

Linear algebra HW 2 - Solutions

Section 4.2

4 Here are some matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & 5 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}, D = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, E = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Find the following if possible and explain why it is not possible if this is the case. $AD, DA, D^T B, D^T BE, E^T D, DE^T$.

- $AD = \begin{pmatrix} 7 \\ 5 \\ -5 \end{pmatrix}$
- DA Undefined.
- $D^T B = (-14 \ 3 \ 6)$
- $D^T BE$ Undefined.
- $E^T D = (11)$
- $DE^T = \begin{pmatrix} -1 & -3 \\ 4 & 12 \end{pmatrix}$

6 Suppose A and B are square matrices of the same size. Which of the following are correct?

a) $(A - B)^2 = A^2 - 2AB + B^2$. *False.* $(A - B)^2 = A^2 - AB - BA + B^2$.

b) $(AB)^2 = A^2B^2$. *False.* $(AB)^2 = ABAB$.

c) $(A + B)^2 = A^2 + 2AB + B^2$. *False.* $(A + B)^2 = A^2 + AB + BA + B^2$.

d) $(A + B)^2 = A^2 + AB + BA + B^2$. *True.*

e) $A^2B^2 = A(AB)B$. *True.*

f) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$. *False.*

$$\begin{aligned}(A + B)^3 &= (A + B)(A^2 + AB + BA + B^2) \\ &= A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3\end{aligned}$$

g) $(A+B)(A-B) = A^2 - B^2$. *False.* $(A+B)(A-B) = A^2 - AB + BA - B^2$.

9] Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Clearly we choose $k = 4$ since every matrix commutes with itself.

12] Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix. Hint: Show that $\frac{1}{2}(A^T + A)$ is symmetric and then consider using this as one of the matrices.

First note that $\frac{1}{2}(A^T + A)$ is symmetric since

$$\left(\frac{1}{2}(A^T + A)\right)^T = \frac{1}{2}((A^T)^T + A^T) = \frac{1}{2}(A + A^T)$$

If this is to be the symmetric part we can solve for the skew symmetric part (after all, they add up to A),

$$A - \frac{1}{2}(A^T + A) = \frac{1}{2}(A - A^T)$$

This is skew symmetric as required:

$$\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - A) = -\left(\frac{1}{2}(A - A^T)\right)$$

22 Find your own examples:

a) 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ with $AB \neq BA$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \text{ versus } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$$

b) 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ with $AB = 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

c) 2×2 matrices A , D , and C with $A \neq 0$, $C \neq D$ such that $AC = AD$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, let $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$. Then

$$AC = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = AD$$

27 Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$. Find A^{-1} if possible. If A^{-1} does not exist, determine why.

Setup the augmented matrix and use row operations,

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{7}{2} & \frac{1}{2} & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} \end{array} \right)$$

[30] Let $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$. Find A^{-1} if possible. If A^{-1} does not exist, determine why.

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & 1 \end{array} \right)$$

It is clear that the row reduced echelon form of our matrix is *not* the identity matrix I_2 , so it is impossible to use a series of row operations to “cancel out” the matrix and solve the equation

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} B = I_2$$

Thus there is no inverse.

[32] Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}$. Find A^{-1} if possible. If A^{-1} does not exist, determine why.

We solve this the same as the inverse problems for the smaller 2×2 matrices.

$$\begin{aligned} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right) &\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & -2 & 3 \end{array} \right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 & 3 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4 & -5 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 & 3 \end{array} \right) \end{aligned}$$

39] Using the inverse of the matrix, find the solution to the systems

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Using the same methods as above, we compute the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}$$

Then via the marvels of matrix multiplication we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{4}{3} \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ \frac{1}{3} \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{5}{3} \\ 1 \end{pmatrix}$$

Now give a solution in terms of a, b, c to

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We perform the same computation but with indeteriminants,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a + 3c \\ \frac{1}{3}b - \frac{2}{3}c \\ a - c \end{pmatrix}$$

48 If A is invertible, show that $(A^T)^{-1} = (A^{-1})^T$.

This one is a little tricky at first glance. Starting with what we know is true, $AA^{-1} = I_n$ we take the transpose of both sides

$$(AA^{-1})^T = I_n^T = I_n$$

Transpose reverse the order of products, so

$$(A^{-1})^T A^T = I_n$$

Right multiplying by $(A^T)^{-1}$ gives us $(A^{-1})^T = (A^T)^{-1}$ as required.

49 If A is invertible, show that $(A^2)^{-1} = (A^{-1})^2$.

Since $A^2(A^{-1})^2 = A^2(A^{-1})(A^{-1}) = A(AA^{-1})A^{-1} = I_n$, and the inverse of A^2 is unique, $(A^2)^{-1} = (A^{-1})^2$.