Linear algebra HW 7

Jordan canonical form

[1] Compute the Jordan canonical form for $A = \begin{pmatrix} 0 & 1 & -1 \\ -11 & 5 & -7 \\ -2 & -1 & -1 \end{pmatrix}$. You should be able to find the roots of the minimal polynomial by looking at it.

We need to find the eigenvalues,

$$0 = \det \begin{pmatrix} 0-\lambda & 1 & -1 \\ -11 & 5-\lambda & -7 \\ -2 & -1 & -1-\lambda \end{pmatrix} = \lambda^3 - 4\lambda^2 - 3\lambda + 18 = (\lambda+2)(\lambda-3)^2$$

So $\lambda = -2, 3, 3$. (The solutions to the polynomial can be found by checking that 3 is a root and using synthetic division). We compute the eigenvectors associated to the eigenvalues,

$$\ker\left(\begin{pmatrix} 0 & 1 & -1 \\ -11 & 5 & -7 \\ -2 & -1 & -1 \end{pmatrix} + 2I_3\right) = \ker\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 0 & 1 & -1 \\ -11 & 5 & -7 \\ -2 & -1 & -1 \end{pmatrix} - 3I_3\right) = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right\}$$

A is defective, so we need to add a generalized eigenvector for $\lambda = 3$ in order to find J.

$$\ker\left(\begin{pmatrix} 0 & 1 & -1 \\ -11 & 5 & -7 \\ -2 & -1 & -1 \end{pmatrix} - 3I_3\right)^2 = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

Forming the matrix of generalized eigenvectors,

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

we have

$$J = P^{-1}AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$\boxed{2}$ Use Matlab or another computer algebra system to find the Jordan canonical form of

$$A = \begin{pmatrix} 1 & -12 & 0 & 8 & 3.5 \\ 0 & -6 & 1 & 4 & 2.5 \\ -2 & -3 & 2 & 4 & 2 \\ -0.5 & -8.5 & 1.5 & 7 & 2.75 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

First we find the eigenvalues, which are $\lambda = -1, -1, 3, 3, 3$. We can compute that

$$\ker\left(\begin{pmatrix} 1 & -12 & 0 & 8 & 3.5 \\ 0 & -6 & 1 & 4 & 2.5 \\ -2 & -3 & 2 & 4 & 2 \\ -0.5 & -8.5 & 1.5 & 7 & 2.75 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} + I_5\right) = \ker\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Therefore the matrix is defective and we need generalized eigenvectors,

$$\ker\left(\begin{pmatrix} 1 & -12 & 0 & 8 & 3.5 \\ 0 & -6 & 1 & 4 & 2.5 \\ -2 & -3 & 2 & 4 & 2 \\ -0.5 & -8.5 & 1.5 & 7 & 2.75 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} + I_5\right)^2 = \ker\begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

Performing similar computations[1] for the other eigenvalue we see that

$$\ker\left(\begin{pmatrix} 1 & -12 & 0 & 8 & 3.5 \\ 0 & -6 & 1 & 4 & 2.5 \\ -2 & -3 & 2 & 4 & 2 \\ -0.5 & -8.5 & 1.5 & 7 & 2.75 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} - 3 * I_5\right) = \ker\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -0.5 & 0 \\ 0 & 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 1 & -12 & 0 & 8 & 3.5 \\ 0 & -6 & 1 & 4 & 2.5 \\ -2 & -3 & 2 & 4 & 2 \\ -0.5 & -8.5 & 1.5 & 7 & 2.75 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} - 3 * I_5\right)^2 = \ker\begin{pmatrix} 1 & 0 & 1 & -1.5 & 0 \\ 0 & 1 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 1 & -12 & 0 & 8 & 3.5 \\ 0 & -6 & 1 & 4 & 2.5 \\ -2 & -3 & 2 & 4 & 2 \\ -0.5 & -8.5 & 1.5 & 7 & 2.75 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} - 3 * I_5\right)^3 = \ker\begin{pmatrix} 1 & 0 & 1 & -1.5 & -1/4 \\ 0 & 1 & 0 & -0.5 & -1/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}\right\}$$

[1] In order to build up the basis of generalized eigenvectors as we go, I've choosen to keep the previous basis vectors, even if it means that the span might look a little less natural. Technically we could just computer a high enough power of $A - \lambda I_n$ and write down a basis for that matrix.

Therefore $P = \begin{pmatrix} 2 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ and conjugating by the matrix of generalized eigenvectors yields

$$J = P^{-1}AP = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$\boxed{3}$ Determine which of the following matrices are in Jordan canonical form:

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Yes, no, no

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Yes, yes, yes

$\boxed{4}$ Let $A = \begin{pmatrix} 10 & -2 \\ 3 & 3 \end{pmatrix}$, diagonalize A and use your results to find a matrix B such that $B^2 = A$.

A has eigenvalues $\lambda = 9, 4$, so diagonalization is possible. Finding the eigenvectors,

$$\ker \left(\begin{pmatrix} 10 & -2 \\ 3 & 3 \end{pmatrix} - 9I_3 \right) = \ker \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$\ker \left(\begin{pmatrix} 10 & -2 \\ 3 & 3 \end{pmatrix} - 4I_3 \right) = \ker \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

Therefore our matrix P is $P = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $P^{-1}AP = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$ as required. Let

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3.2 & -0.4 \\ 0.6 & 1.8 \end{pmatrix}$$

and we can easily check that $B^2 = A$.

5 What information about a matrix can you read off of it's Jordan canonical form? Can you think of something which you can't see from the Jordan canonical form?

From the Jordan canonical form you tell the eigenvalues, and the algebraic and geometric multipliticies of each eigenvalue. From that information you can tell the characteristic polynomial of the matrix and whether it is invertible (as well as other stuff). The JCF does not tell you the actual eigenvectors of a matrix, though once you know the eigenvalues it is relatively easy to find the eigenvectors.

6 If A is an 2×2 matrix with eigenvalues $2, 5$ and B is a 2×2 matrix with eigenvalues $1, 10$, is it possible that $A = C^{-1}BC$ for some matrix C? Justify your answer.

No. Both matrices are diagonalizable since they algebraic multiplicity 1 on all of their eigenvalues, so the Jordan form of A is $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 5 \end{smallmatrix}\right)$ and the Jordan form of B is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 10 \end{smallmatrix}\right)$. Since the Jordan form of a matrix is unique (up to reordering the blocks), and the Jordan forms of A and B are different, the matrices are not similar.