Linear algebra HW 6 - Solutions

Section 12.4

Note: when computing characteristic polynomials I always divided out by the leading coefficent to make them all monic. This is legitimate for finding the roots (after all, we are setting the polynomial equal to 0), but it is slightly imprecise to say that $\det(A-\lambda I_n) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$, *since the coefficent of* λ^n *might not actually be equal to 1.*

[3] Suppose T is a linear transformation and it satisfies $T^2 = T$ and $T(x) = x$ for all x in a certain (non-zero!) subspace, V. Show that 1 is an eigenvalue for T and show that all eigenvalues have absolute values no larger than 1.

First, let v be any nonzero vector in the subspace V. By assumption, $T(v) = v = 1v$, so $\lambda = 1$ is an eigenvalue of T. Now suppose that $T(w) = \lambda w$ for some vector w and eigenvalue λ. We know that

$$T^2(w) = T(T(w)) = T(\lambda w) = \lambda T(w) = \lambda^2 w$$

however $T^2 = T$ so $T^2(w) = T(w) = \lambda w$. Therefore $\lambda^2 w = \lambda w$, leaving only the possibilities that $\lambda = 0$ or $\lambda = 1$.

[4] Show that if $Ax = \lambda x$ and $Ay = \lambda y$, then whenever a, b are scalars,

$$A(ax + by) = \lambda(ax + by)$$

Does this imply that $ax + by$ is an eigenvector? Explain.

The first property is clear since

$$A(ax + by) = Aax + Aby = aAx + bAy = a\lambda x + b\lambda y = \lambda(ax + by)$$

Provided that $ax + by \neq 0$, then $ax + by$ is indeed an eigenvector (remember, 0 is never an eigenvector).

$\boxed{5}$ Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -19 & -14 & -1 \\ 8 & 4 & 8 \\ 15 & 30 & -3 \end{pmatrix}$$

Determine whether the matrix is defective.

First we need to find the eigenvalues,

$$\begin{aligned} 0 &= \begin{vmatrix} -19-\lambda & -14 & -1 \\ 8 & 4-\lambda & 8 \\ 15 & 30 & -3-\lambda \end{vmatrix} = \lambda^3 + 18\lambda^2 - 144\lambda - 2592 \\ &= (\lambda+12)(\lambda-12)(\lambda+18) \\ \lambda &= 12, -12, -18 \end{aligned}$$

We find the eigenvectors by determining a basis for $\ker(A - \lambda I)$ for each eigenvalue λ. (Remember that since the kernel is the solution to $Ax = 0$, doing row operations to a matrix doesn't change it's kernel).

$$\ker\left(\begin{pmatrix} -19 & -14 & -1 \\ 8 & 4 & 8 \\ 15 & 30 & -3 \end{pmatrix} - 12I_3\right) = \ker\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} -19 & -14 & -1 \\ 8 & 4 & 8 \\ 15 & 30 & -3 \end{pmatrix} + 12I_3\right) = \ker\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} -19 & -14 & -1 \\ 8 & 4 & 8 \\ 15 & 30 & -3 \end{pmatrix} + 18I_3\right) = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

The matrix is not defective since it has three eigenvalues, each of which has algebraic and geometric multiplicity 1.

$\boxed{6}$ Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -3 & -30 & 15 \\ 0 & 12 & 0 \\ 15 & 30 & -3 \end{pmatrix}$$

Determine whether the matrix is defective.

First we need to find the eigenvalues,

$$\begin{aligned} 0 &= \begin{vmatrix} -3-\lambda & -30 & 15 \\ 0 & 12-\lambda & 0 \\ 15 & 30 & -3-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 - 288\lambda + 2592 \\ &= (\lambda - 12)^2(\lambda + 18) \\ \lambda &= 12, 12, -18 \end{aligned}$$

We find the eigenvectors by determining a basis for $\ker(A - \lambda I)$ for each eigenvalue λ.

$$\ker\left(\begin{pmatrix} -3 & -30 & 15 \\ 0 & 12 & 0 \\ 15 & 30 & -3 \end{pmatrix} - 12I_3\right) = \ker\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} -19 & -14 & -1 \\ 8 & 4 & 8 \\ 15 & 30 & -3 \end{pmatrix} + 18I_3\right) = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

The eigenvalue 12 has algebraic and geometric multiplicity 2, and the eigenvalue -18 has algebraic and geometric multiplicity 1, so the matrix is not defective.

