Math 215 Project 2: 2D Diffusion

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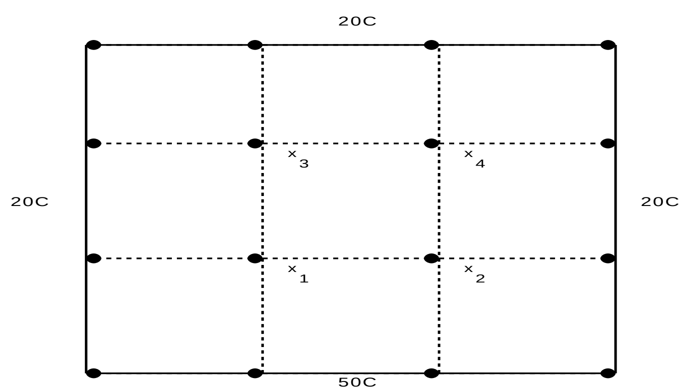
Problem Description: Let us consider a rectangular plate with some heat sources on each edge. Assuming these heat sources remain constant, the plate will eventually reach an equilibrium temperature. You can think of this, in 3D, as a cubic heat sink, perhaps on a CPU. The CPU is releasing what we will assume is a constant heat source, while the ambient air is staying at a constant temperature. This would make 1 side of the cube a very high temperature (from the CPU) and the other 5 sides a relatively low temperature (room temperature). We are, of course, ignoring any fans or buildup of heat within the enclosure. The 2D analog to this problem would be some cross section cut out from this cube.

We make the assumption that there is an underlying grid on our rectangular plate. To determine the equilibrium, we may assume that if the plate is at equilibrium and x_i is a grid point not on the boundary, then the temperature at x_i is given by the average of the temperatures of the four closest grid points to x_i. This creates a linear system for x_i, which we can then solve using the linear algebra techniques learned in class. If you have also looked at the 1D Diffusion problem, you'll notice this is quite similar. This problem actually corresponds to solving the differential equation

$$\Delta u\,(x, y) = 0,\ 0 < x, y < 1$$
$$u\,(x, 0) = u_0,\ u\,(x, 1) = u_1,\ u\,(0, y) = u_2,\ u\,(1, y) = u_3$$

using finite differences in two dimensions. The details for this technique will not be mentioned here.

Example Problem: We consider a rectangular plate with temperature $50°C$ on the bottom boundary and $20°C$ on the other three boundaries. We create a *uniform* grid with 4 interior grid points, and wish to solve for the temperature at each x_i, which we will denote by $u_i, i = 1, \ldots, 4$. The problem is illustrated in the graphic below:



The idea is that we must determine equations for each u_i. Let us start with u_1. We know that u_1 will be given by the average of each of the 4 temperatures around it, thus

$$u_1 = \frac{20 + 50 + u_2 + u_3}{4}.$$

Using the same idea, we have

$$u_2 = \frac{20 + 50 + u_1 + u_4}{4},\ u_3 = \frac{20 + 20 + u_1 + u_4}{4},\ u_4 = \frac{20 + 20 + u_2 + u_3}{4}.$$

Moving all of the constants to the right hand side and all of the unknowns to the left, we have the following system

$$\begin{aligned} 4u_1 - u_2 - u_3 &= 70 \\ -u_1 + 4u_2 - u_4 &= 70 \\ -u_1 + 4u_3 - u_4 &= 40 \\ -u_2 - u_3 + 4u_4 &= 40 \end{aligned}$$

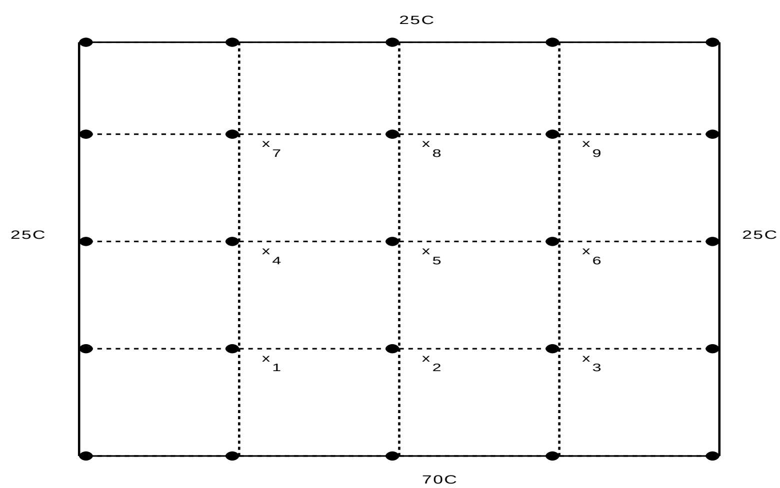
which we can write in augmented matrix form as

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 70 \\ -1 & 4 & 0 & -1 & 70 \\ -1 & 0 & 4 & -1 & 40 \\ 0 & -1 & -1 & 4 & 40 \end{array}\right].$$

When solved, we have

$$\mathbf{u} = \begin{bmatrix} 31.25 \\ 31.25 \\ 23.75 \\ 23.75 \end{bmatrix}.$$

Assigned Problem: Consider a rectangular plate with temperature $70°C$ on the bottom boundary and $25°C$ on the other three boundaries. Create a *uniform* grid with 9 interior grid points, 3 in each direction. The problem is illustrated in the graphic below:



Using the same idea as the previous example, derive a matrix A and a right hand side $\mathbf{b}$ so that the solution to $A\mathbf{u} = \mathbf{b}$ returns the temperatures given by $\mathbf{u} = [u_1, u_2, \ldots, u_9]^T$. Solve the system using Matlab. Comment on what A looks like (but do NOT print out A) if we have 16 interior points (4×4). What about if we have 64 interior points?