Linear algebra HW 5 - Solutions

Section 9.3

$\boxed{1}$ Study the definition of a linear transformation. State it from memory.

A linear transformation is a function $T : V \to W$ between vector spaces such that $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all scalars α and β and all vectors u and v.

$\boxed{4}$ Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$.

Let T be such a linear transformation, we look at the action of T on the standard basis vectors e_1 and e_2. By trig, $T(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ and $T(e_2) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$. The matrix of T is

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$\boxed{11}$ Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.

The matrix for the linear transformation which reflects across the x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the matrix for reflection across the y-axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (in both cases just look at the effect on the standard basis vectors). The matrix for the composition of all 3 transformations is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$$

$\boxed{15}$ Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/6$.

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$\boxed{17}$ Find the matrix for $\text{proj}_u(v)$ where $u = (1, -2, 3)$.

Look at the action on the standard basis vectors.

$$\text{proj}_u(e_1) = \frac{e_1 \cdot (1, -2, 3)}{(1, -2, 3) \cdot (1, -2, 3)}(1, -2, 3)^T = \frac{1}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\text{proj}_u(e_2) = \frac{e_2 \cdot (1, -2, 3)}{(1, -2, 3) \cdot (1, -2, 3)}(1, -2, 3)^T = \frac{-1}{7} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\text{proj}_u(e_3) = \frac{e_3 \cdot (1, -2, 3)}{(1, -2, 3) \cdot (1, -2, 3)}(1, -2, 3)^T = \frac{3}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

The matrix for $\text{proj}_u(v)$ is thus

$$\begin{pmatrix} \frac{1}{14} & -\frac{1}{7} & \frac{3}{14} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{3}{7} \\ \frac{3}{14} & -\frac{3}{7} & \frac{9}{14} \end{pmatrix}$$

$\boxed{20}$ Show that the function T_u defined by $T_u(v) := v - \text{proj}_u(v)$ is also a linear transformation.

Let α be any scalar and v, w be any vectors.

$$\begin{aligned} T_u(v+w) &= v + w - \text{proj}_u(v+w) = v + w - \frac{(v+w)\cdot u}{u \cdot u} u \\ &= v + w - \frac{(v\cdot u) + (w \cdot u)}{u\cdot u} u = v - \text{proj}_u(v) + w - \text{proj}_u(w) \\ &= T_u(v) + T_u(w) \end{aligned}$$

Similarly

$$T_u(\alpha v) = \alpha v - \frac{(\alpha v)\cdot u}{u \cdot u} u = \alpha \left(v - \frac{v \cdot u}{u \cdot u} u \right) = \alpha T_u(v)$$

$\boxed{21}$ If $u = (1, 2, 3)^T$, as in Example 17.5.20 and T_u is given in the above problem, find the matrix, A_u which satisfies $A_u x = T_u(x)$.

We could just compute $T_u(e_1), T_u(e_2), T_u(e_3)$ and be done. But if want to avoid doing more work than necessary, we can observe that the matrix of the sum of two linear transformations is the sum of the matrices of the transformations. Thus

$$\begin{aligned} A_u &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \\ -\frac{1}{7} & -\frac{2}{7} & -\frac{3}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{14} & -\frac{1}{7} & -\frac{3}{14} \\ \frac{1}{7} & 1 + \frac{2}{7} & \frac{3}{7} \\ -\frac{3}{14} & -\frac{3}{7} & 1 - \frac{9}{14} \end{pmatrix} \\ &= \begin{pmatrix} \frac{13}{14} & -\frac{1}{7} & -\frac{3}{14} \\ \frac{1}{7} & \frac{9}{7} & \frac{3}{7} \\ -\frac{3}{14} & -\frac{3}{7} & \frac{5}{14} \end{pmatrix} \end{aligned}$$

$\boxed{22}$ Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to finding a basis for the kernel of the matrix. Computing the rref of the augmented matrix we find that

$$\begin{pmatrix} 1 & -1 & 2 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 3 & -4 & 5 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Thus $x = -3z$ and $y = -z$, so all solutions are of the form $\begin{pmatrix} -3z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$. The solution set is Span $\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$.

[23] Using Problem 22 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

By using the rref of the augemented matrix we find that

$$\begin{pmatrix} 1 & -1 & 2 & | & 1 \\ 1 & -2 & 1 & | & 2 \\ 3 & -4 & 5 & | & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & | & 0 \\ 0 & 1 & 1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Choosing $z = 0$ we find (one of many) solutions, $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. The general solution to the system is the set of all vectors of the form

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \text{Span} \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

[24] Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Compute the rref of the augmented matrix:

$$\begin{pmatrix} 0 & -1 & 2 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & -4 & 5 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Thus the kernel is Span $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

[25] Using Problem 24 find the general solution to the following linear system.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Again, use the rref to find a specific solution:

$$\begin{pmatrix} 0 & -1 & 2 & | & 1 \\ 1 & -2 & 1 & | & -1 \\ 1 & -4 & 5 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & | & -3 \\ 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Taking $z = 0$ we have a particular solution: $\begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$ and the general solution system is

$$\begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} + \text{Span} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$\boxed{43}$ Show that if A is an $m \times n$ matrix, then $\ker(A)$ is a subspace.

Let $u, v \in \ker(A)$ and let α be any scalar. By assumption, $Au = Av = 0$. So $A(u + v) = 0$ which implies $u + v \in \ker(A)$. Similarly, $A(\alpha u) = \alpha Au = 0$.