Linear algebra HW 1 - Solutions

Section 2.6

[2] Compute $5(1, 2 + 3i, 3, -2) + 6(2 - i, 1, -2, 7)$

$$\begin{aligned} 5(1, 2 + 3i, 3, -2) + 6(2 - i, 1, -2, 7) &= (5, 10 + 15i, 15, -10) + (12 - 6i, 6, -12, 42) \\ &= (17 - 6i, 16 + 15i, 3, 32) \end{aligned}$$

[3] Draw a picture of the points in $\mathbb{R}^2$ which are determined by the following ordered pairs.

Just plot the points in $\mathbb{R}^2$ as usual.

[4] Does it make sense to write $(1, 2) + (2, 3, 1)$? Explain.

No, since one is an element of $\mathbb{R}^2$ and the other is an element of $\mathbb{R}^3$ (presumably the field is $\mathbb{R}$, but it doesn't matter for the problem).

[5] Draw a picture of the points in $\mathbb{R}^3$ which are determined by the following ordered triples.

Just plot the points in $\mathbb{R}^3$ as usual.

[Not in book] Find the vector v associated to the arrow with head at $Q = (7, 6, -1)$ and tail at $P = (0, -3, 8)$, and then compute the length of v. What's the distance between P and Q?

The vector we want will be the solution of the equation $P + v = Q$, or in other words

$$v = Q - P = (7, 6, -1) - (0, -3, 8) = (7, 6 + 3, -1 - 8) = (7, 9, -9)$$

The length of v is the distance between the two points (since after all it starts at one and ends at the other).

$$|v| = \sqrt{7^2 + 9^2 + (-9)^2} = \sqrt{211}$$

Section 3.3

$\boxed{1}$ Find the point, (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$.

We have

$$\begin{aligned}(x + 3y) + 3(4x - y) &= 1 + 3(3) \\ 13x &= 10 \\ x &= \frac{10}{13}\end{aligned}$$

from which

$$y = 4x - 3 = 4\frac{10}{13} - 3 = \frac{1}{13}$$

$\boxed{2}$ Solve Problem 1 graphically. That is, graph each line and see where they intersect.

Grab your favorite graphing calculator, plot the two lines and computer the intersection numerically.

$\boxed{7}$ You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

a) No solution. The k lines have no common intersection.

b) A unique solution. All k lines meet in a single point.

c) An infinite number of solutions. All of the lines are actually the same line.

[12] Suppose a system of equations has fewer equations than variables. Must such a system be consistent? If so, explain why and if not, give an example which is not consistent.

Such a system does not have to be consistent, consider the system of equations describing the intersection of two parallel planes

$$x + y + z = 1$$
$$x + y + z = 2$$

However, if such a system has solution it must have infinitely many solutions.

[13] If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

Such a system can have a solution, consider the redundant equations

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \\ 3x + 3x &= 3 \end{aligned}$$

The solutions to the system of equations are clearly the same as the solutions of the single equation $x + y = 1$.

[19] Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 1 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 5 \end{aligned}$$

We write the augemented matrix and then compute a row reduced echelon form,

$$\begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 1 & -1 & 1 & 1 & | & 1 \\ 2 & 1 & -1 & 0 & | & 1 \\ 4 & 2 & 1 & 0 & | & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 0 & -3 & 0 & 2 & | & -1 \\ 0 & -3 & -3 & 2 & | & -3 \\ 0 & -6 & -3 & 4 & | & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 0 & -3 & 0 & 2 & | & -1 \\ 0 & 0 & -3 & 0 & | & -2 \\ 0 & 0 & -3 & 0 & | & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 0 & -3 & 0 & 2 & | & -1 \\ 0 & 0 & -3 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}$$

The bottom row indicates that the system is inconsistent since $0 \neq 1$.

[20] Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 0 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 3 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 1 & -1 & 1 & 1 & | & 0 \\ 2 & 1 & -1 & 0 & | & 1 \\ 4 & 2 & 1 & 0 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 1 & -1 & 1 & 1 & | & 0 \\ 2 & 1 & -1 & 0 & | & 1 \\ 4 & 2 & 1 & 0 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 0 & -3 & 0 & 2 & | & -2 \\ 0 & -3 & -3 & 2 & | & -3 \\ 0 & -6 & -3 & 4 & | & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & -1 & | & 2 \\ 0 & -3 & 0 & 2 & | & -2 \\ 0 & 0 & -3 & 0 & | & -1 \\ 0 & 0 & -3 & 0 & | & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & | & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & | & \frac{2}{3} \\ 0 & 0 & 1 & 0 & | & \frac{1}{3} \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The system is consistent, but the solution is not unique since w is a free variable.

21 Find the general solution of the system whose augmented matrix is

$$\begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 1 & 3 & 4 & | & 2 \\ 1 & 0 & 2 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 0 & 1 & 4 & | & 0 \\ 0 & -2 & 2 & | & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 0 & 1 & 4 & | & 0 \\ 0 & 0 & 10 & | & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & \frac{6}{5} \\ 0 & 1 & 0 & | & \frac{2}{5} \\ 0 & 0 & 1 & | & -\frac{1}{10} \end{pmatrix}$$

Thus $x = \frac{6}{5}$, $y = \frac{2}{5}$, and $z = -\frac{1}{10}$, and there are no other solutions.

22 Find the general solution of the system whose augmented matrix is

$$\begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 2 & 0 & 1 & | & 1 \\ 3 & 2 & 1 & | & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 2 & 0 & 1 & | & 1 \\ 3 & 2 & 1 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 0 & -4 & 1 & | & -3 \\ 0 & -4 & 1 & | & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} & | & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & | & \frac{3}{4} \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The solution set has one free variable, z. Viewing x and y as function of the free variable we can describe all solutions

$$x(z) = \frac{1}{2} - \frac{1}{2}z$$
$$y(z) = \frac{3}{4} + \frac{1}{4}z$$

26 Give the complete solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

$$\begin{pmatrix} 7 & 14 & 15 & | & 22 \\ 2 & 4 & 3 & | & 5 \\ 3 & 6 & 10 & | & 13 \end{pmatrix} \overset{\text{rref...}}{\sim} \begin{pmatrix} 1 & 2 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The solution set has one free variable, y. Viewing x and z as function of the free variable we can describe all solutions

$$\begin{aligned} x(y) &= 1 - 2y \\ z(y) &= 1 \end{aligned}$$

$\boxed{27}$ Give the complete solution to the system of equations, $3x-y+4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

$$\begin{pmatrix} 3 & -1 & 4 & | & 6 \\ 0 & 1 & 8 & | & 0 \\ -2 & 1 & 0 & | & -4 \end{pmatrix} \overset{\text{rref...}}{\sim} \begin{pmatrix} 1 & 0 & 4 & | & 2 \\ 0 & 1 & 8 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The solution set has one free variable, z. Viewing x and y as functions of z we can describe all of the solutions

$$\begin{aligned} x(z) &= 2 - 4z \\ y(z) &= -8z \end{aligned}$$