Linear algebra HW 4 - Solutions

Section 8.6

4 Determine which matrices are in row reduced echelon form.

a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{pmatrix}$ Not in rref.

b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ In rref.

c) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$ In rref.

5 Row reduce the following matrices to obtain the row reduced echelon form. List the pivot columns in the original matrix.

a) $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{pmatrix}$ has rref $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$. Columns 1, 2, and 3 of the original matrix are pivot columns.

b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ has rref $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Columns 1, 2, and 3 of the original matrix are pivot columns.

c) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}$ has rref $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Columns 1, 2, and 4 of the original matrix are pivot columns.

6 Find the rank of the following matrices. If the rank is r, identify r columns **in the original matrix** which have the property that every other column may be written as a linear combination of these. Also find a basis for the row and column spaces of the matrices.

a) $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ has rref $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ has rref $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

c) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 3 & 2 & 12 & 1 & 6 & 8 \\ 0 & 1 & 1 & 5 & 0 & 2 & 3 \\ 0 & 2 & 1 & 7 & 0 & 3 & 4 \end{pmatrix}$ has rref $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 6 & 0 & 5 & 4 \\ 0 & 1 & 1 & 2 & 0 & 2 & 2 \\ 0 & 2 & 1 & 4 & 0 & 3 & 2 \end{pmatrix}$ has rref $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

e) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 3 & 2 & 6 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 4 & 0 & 3 & 1 \end{pmatrix}$ has rref $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$\boxed{7}$ Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

Because the number of linearly independent rows in the rref is the same as the number of linearly independent columns in the rref. There cannot be more linearly independent columns than the total number of columns in the matrix, and likewise for rows. Thus the rank is limited by whichever number is smaller (note: this doesn't mean that $\text{Rank}(A) = \min(m, n)$, only that $\text{Rank}(A) \leq \min(m, n)$).

$\boxed{8}$ Let H denote $\text{Span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$. Find the dimension of H and determine a basis.

We write the matrix consisting of the vectors generating the span, and then row reduce that matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There are two linearly independent columns, so $\dim(H) = 2$. Columns 1 and 3 of the rref generate column 2, and this relationship is preserved under row operations, so

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

is a basis for H.

$\boxed{11}$ Let $M = \{u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_3 = u_1 = 0\}$. Is M a subspace? Explain.

Let $u, v \in M$ and let α be any scalar.

$$\alpha u = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \\ \alpha u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha u_2 \\ 0 \\ \alpha u_4 \end{pmatrix} \in M$$

and

$$u + v = \begin{pmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 + v_2 \\ 0 \\ u_4 + v_4 \end{pmatrix} \in M$$

So M is a subspace.

$\boxed{12}$ Let $M = \{u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_3 \geq u_1\}$. Is M a subspace? Explain.

No, M is not a subspace. Consider the vector $(1, 2, 3, 4)^T \in M$. If M were a subspace then $-(1, 2, 3, 4)^T = (-1, -2, -3, -4)^T$ would also have to be in M. But $-3 \not\geq -1$.

$\boxed{13}$ Let $w \in \mathbb{R}^4$ and let $M = \{u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : w \cdot u = 0\}$. Is M a subspace? Explain.

Let $u, v \in M$ and let α be any scalar. We check both properties of being a subspace:

$$w \cdot (\alpha u) = \alpha(w \cdot u) = \alpha(0) = 0$$

$$w \cdot (u + v) = w \cdot u + w \cdot v = 0 + 0 = 0$$

So αu and $u + v$ are both in M. Therefore M is a subspace.

$\boxed{14}$ Let $M = \{u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_i \geq 0 \text{ for each } i = 1, 2, 3, 4\}$. Is M a subspace? Explain.

No. Like with a previous problem, if $u = (1, 2, 3, 4)$ then $u \in M$. But $-u = (-1, -2, -3, -4) \notin M$.

$\boxed{19}$ Suppose $\{x_1, \ldots, x_k\}$ is a set of vectors from $\mathbb{F}^n$. Show that 0 is in $\text{Span}(x_1, \ldots, x_k)$.

This is trivial, just take all of the scalars multiples to be 0.

$$0 = \sum_{i=1}^{k} 0x_i$$

$\boxed{23}$ Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

It can be checked that

$$\begin{pmatrix} 4 & 2 & 3 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The cols of the rref are linearly independent, so the original columns were linearly independent.

$\boxed{24}$ Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\1\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}$$

It can be checked that

$$\begin{pmatrix}1&4&3\\2&5&1\\3&1&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

The cols of the rref are linearly independent, so the original columns were linearly independent.

25 Here are four vectors. Determine whether they span $\mathbb{R}^3$. Are these vectors linearly independent?

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\3\\3\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}$$

There cannot be 4 linearly independent vectors in $\mathbb{R}^3$, so we know the answer to the second part is "no" before doing any work. We compute the rref of the matrix of columns,

$$\begin{pmatrix}1&4&3&2\\2&3&1&4\\3&3&0&6\end{pmatrix} \sim \begin{pmatrix}1&0&-1&2\\0&1&1&0\\0&0&0&0\end{pmatrix}$$

The dimension of the span of a set of vectors is equal to the rank of the matrix formed from those vectors. In this case the span is only 2-dimensional, so it cannot span $\mathbb{R}^3$.

26 Here are four vectors. Determine whether they span $\mathbb{R}^3$. Are these vectors linearly independent?

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\3\\3\end{pmatrix}, \begin{pmatrix}3\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}$$

There cannot be 4 linearly independent vectors in $\mathbb{R}^3$, so we know the answer to the second part is "no" before doing any work. It can be checked that

$$\begin{pmatrix}1&4&3&2\\2&3&2&4\\3&3&0&6\end{pmatrix} \sim \begin{pmatrix}1&0&0&2\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

from which it follows that the dimension of the span is 3, so the vectors span all of $\mathbb{R}^3$.