$\boxed{7}$ Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 8 & 4 & 5 \\ 0 & 12 & 9 \\ -2 & 2 & 10 \end{pmatrix}$$

Determine whether the matrix is defective.

Same procedure as the previous two problems. First we need to find the eigenvalues,

$$\begin{aligned} 0 &= \begin{vmatrix} 8-\lambda & 4 & 5 \\ 0 & 12-\lambda & 9 \\ -2 & 2 & 10-\lambda \end{vmatrix} = \lambda^3 - 30\lambda^2 + 288\lambda - 864 \\ &= (\lambda-12)^2(\lambda-6) \\ \lambda &= 12, 12, 6 \end{aligned}$$

$$\ker\left(\begin{pmatrix} 8 & 4 & 5 \\ 0 & 12 & 9 \\ -2 & 2 & 10 \end{pmatrix} - 12I_3\right) = \ker\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 8 & 4 & 5 \\ 0 & 12 & 9 \\ -2 & 2 & 10 \end{pmatrix} - 6I_3\right) = \ker\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1/2 \\ -3/2 \\ 1 \end{pmatrix}\right\}$$

This time the matrix is defective since the eigenvalue 12 has algebraic multiplicity 2 but only geometric multiplicity 1.

8 Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 7 & -2 & 0 \\ 8 & -1 & 0 \\ -2 & 4 & 6 \end{pmatrix}$$

Can you find three independent eigenvectors?

First we find the eigenvalues[1].

$$0 = \begin{vmatrix} 7-\lambda & -2 & 0 \\ 8 & -1-\lambda & 0 \\ -2 & 4 & 6-\lambda \end{vmatrix} = \lambda^3 - 12\lambda^2 + 45\lambda - 54$$
$$= (\lambda-6)(\lambda-3)^2$$
$$\lambda = 6, 3, 3$$

$$\ker\left(\begin{pmatrix} 7 & -2 & 0 \\ 8 & -1 & 0 \\ -2 & 4 & 6 \end{pmatrix} - 6I_3\right) = \ker\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 7 & -2 & 0 \\ 8 & -1 & 0 \\ -2 & 4 & 6 \end{pmatrix} - 3I_3\right) = \ker\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}\right\}$$

There are not 3 linearly independent vectors, hence the matrix is defective.

9 Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

Can you find three independent eigenvectors?

[1] We're just skipping a bunch of horrible algebra with row-column expansion. You should try to work out the polynomial though. It builds character. Although you could find the roots for these examples by hand since they're rational, that borders on masochism. If you did want to do it, you should use the rational root test.

$$0 = \begin{vmatrix} 3-\lambda & -2 & -1 \\ 0 & 5-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = \lambda^3 - 12\lambda^2 + 45\lambda - 54$$
$$= (\lambda-6)(\lambda-3)^2$$
$$\lambda = 6, 3, 3$$

$$\ker\left(\begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix} - 6I_3\right) = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix} - 3I_3\right) = \ker\begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}\right\}$$

Yes, there are 3 linearly independent eigenvectors (and the matrix is diagonalizable).

$\boxed{19}$ Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 & -6 \\ 7 & -5 & -6 \\ -1 & 7 & 2 \end{pmatrix}$

Determine whether the matrix is defective.

