Math 215 - Section 7
Exam #1

Instructors Name: Ryan Smith
Students Name (please print): Solutions ______________________

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ______________________

Date: ______________________

IMPORTANT: SOME OF THE PROBLEMS ALLOW YOU TO CHOOSE WHICH QUESTION YOU DO. IT MUST BE CLEAR WHICH ONE YOU ARE SOLVING. DO NOT DO BOTH PARTS OR YOUR SCORE FROM THE LOWER ONE WILL BE USED.

You should work alone and show all of your work on all the problems. No credit will be awarded for the correct answer without any supporting work.

1. If A is an $n \times n$ matrix, define the (i, j)th *cofactor* of A (in class we wrote $\text{cof}(A)_{ij}$). Be careful about the sign.

   `The` (i,j)`th cofactor is` $(-1)^{i+j}$ `times the determinant of the` $(n-1)\times(n-1)$ `matrix of` A `with the` i`th row and the` j`th column removed.`

2. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{pmatrix}$. Find the determinant using row operations.

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \\ 1 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \\ 0 & \frac{7}{2} & -\frac{13}{2} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & -\frac{16}{3} \end{vmatrix} = 2(3)\left(-\frac{16}{3}\right) = -32$$

3. Carry out the matrix operations or explain why they cannot be done.

(a) $\begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 4 & 1 \end{pmatrix} \left(\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$

$$\begin{aligned} \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 4 & 1 \end{pmatrix} \left(\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) &= \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 4 & 1 \end{pmatrix} \left(\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ -1 & -2 \\ 14 & 3 \end{pmatrix} \end{aligned}$$

(b) $\begin{pmatrix} 4 & 0 & -1 \\ 2 & 2 & -1 \end{pmatrix}^{-1}$

`Only square matrices can have an inverse.`

(c) $\begin{pmatrix} 4 & 0 & -1 \\ 2 & 2 & -1 \end{pmatrix}^{2}$

`We can't multiply a` 2×3 `matrix by a` 2×3 `matrix since the inner dimensions don't agree.`

(d) $\begin{pmatrix} 1 & -2 & 7 \end{pmatrix}^T \begin{pmatrix} 0 & 5 & 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 & 7 \end{pmatrix}^T \begin{pmatrix} 0 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} \begin{pmatrix} 0 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 2 \\ 0 & -10 & -4 \\ 0 & 35 & 14 \end{pmatrix}$$

(e) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}^{3}$

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}^{3} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

4. You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

   (a) No solutions.

   `Each of the equations defines a line. If there is no solution then those don't lines don't all meet at a single point.`

   (b) A unique solution.

   `All of the lines meet at a single point.`

   (c) An infinite number of solutions.

   `All of the equations were actually defining the same line, so there is only one line.`

5. If the row reduced echelon form of a matrix looks like

$$\begin{pmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 4 & | & 6 \end{pmatrix}$$

what can you say about the solutions to the corresponding system of equations? In particular, is there no solution, a unique solution, a line of solutions, a plane of solutions, etc.? Justify your answer.

`There are fewer leading entries than variables, and the system is consistent, so there will be free variables. The third variable (which we can choose to call` z`) does not have a leading entry in it's column, so it corresponds to a free variable. A single free variable means the solution space is 1-dimension thus the solutions form a line.`

6. Compute the determinant

$$\begin{vmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{vmatrix}$$

without using the explicit formula for the determinant of a 3×3 matrix we discussed in class. Fully simplify your answer. Describe the effect of multiplying a vector in $\mathbb{R}^3$ by this matrix.

`We can use cofactor expansion along the first column.`

$$\begin{vmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{vmatrix} = (-1)^{1+1}\cos(\theta)\begin{vmatrix} 1 & 0 \\ 0 & \cos(\theta) \end{vmatrix} + (-1)^{2+1}0\begin{vmatrix} 0 & \sin(\theta) \\ 0 & \cos(\theta) \end{vmatrix}$$
$$+ (-1)^{3+1} - \sin(\theta)\begin{vmatrix} 0 & \sin(\theta) \\ 1 & 0 \end{vmatrix}$$
$$= \cos^2(\theta) + 0 + \sin^2(\theta) = 1$$

Observe that any vector of the form $\begin{pmatrix}0 & b & 0\end{pmatrix}^T$ is fixed by this matrix. This is our first clue since it means that a vector lying along the y-axis is not moved. Choosing a convenient value of θ such as π we see that $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ goes to $\begin{pmatrix}-1 & 0 & 0\end{pmatrix}^T$ and similarly for the z-axis. Finally, if we take $\theta = \frac{\pi}{2}$ we see that the matrix interchanges the x and z axis. Thus the matrix rotates vectors about the y-axis.

