been 300 Pandas left by $300 = 1200e^{-0.36860t}$. Solve to get $t \approx 37.6$ years, so the year 2013.

6. A mathematically minded tourist in New York notices that from where he is standing the tops of two skyscrapers line up. He consults his guide book to find that the first is merely 560 ft high, but the building behind it is 910 ft high. If he is looking up at an angle of $\frac{5\pi}{37}$ to see the tops of the two buildings, what is the distance S between the two buildings?



Let x be the distance from the observer to the first building, then $\tan\frac{5\pi}{37} = \frac{560}{x}$, so we solve to get $x = \frac{560}{\tan\frac{5\pi}{37}} \approx 1238.8$ feet. The distance to the far away building is $x + S$, so by the same logic $\tan\frac{5\pi}{37} = \frac{910}{x+S}$. We solve to find

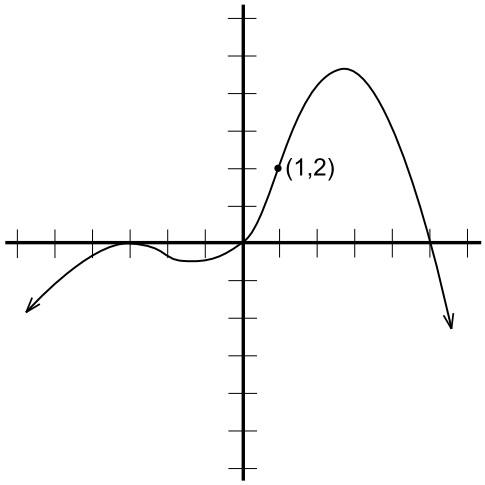
$$S = \frac{910}{\tan\frac{5\pi}{37}} - x \approx 774.2 \text{ feet}$$

7. Solve for x, where $(\ln x)^2 - \ln(x^3) + 2 = 0$. Hint: think about properties of logs before causing yourself a lot of pain.
We would be well advised to follow the hint for this problem. $\ln(x^3) = 3\ln(x)$ by the properties of logs, so

$$(\ln x)^2 - \ln(x^3) + 2 = (\ln x)^2 - 3\ln(x) + 2 = (\ln(x) - 1)(\ln(x) - 2) = 0$$

This gives $\ln(x) = 1$ and $\ln(x) = 2$, from which we find $x = e^1, e^2$.

8. Find the equation for the polynomial shown in the graph:



We can find the basic equation from the zeros of the graph, $f(x) = a(x+3)(x+0)(x-5)$. However, $x = -3$ is a turning point of the graph, so that factor must have even multiplicity: $f(x) = a(x+3)^2(x+0)(x-5)$. It only remains then to find the correct a so that $f(1) = 2$: $2 = a(1+3)^2(1+0)(1-5)$, so $a = -\frac{1}{32}$. Our final answer is $f(x) = -\frac{1}{32}x(x+3)^2(x-5)$.

9. Compute $\lim_{x\to 25} \dfrac{25-x}{5-\sqrt{x}}$

To simplify the limit we need to factor the numerator, after which everything cancels out nicely:

$$\lim_{x\to 25} \frac{25-x}{5-\sqrt{x}} = \lim_{x\to 25} \frac{(5-\sqrt{x})(5+\sqrt{x})}{5-\sqrt{x}} = \lim_{x\to 25} 5+\sqrt{x} = 5+\sqrt{25} = 10$$

10. Evaluate $\lim_{x\to\infty}\left[\frac{1}{2x}+\frac{5x^2-x+1}{2x^2+2}+3e^{-x}\right]$

Each of the limits exists on its own, so we can rewrite the limit of the sum as the sum of the limits:

$$\begin{aligned}\lim_{x\to\infty}\left[\frac{1}{2x}+\frac{5x^2-x+1}{2x^2+2}+3e^{-x}\right] &= \lim_{x\to\infty}\frac{1}{2x}+\lim_{x\to\infty}\frac{5x^2-x+1}{2x^2+2}+\lim_{x\to\infty}3e^{-x}\\ &= 0+\lim_{x\to\infty}\frac{5-\frac{1}{x}+\frac{1}{x^2}}{2+\frac{2}{x^2}}+0\\ &= \frac{\lim_{x\to\infty}\left(5-\frac{1}{x}+\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(2+\frac{2}{x^2}\right)}\\ &= \frac{\lim_{x\to\infty}5-\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}2+2\lim_{x\to\infty}\frac{1}{x^2}}\\ &= \frac{5}{2}\end{aligned}$$

11. For each function determine the domain and find the limit at the points where the function is not defined (if the limits exists). If the limit does not exist, say so.

(a) $g(x)=\dfrac{2x^2-6x-20}{x^2-2}$

The function will be undefined when the denominator is zero, so $x=\pm\sqrt{2}$. Therefore the domain is $(-\infty,-\sqrt{2})\cup(-\sqrt{2},\sqrt{2})\cup(\sqrt{2},\infty)$. If we try to take either of the limits we get some number over zero, so neither limit will exist. Alternatively, we could look at the graph and see that the function has vertical asymptotes at $x=\pm\sqrt{2}$, so the limit can't exist at those points.

(b) $h(x)=\dfrac{x^2-6x}{x-6}$

The function is except when $x=6$, so the domain is $(-\infty,6)\cup(6,\infty)$. At $x=6$ the limit will be

$$\lim_{x\to6}\frac{x^2-6x}{x-6}=\lim_{x\to6}\frac{x(x-6)}{x-6}=\lim_{x\to6}x=6$$

12. Find the equation of the line which is tangent to $f(x) = \dfrac{3}{x}$ at $(3, 1)$. Use the limit definition of the slope; do not use any formulas you might know for the derivative.
First we need to find the slope of the tangent line at $x = 3$. Just using the definition:

$$f'(3) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h} - \frac{3}{3}}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h} - \frac{3+h}{3+h}}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h(3+h)} = \lim_{h\to 0} \frac{-1}{3+h} = \frac{-1}{3}$$

The equation of the tangent line will be of the form $y = -\frac{1}{3}x + b$, so we just need to find b such that it passes through the point $(3, 1)$:

$$1 = -\frac{1}{3}(3) + b$$
$$2 = b$$

So $y = -\frac{1}{3}x + 2$