Linear algebra HW 3 - Solutions

Section 6.3

$\boxed{1}$ Find the determinants of the following matrices.

a) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{pmatrix}$

Use cofactor expansion along the first column (since it has a 0):

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{vmatrix} &= 1(-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 9 & 8 \end{vmatrix} + 3(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 9 & 8 \end{vmatrix} + 0(-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} \\ &= 2(8) - 2(9) - 3(2(8) - 3(9)) = 31 \end{aligned}$$

b) $\begin{pmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{pmatrix}$ Use cofactor expansion along the first column (though any other would be ok too)

$$\begin{aligned} \begin{vmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{vmatrix} &= 4(-1)^{1+1} \begin{vmatrix} 7 & 8 \\ -9 & 3 \end{vmatrix} + 1(-1)^{2+1} \begin{vmatrix} 3 & 2 \\ -9 & 3 \end{vmatrix} + 3(-1)^{3+1} \begin{vmatrix} 3 & 2 \\ 7 & 8 \end{vmatrix} \\ &= 4(7(3) - 8(-9)) - (3(3) - 2(-9)) + 3(3(8) - 7(2)) = 375 \end{aligned}$$

c) $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$

Although we could do this with cofactor expansion, we would have to use it twice, which would produce a huge number of terms. So row operations are a better choice here

$$\begin{vmatrix}1&2&3&2\\1&3&2&3\\4&1&5&0\\1&2&1&2\end{vmatrix}=\begin{vmatrix}1&2&3&2\\0&1&-1&1\\0&-7&-7&-8\\0&0&-2&0\end{vmatrix}=\begin{vmatrix}1&2&3&2\\0&1&-1&1\\0&0&-14&-1\\0&0&-2&0\end{vmatrix}=\begin{vmatrix}1&2&3&2\\0&1&-1&1\\0&0&0&-1\\0&0&-2&0\end{vmatrix}$$

$$=-\begin{vmatrix}1&2&3&2\\0&1&-1&1\\0&0&-2&0\\0&0&0&-1\end{vmatrix}=-(1)(1)(-2)(-1)=-2$$

$\boxed{7}$ Find the determinant using row operations.

$$\begin{vmatrix}2&1&3\\2&4&2\\1&4&-5\end{vmatrix}$$

$$\begin{vmatrix}2&1&3\\2&4&2\\1&4&-5\end{vmatrix}=2\begin{vmatrix}1&\frac{1}{2}&\frac{3}{2}\\2&4&2\\1&4&-5\end{vmatrix}=2\begin{vmatrix}1&\frac{1}{2}&\frac{3}{2}\\0&3&-1\\0&\frac{7}{2}&-\frac{13}{2}\end{vmatrix}=2(3)\left(\frac{7}{2}\right)\begin{vmatrix}1&\frac{1}{2}&\frac{3}{2}\\0&1&-\frac{1}{3}\\0&1&-\frac{13}{7}\end{vmatrix}$$

$$=21\begin{vmatrix}1&\frac{1}{2}&\frac{3}{2}\\0&1&-\frac{1}{3}\\0&0&-\frac{32}{21}\end{vmatrix}=21(1)(1)\left(-\frac{32}{21}\right)=-32$$

$\boxed{8}$ Find the determinant using row operations.

$$\begin{vmatrix}1&2&1&2\\3&1&-2&3\\-1&0&3&1\\2&3&2&-2\end{vmatrix}$$

$$\begin{aligned}\begin{vmatrix}1&2&1&2\\3&1&-2&3\\-1&0&3&1\\2&3&2&-2\end{vmatrix} &= \begin{vmatrix}1&2&1&2\\0&-5&-5&-3\\0&2&4&3\\0&-1&0&-6\end{vmatrix} = \begin{vmatrix}1&2&1&2\\0&0&-5&27\\0&0&4&-9\\0&-1&0&-6\end{vmatrix} = \begin{vmatrix}1&2&1&2\\0&-1&0&-6\\0&0&-5&27\\0&0&4&-9\end{vmatrix}\\ &= \begin{vmatrix}1&2&1&2\\0&-1&0&-6\\0&0&-1&18\\0&0&4&-9\end{vmatrix} = \begin{vmatrix}1&2&1&2\\0&-1&0&-6\\0&0&-1&18\\0&0&0&63\end{vmatrix} = (1)(-1)(-1)(63)\\ &= 63\end{aligned}$$

$\boxed{17}$ Show $\det(aA) = a^n \det(A)$ where here A is an $n \times n$ matrix and a is a scalar.

Observe that $\det(aI_n) = a^n$ since aI_n is an upper triangular matrix with a along the main diagonal. Then $\det(aA) = \det(aI_nA) = \det(aI_n)\det(A) = a^n\det(A)$.