7. Solve the system of equations

$$x - 4y = 3$$
$$2x - 5y = 7$$

using the inverse matrix, or explain why there is no solution.

`Let` $A = \begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix}$`, it falls to us to compute` A^{-1}`. Fortunately, we know a formula for the inverse of a` 2×2 `matrix (otherwise we could just setup the augmented matrix with` I_2`).`

$$A^{-1} = \frac{1}{(-5)1 - (-2)4} \begin{pmatrix} -5 & 4 \\ -2 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -5 & 4 \\ -2 & 1 \end{pmatrix}$$

`We can solve the system by observing that` $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$`. So`

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -5 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{13}{3} \\ \frac{1}{3} \end{pmatrix}$$

`So` $x = \frac{13}{3}$ `and` $y = \frac{1}{3}$`.`

8. Choose 2 of the 3 statements below. For each statement you choose, write true or false. If it is true **prove** it, if it is false write down a counter example.

   (a) If A, B, and C are all nonzero $n \times n$ matrices and $AB = AC$, then $B = C$.

   `False. Take` $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$`,` $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ `and` $C = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$`. Then`

$$AB = AC = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

   (b) If A and B are $n \times n$ matrices and B is invertible then $\det(B^{-1}AB) = \det(A)$.

   `True.`

$$\det(B^{-1}AB) = \det(B^{-1}) \det(A) \det(B) = \det(B)^{-1} \det(A) \det(B) = \det(A)$$

   (c) If A is symmetric and invertible, then A^{-1} is also symmetric.

   `True. Transpose commutes with inverses, so` $(A^{-1})^T = (A^T)^{-1} = A^{-1}$`.`

9. Find a complete solution set to the system of equations:

$$\begin{aligned} 2x - 3y + 4z &= 4 \\ x - y + 2z &= 2 \\ x - 3y + 2z &= 2 \end{aligned}$$

`Setup the augmented matrix,`

$$\begin{pmatrix} 2 & -3 & 4 & | & 4 \\ 1 & -1 & 2 & | & 2 \\ 1 & -3 & 2 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & | & 0 \\ 1 & -1 & 2 & | & 2 \\ 1 & -3 & 2 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & | & 0 \\ 1 & 0 & 2 & | & 2 \\ 1 & 0 & 2 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & | & 0 \\ 1 & 0 & 2 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 2 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

`From the row reduced echelon form we see that` $x + 2z = 2$ `and` $y = 0$. z `is a free variable, so it parameterizes the solutions:`

$$\begin{aligned} x(z) &= 2 - 2z \\ y(z) &= 0 \end{aligned}$$

10. If A, B and C are $n \times n$ matrices, and λ and μ are scalars, prove that $A(\lambda B + \mu C) = \lambda AB + \mu AC$.

$$\begin{aligned} (A(\lambda B + \mu C))_{ij} &= \sum_{k=1}^{n} A_{ik} (\lambda B + \mu C)_{kj} = \sum_{k=1}^{n} A_{ik} (\lambda B_{kj} + \mu C_{kj}) \\ &= \lambda \sum_{k=1}^{n} A_{ik} B_{kj} + \mu \sum_{k=1}^{n} A_{ik} C_{kj} = (\lambda AB)_{ij} + (\mu AC)_{ij} \\ &= (\lambda AB + \mu AC)_{ij} \end{aligned}$$

11. **Challenge problem:** Recall that a matrix A which has the property that $A^T = -A$ is call skew-symmetric. Let n be an odd number. Show that no $n \times n$ skew-symmetric matrix has an inverse.

   `Take the determinant of both sides,` $\det(A^T) = \det(-A)$ `and use the fact that` $\det(A^T) = \det(A)$ `to get` $\det(A) = \det(-A)$. `Since` $\det(cA) = c^n \det(A)$, `we have` $\det(A) = (-1)^n \det(A)$. `But` n `is odd so` $(-1)^n = -1$, `thus` $\det(A) = -\det(A)$, `i.e.` $2\det(A) = 0$. `This implies that` $\det(A) = 0$, `which is synonymous with having no inverse.`