Math 215 Project 2: Orbits

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Problem Description: Given two points in a plane, we know how to find a line going through them by solving a system of linear equations

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0. \end{aligned}$$

Similarly, given three points in a plane, we can find a circle going through them by examining a system of equations. If *(x,y)* is a point on a circle, then we know that for some constants a, b, and c, we have

$$a\left(x^2+y^2\right) + bx + cy + d = 0.$$

Taking multiple points on a circle, we have a system such as

$$\begin{aligned} a\left(x^2+y^2\right) + bx + cy + d &= 0 \\ a\left(x_1^2+y_1^2\right) + bx_1 + cy_1 + d &= 0 \\ a\left(x_2^2+y_2^2\right) + bx_2 + cy_2 + d &= 0 \\ a\left(x_3^2+y_3^2\right) + bx_3 + cy_3 + d &= 0. \end{aligned}$$

We cannot simply put this system into a matrix and solve as we normally do, because these are not linear equations, but we do have some useful information. Since there are infinitely many points on a circle, there are infinitely many solutions to this system (if a, b, c and d solve the equations then ka, kb, kc and kd also solve them for any $k \neq 0$). This means that the determinant of the matrix must be 0. We have

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Recalling properties of the determinant, we have

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = \left(x^2+y^2\right)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - x\begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix} + y\begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix} - \begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix}.$$

Since the determinant of each of these matrices is just a number, we have the equation for a circle.

Example Problem: Let us consider a circle that contains the points $(0,0)$, $(4,5)$ and $(9,2)$. We can plug in these points to obtain:

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 0 & 0 & 0 & 1 \\ 41 & 4 & 5 & 1 \\ 85 & 9 & 2 & 1 \end{vmatrix} = -37\left(x^2+y^2\right) + 343x + 29y.$$

A nice way to reduce the complexity of this problem in Matlab would be to program the lower rows of the system as a matrix. For example, we write

A=[0,0,0,1 ; 41,4,5,1 ; 85,9,2,1]

Then we can simply tell Matlab to compute the determinant of each submatrix:

det(A(:,[2 3 4])), det(A(:,[1 3 4])), det(A(:,[1 2 4])), det(A(:,[1 2 3]))

Here *det(A(:,[2 3 4]))* asks Matlab to compute the determinant using the 2^{nd}, 3^{rd} and 4^{th} columns. The other determinants use the same idea.

Assigned Problem: A similar technique can be used to find an equation approximating the orbit of an asteroid. If an astronomer observes an asteroid at five different times, and records the coordinates, this technique can be used to find an equation for the orbit. Suppose the coordinates recorded are: $(6.02, 5.44)$, $(7.23, 3.01)$, $(8.44, 0.94)$, $(10.03, -1.37)$, $(12.15, -3.72)$. Use the following equation for an ellipse:

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 = 0.$$

Plot your solution along with the 5 points.