$\boxed{18}$ Prove by doing computations that $\det(AB) = \det(A)\det(B)$ if A and B are 2×2 matrices.

$$\begin{aligned}\det\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}e&f\\g&h\end{pmatrix}\right) &= \det\begin{pmatrix}ae+bg&af+bh\\ce+dg&cf+dh\end{pmatrix}\\ &= (ae+bg)(cf+dh)-(ce+dg)(af+bh)\\ &= aecf+aedh+bgcf+bgdh-(ceaf+cebh+dgaf+dgbh)\\ &= adeh-adfg-bceh+bcfg\\ \det\begin{pmatrix}a&b\\c&d\end{pmatrix}\det\begin{pmatrix}e&f\\g&h\end{pmatrix} &= (ad-bc)(eh-fg)\\ &= adeh-adfg-bceh+bcfg\end{aligned}$$

$\boxed{22}$ A matrix is said to be orthogonal if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

If A is orthogonal, then taking the determinant of $A^T A = I$ tells us that $\det(A^T)\det(A) = 1$. For any matrix $\det(A^T) = \det(A)$, so in this case $\det(A)^2 = 1$. Thus $\det(A) = \pm 1$ for orthogonal matrices.

$\boxed{27}$ Tell whether the statement is true or false.

a) If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column. **False**. One column could be a linear combination of the other two columns and the matrix would still have determinant zero.

b) If any two columns of a square matrix are equal, then the determinant of the matrix equals zero. **True**.

c) For A and B two $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$. **False**.

d) For A an $n \times n$ matrix, $\det(3A) = 3\det(A)$. **False**. What is true is that $\det(3A) = 3^n \det(A)$.

e) If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$. **True**.

f) If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4\det(A)$. **True**.

g) For A an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$. **True**.

h) If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$. **True**.

i) Cramers rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions. **False**. Cramer's rule only applies to systems of equations where there are as many variables as equations (ie square matrices), and such a system either has a unique solution or no solution.

j) If $A^k = 0$ for some positive integer, k, then $\det(A) = 0$. **True**.

k) If $Ax = 0$ for some $x \neq 0$, then $\det(A) = 0$. **True**.

$\boxed{29}$ Use Cramers rule to find the solution to

$$\begin{aligned} x + 2y + z &= 1 \\ 2x - y - z &= 2 \\ x + z &= 1 \end{aligned}$$

$\boxed{30}$ Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

$\boxed{31}$ Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

$\boxed{37}$ Here is a matrix,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

We have to compute the determinant, set it equal to zero and try to solve to see if there is such a t. Using cofactor expansion along the first first column we see that

$$\begin{aligned} \det \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix} &= 1(-1)^{1+1} \det \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix} + 0(-1)^{2+1} \begin{pmatrix} t & t^2 \\ 0 & 2 \end{pmatrix} + t(-1)^{3+1} \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} \\ &= 2 + t(2t^2 - t^2) = t^3 + 2 \end{aligned}$$

The determinant is zero if and only if $t^3 + 2 = 0$. So when $t = -\sqrt[3]{2}$ the matrix has no inverse.

$\boxed{44}$ Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Is everything the same for lower triangular matrices?

The determinant of an upper triangular matrix is the product of the entries on the main diagonal, and A is invertible if and only if $\det(A) \neq 0$. So A has an inverse if and only if all of the main diagonal entries are nonzero.

The inverse of an upper triangular matrix will be upper triangular, to see this write down the augmented matrix $(A|I_n)$. To row reduce A to the identity matrix, we first add appropriate scalar multiples of the bottom row to those above it so that the right column of A is zero exact for A_{nn}. Then we add multiples of the second to last row to the rows above it, etc. On the right, I_n is upper triangular, and each of the operations on the left-hand-side keep the right-side upper triangular.

For lower triangular matrices, we could repeat the same arguement we made (starting with the first row instead of the last). Instead of repeating we could also just note that if A is lower triangular then A^T is upper triangular, so $(A^T)^{-1}$ is upper triangular. As we showed in class, $(A^T)^{-1} = (A^{-1})^T$, so that matrix is upper triangular. Thus $((A^{-1})^T)^T = A^{-1}$ is lower triangular.

$\boxed{45}$ If A, B, and C are each $n \times n$ matrices and ABC is invertible, why are each of A, B, and C invertible.

ABC is invertible if and only if $\det(ABC) \neq 0$. But $\det(ABC) = \det(A)\det(B)\det(C)$, from which we conclude that $\det(A)$, $\det(B)$, and $\det(C)$ must all be nonzero. Therefore A, B, and C are all invertible.

Section 6.4

Solutions are already in the book.