27 Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}4\\3\\3\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}$$

There cannot be 4 linearly independent vectors in $\mathbb{R}^3$, so they cannot be a basis. It can be checked that

$$\begin{pmatrix}1&4&1&2\\0&3&2&4\\3&3&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\end{pmatrix}$$

from which it follows that the dimension of the span is 3, so the vectors span all of $\mathbb{R}^3$.

28 Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}$$

The matrix of the vectors has I_3 for it's rref, so they are linearly independent. The dimension of the span is 3, so they span $\mathbb{R}^3$. So they are a basis.

34 If you have 5 vectors in $\mathbb{F}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{F}^5$? Explain.

Yes. If they did not span all of $\mathbb{F}^5$ then it would be possible to extend it to a basis of $\mathbb{F}^5$ (since any linearly independent set can be extended to a basis). But this would produce a basis for a 5-dimensional vector space with more than 5 elements, which is impossible.

35 If you have 6 vectors in $\mathbb{F}^5$, is it possible they are linearly independent? Explain.

No. If they were linearly independent then their span would be 6 dimensional. But a 5-dimensional vector space cannot contain a 6-dimensional subspace. (If you want to be a little more rigorous you could replace the last line with "it would be possible to extend the set to a basis of $\mathbb{F}^5$ with 6 or more elements, which is impossible since $\dim(\mathbb{F}^5) = 5$).

40 If $b \neq 0$, can the solution set of $Ax = b$ be a plane through the origin? Explain.

No. $A0 = 0$ and b is not zero, so the solution set can't be a plane through the origin. (As we know from a later section, it is however a plane through the origin shifted by some vector).

41 Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

No. If A is matrix of coefficients for the system of equations, then the columns of A are linearly dependent (since there is one column for each variable and one row for each equation). This implies that $\ker(A) \neq \{0\}$. If there is a solution to the system $Ax = b$, then $A(x + v) = b$ for every vector $v \in \ker(A)$.

42 Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

No. If you want a low-tech explanation just consider that this means there is a row which is 0 except for the right most entry which is 1, which says $0 = 1$ (remember, this is an augmented matrix). For a higher tech explanation, we can say that the 4th column is not in the span of the first 3 columns, which in turn means the system of equations cannot be satisfied.

43 Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

Yes and yes. If every column is a pivot column then $\text{Rank}(A) = n$. This is equivalent to saying that A is invertible, so $Ax = b$ has a solution for any choice of b, namely $x = A^{-1}b$. Since A is invertible, $Av = Au \Rightarrow u = v$, so the solutions are unique.

44 Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix.

Let A be the coefficient matrix of the system, then the solution to $Ax = b$ is unique (provided it exists) if and only if $\ker(A) = \{0\}$. By the rank nullity theorem, $\text{Rank}(A) + \dim(\ker(A)) = n$ (where A is an $m \times n$ matrix), so for a unique solution $\text{Rank}(A) = n$. This means that every column is a pivot column.

46 Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that A maps $\mathbb{F}^n$ onto $\mathbb{F}^m$. **Hint:** The vectors $e_1, \ldots, e_m$ occur as columns in the row reduced echelon form for A.

Observe that the image of A (recall that $\text{Im }(A) = \{Av | v \in F^n\}$, in other words the set of all of the vectors which multiplication by A can produce) is a subspace of $\mathbb{F}^m$ since if $Au_1 = v_1$ and $Au_2 = v_2$ then $A(u_1 + u_2) = v_1 + v_2$ and $A(\alpha u_1) = \alpha A u_1 = \alpha v_1$. The image is the same as the row space of A, and the dimension of the row space is always $\text{Rank}(A)$. Therefore $\dim(\text{Im }(A)) = \text{Rank}(A) = m$, so $\text{Im }(A) = \mathbb{F}^m$.

47 Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. **Hint:** If not, there exists a vector, x such that $Ax = 0$, and this implies at least one column of A is a linear combination of the others. Show this would require the column rank to be less than n.

We will prove this in a slightly different way. A is one to one if and only if $\ker(A) = \{0\}$. We can find a basis for the kernel of A by solving the equation $Ax = 0$, which is the same thing as computing the rref of the augmented matrix of A with an extra column of zeros. Under row operations the column of zeros remains zero, so the rref of the augmented matrix is $\begin{pmatrix} e_1 & e_2 & \dots & e_n & 0 \end{pmatrix}$. There are no free variables in the rref, so $\ker(A) = \{0\}$ as required.

$\boxed{48}$ Explain why an $n \times n$ matrix, A is both one to one and onto if and only if its rank is n.

Observe that for an $n \times n$ matrix, having rank n is equivalent to being invertible. If A is invertible then it is clearly one to one since if $Av = Au$ then $A^{-1}Av = A^{-1}Au$ implies $u = v$. Onto is also obvious, since if v is any vector then $A^{-1}v$ is a vector which is mapped onto v: $A(A^{-1}v) = v$.

By the rank nullity theorem, $\text{Rank}(A) + \dim(\ker(A)) = n$. Since A is one to one, it has trivial kernel, $\ker(A) = \{0\}$. This means that $\text{Rank}(A) = n$.[1]

[1] An astute reader may have noticed that we didn't use the assumption that A is onto. This was unnecessary since for $n \times n$ matrices, a matrix is onto if and only if it is one to one (think about why this is true).