Math 215 - Section 7
Exam #2

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Students Name (please print): Solutions

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ______________________________

Date:______________________________

IMPORTANT: SOME OF THE PROBLEMS ALLOW YOU TO CHOOSE WHICH QUESTION YOU DO. IT MUST BE CLEAR WHICH ONE YOU ARE SOLVING. DO NOT DO BOTH PARTS OR YOUR SCORE FROM THE LOWER ONE WILL BE USED.

You should work alone and show all of your work on all the problems. No credit will be awarded for the correct answer without any supporting work.

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Points						

1. Carefully define what it means for a set of vectors $\{v_1, \ldots, v_k\}$ to be linearly independent.

   If $\sum_{i=1}^{k} c_i v_i = 0$ then $c_1 = c_2 = \cdots = c_k = 0$.

2. Let B be the matrix

$$\begin{pmatrix} 2 & 3 & -1 & 2 & 5 \\ 0 & 3 & 1 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 2 & 6 & 0 & 2 & 4 \end{pmatrix}$$

   It has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Find a basis for the column space of B. You do not have to show any work.

$$\left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

   (b) Find a basis for the row space of B. You do not have to show any work.

$$\left\{ \begin{pmatrix} 2 \\ 3 \\ -1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\} \text{ are both OK}$$

3. If A is a matrix whose reduced row echelon form is

$$\begin{pmatrix} 0 & 1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determine the rank of A and the dimension of the kernel of A.

Counting the number of pivot columns we see that there are 4 linearly independent columns in the matrix, so $\text{Rank}(A) = 4$. By the rank-nullity theorem, $\text{Rank}(A) + \dim(\ker(A)) = 7$, so $\dim(\ker(A)) = 3$.

4. Use that fact that $v = \begin{pmatrix} 2372 \\ -1028 \\ 4374 \end{pmatrix}$ is a solution to

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4690 \\ -5402 \\ -15494 \end{pmatrix}$$

to find the general system of solutions.

To find the general system of solutions we have to solve the homogenous system of equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or in other words find a basis for the kernel. Using the augmented matrix for the system,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus the kernel is $\text{Span}\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, and the general system of solutions can be written as

$$\begin{pmatrix} 2372 \\ -1028 \\ 4374 \end{pmatrix} + \text{Span}\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

5. Let $V = \left\{(a_1, a_2, a_3)^T \in \mathbb{R}^3 | a_1 \geq 0\right\}$. Is V a vector subspace of $\mathbb{R}^3$? Justify your answer.

No. The vector $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in V, but $(-1)v = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ is not in V, so it is not closed under scalar multiplication.

6. Compute $\dim \left(\operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\} \right)$.

To do this we write down the rref of the matrix formed from these vectors since it will show us any relationships between the columns.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first and third column form a basis for the column space, so the dimension of the span is 2.

7. Find the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

To find the eigenvalues we have to solve

$$\det\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda I_2\right) = 0$$

or in other words,

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)(1-\lambda) - 4 &= 0 \\ 1 - 2\lambda + \lambda^2 - 4 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0 \\ \lambda &= 3, -1 \end{aligned}$$

8. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right)$ and $T\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 7 \\ 3 \end{smallmatrix}\right)$ find the 2×2 matrix A such that $T(x) = Ax$ for all vectors x. *Hint: this problem is much easier if you think about the properties of linear transformations than if you jump into computations.*

In order to find the matrix of the transformation we need to know $T(e_1)$ and $T(e_2)$. Since $T\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right)$, we can write

$$T\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) = T(2e_1) = 2T(e_1) = \left(\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right)$$

Thus $T(e_1) = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$. We also know that $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) + e_2$, so

$$T\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = T\left(\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) + e_2\right) = T\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) + T(e_2) = \left(\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right) + T(e_2) = \left(\begin{smallmatrix} 7 \\ 3 \end{smallmatrix}\right)$$

Therefore $T(e_2) = \left(\begin{smallmatrix} 3 \\ -3 \end{smallmatrix}\right)$ and the matrix is

$$\begin{pmatrix} 2 & 3 \\ 3 & -3 \end{pmatrix}$$

9. Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane.

(a) Find the matrix for T.

$T(e_1) = e_1$ and $T(e_2) = e_2$. The only interesting thing happens with $T(e_3) = -e_3$. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(b) Find the eigenvalues of T.

The matrix is upper triangular so the eigenvalues are just the main diagonal entries. Thus $\lambda = 1, -1$ are the eigenvalues.

(c) Find the eigenvectors of T.

We do not need to use the matrix form of T to answer the questions, since $T(e_1) = e_1$ and $T(e_2) = e_2$ implies that e_1 and e_2 are eigenvectors of T with eigenvalue 1. Similarly, $T(e_3) = -e_3$ means that e_3 is an eigenvector for the eigenvalue -1.

10. Let A be an $n \times n$ matrix and let $v_1, \ldots, v_n$ be a basis for $\mathbb{R}^n$. If $Av_i = 0$ for $i = 1, 2, \ldots, n$, prove that $Aw = 0$ for every vector w in $\mathbb{R}^n$.

    Let w be any vector in $\mathbb{R}^n$. We can write $w = \sum_{i=1}^n c_i v_i$ for some choice of c_i. Then

$$Aw = A\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n A c_i v_i = \sum_{i=1}^n c_i A v_i = \sum_{i=1}^n c_i 0 = 0$$

11. **Challenge problem:** Continuing the setup from problem 10, show that A must actually be the zero matrix.

    Observe that for any matrix, Ae_1 gives back the first column of A, Ae_2 is equal to the second column of A, etc. By the previous part of the problem, we know that $Ae_1 = 0$, so the first column of A is all 0. Continuing, $Ae_2 = 0$ so the second column of A is all 0, and so on. Thus every column of A is 0 and A is the zero matrix.

    Alternatively, if you want to use a little less English and a bit more math, observe that for any matrix

$$A = \begin{pmatrix} Ae_1 & Ae_2 & \ldots & Ae_n \end{pmatrix}$$

    But $Ae_i = 0$ for all i, so

$$A = \begin{pmatrix} \vec{0} & \vec{0} & \ldots & \vec{0} \end{pmatrix} = 0$$

12. **Challenge problem:** Recall that a matrix A is called nilpotent if $A^k = 0$ for some value k. Show that the only eigenvalue of a nilpotent matrix is 0.

    Suppose that $Av = \lambda v$ for some eigenvector v and eigenvalue λ. $A^k v = \lambda^k v$, but $A^k = 0$ so $\lambda^k v = 0$. Therefore $\lambda = 0$ since eigenvectors are never 0.