$$0 = \begin{vmatrix} 1-\lambda & 1 & -6 \\ 7 & -5-\lambda & -6 \\ -1 & 7-\lambda & 2 \end{vmatrix} = \lambda^3 - 2\lambda^2 + 16\lambda - 240$$
$$= (\lambda+6)(\lambda-(2+6i))(\lambda-(2-6i))$$
$$\lambda = -6, 2+6i, 2-6i$$

$$\ker\left(\begin{pmatrix}1 & 1 & -6\\ 7 & -5 & -6\\ -1 & 7 & 2\end{pmatrix} + 6I_3\right) = \ker\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix}1 & 1 & -6\\ 7 & -5 & -6\\ -1 & 7 & 2\end{pmatrix} - (2+6i)I_3\right) = \ker\begin{pmatrix}1 & 0 & -i\\ 0 & 1 & -i\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}i\\ i\\ 1\end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix}1 & 1 & -6\\ 7 & -5 & -6\\ -1 & 7 & 2\end{pmatrix} - (2-6i)I_3\right) = \ker\begin{pmatrix}1 & 0 & i\\ 0 & 1 & i\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}-i\\ -i\\ 1\end{pmatrix}\right\}$$

All of the algebraic and geometric multiplicities are 1, so they are all equal and the matrix is not defective.

20 Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix}4 & 2 & 0\\ -2 & 4 & 0\\ -2 & 2 & 6\end{pmatrix}$

Determine whether the matrix is defective.

$$0 = \begin{vmatrix}4-\lambda & 2 & 0\\ -2 & 4-\lambda & 0\\ -2 & 2 & 6-\lambda\end{vmatrix} = \lambda^3 - 14\lambda^2 + 68\lambda - 120$$

$$= (\lambda - 6)(\lambda - (4+2i))(\lambda - (4-2i))$$

$$\lambda = 6, 4+2i, 4-2i$$

$$\ker\left(\begin{pmatrix}4 & 2 & 0\\ -2 & 4 & 0\\ -2 & 2 & 6\end{pmatrix} - 6I_3\right) = \ker\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix}4 & 2 & 0\\ -2 & 4 & 0\\ -2 & 2 & 6\end{pmatrix} - (4+2i)I_3\right) = \ker\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -i\\ 0 & 0 & 0\end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix}1\\ i\\ 1\end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix}4 & 2 & 0\\ -2 & 4 & 0\\ -2 & 2 & 6\end{pmatrix} - (4-2i)I_3\right) = \ker\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & i\\ 0 & 0 & 0\end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix}1\\ -i\\ 1\end{pmatrix}\right\}$$

The matrix is not defective. We could have come to this conclusion as soon as we saw that all of the eigenvalues had algebraic multiplicity 1, since the geometric multiplicity is always less than or equal to the algebraic multiplicity and it can't be 0 (if it were 0 then we didn't have an eigenvalue).

21 Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

Presumably the problem means that T is a linear transformation of $\mathbb{R}^2$, since otherwise it would have specified a plane through which to reflect the vectors. Let e_1 and e_2 be the standard basis vectors, then $T(e_1) = e_1$ and $T(e_2) = -e_2$ and the matrix of T is $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. This has eigenvalues $1, -1$. You can guess the eigenvectors just by looking at the transformation, but doing it algebraicly,

$$\ker\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - I_2\right) = \ker\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix}1\\ 0\end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + I_2\right) = \ker\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix}0\\ 1\end{pmatrix}\right\}$$

$\boxed{22}$ Let T be the linear transformation which rotates all vectors in $\mathbb{R}^2$ counterclock- wise through an angle of $\pi/2$. Find a matrix of T and then find eigenvalues and eigenvectors.

As always, we look at the action of T on the standard basis vectors. $T(e_1) = e_2$ and $T(e_2) = -e_1$. The matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We find the eigenvectors,

$$\begin{aligned} 0 &= \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 + 1 \\ &= (\lambda - i)(\lambda + i) \\ \lambda &= i, -i \end{aligned}$$

$$\ker\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - iI_2\right) = \ker\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + iI_2\right) = \ker\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$$

$\boxed{23}$ Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

Reflecting through the xy-plane fixes any vector in the xy-plane, so $T(e_1) = e_1$ and $T(e_2) = e_2$. The only thing which changes is the z coordinate, $T(e_3) = -e_3$. Thus the matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which has eigenvalues $1, 1, -1$.

$$\ker\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} - I_3\right) = \ker\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

$$\ker\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} + I_3\right) = \ker\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$\boxed{32}$ Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

Yes, the only eigenvalues of nilpotent matrix are 0. For example, $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ only has 0 for an eigenvalue.