# Math 215 Project 2: 1D Diffusion

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***Problem Description:*** We consider the boundary value problem given by

$$\begin{aligned} u''(x) &= f(x),\ x \in (0,1) \\ u(0) &= u_0 \\ u(1) &= u_1. \end{aligned}$$

This corresponds to diffusion along a 1D rod with some given forcing $f(x)$. You can think of this as a wire in which you have some given temperatures at the endpoints $u_0$ and $u_1$, but some source of external heat, given by $f(x)$. While this differential equation looks quite simple (and is easily solved for smooth $f(x)$), the technique we will use to solve it is applicable to more complication differential equations, such as $a(x)u''(x)+b(x)u'(x)+c(x)u(x) = d(x)$ where $a, b, c$ and $d$ are not necessarily *nice* functions.

Recall, from your early calculus classes, how you approximate a derivative. If I have $f(1)$ and $f(1.1)$, how do you approximate $f'(1)$? You write

$$f'(1) \approx \frac{f(1.1) - f(1)}{1.1 - 1}.$$

This uses the slope of the secant line to approximate the slope of the tangent line. For our differential equation, we're assuming there is an underlying uniform grid on $[0,1]$. For example, we may have a grid with 5 points, given by $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. For these 5 points, note that we have a distance of $1/(5-1) = 1/4$ in between each point. Suppose we know the value of $u$ at each of these points. How can we approximate $u'\left(\frac{1}{4}\right)$? By using a secant line! We can use any of the following formulas (corresponding to a backwards, central, and forward approximation):

$$u'\left(\frac{1}{4}\right) \approx \frac{u\left(\frac{1}{4}\right) - u(0)}{1/4} \approx \frac{u\left(\frac{1}{2}\right) - u(0)}{1/2} \approx \frac{u\left(\frac{1}{2}\right) - u\left(\frac{1}{4}\right)}{1/4}.$$

Now, suppose we want to approximate $u''\left(\frac{1}{4}\right)$. Well, we can do this by using differences of first derivatives, BUT each of these derivatives can use the previous approximation for the first derivative:

$$u''\left(\frac{1}{4}\right) \approx \frac{u'\left(\frac{1}{2}\right) - u'(0)}{1/2} = \frac{\frac{u\left(\frac{1}{2}\right) - u\left(\frac{1}{4}\right)}{1/4} - \frac{u\left(\frac{1}{4}\right) - u(0)}{1/4}}{1/4} = \frac{u\left(\frac{1}{2}\right) - 2u\left(\frac{1}{4}\right) + u(0)}{1/16}.$$

This leads us to the definition a *finite difference* approximation to the $2^{nd}$ derivative.

**Definition:** For a grid with $N$ points on $[0,1]$ $\left(x_0 = 0,\ x_1 = \frac{1}{N-1}, \ldots, x_{N-2} = \frac{N-2}{N-1},\ x_{N-1} = 1\right)$ we define the *second order finite difference approximation to the second derivative of* $u(x)$ at a interior point $x_i$, $i = 1, \ldots, N-2$ by:

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\left(1/(N-1)\right)^2}.$$

If we let $h$ denote the step size in between each point, $h = \frac{1}{N-1}$, and define $u_i \equiv u(x_i)$ then we may write this as:

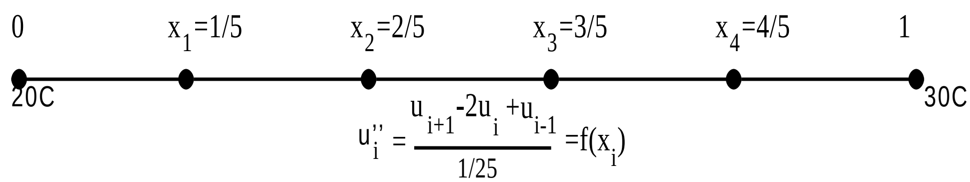
$$u_i'' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

***Example Problem:*** Approximate the solution to

$$\begin{aligned} u''(x) &= 100\sin(4\pi x),\ \ x \in (0,1) \\ u(0) &= 20 \\ u(1) &= 30 \end{aligned}$$

using the second order finite difference approximation and a mesh with 4 interior points.

We must determine equations for each $x_i$ using the finite difference approximation to the 2nd derivative. First, let us look at an illustration of the problem:




Let us start with $x_1$. We have

$$u_1'' = f(x_1) = 100\sin(4\pi x_1) = 100\sin\left(4\pi\frac{1}{5}\right) = 100\sin\left(\frac{4}{5}\pi\right).$$

Putting in our approximation for $u_1''$ we have:

$$u_1'' = \frac{u_2 - 2u_1 + u_0}{\frac{1}{25}} = \frac{u_2 - 2u_1 + 20}{\frac{1}{25}} = 100\sin\left(\frac{4}{5}\pi\right).$$

Doing that same at points $x_2$,$x_3$ and $x_4$ we get:

$$\begin{aligned} u_2'' &= \frac{u_3 - 2u_2 + u_1}{\frac{1}{25}} = 100\sin\left(\frac{8}{5}\pi\right) \\ u_3'' &= \frac{u_4 - 2u_3 + u_2}{\frac{1}{25}} = 100\sin\left(\frac{12}{5}\pi\right) \\ u_4'' &= \frac{30 - 2u_4 + u_3}{\frac{1}{25}} = 100\sin\left(\frac{16}{5}\pi\right). \end{aligned}$$

Take each equation, multiply by 1/25, and bring the constants to the right hand side. We get:

$$\begin{aligned} -2u_1 + u_2 &= 4\sin\left(\frac{4}{5}\pi\right) - 20 \\ u_1 - 2u_2 + u_3 &= 4\sin\left(\frac{8}{5}\pi\right) \\ u_2 - 2u_3 + u_4 &= 4\sin\left(\frac{12}{5}\pi\right) \\ u_2 - 2u_4 &= 4\sin\left(\frac{16}{5}\pi\right) - 30. \end{aligned}$$

In matrix form, we wish to find $A\mathbf{u} = \mathbf{b}$ with

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4\sin\left(\frac{4}{5}\pi\right) - 20 \\ 4\sin\left(\frac{8}{5}\pi\right) \\ 4\sin\left(\frac{12}{5}\pi\right) \\ 4\sin\left(\frac{16}{5}\pi\right) - 30 \end{bmatrix}.$$

How can we program this in Matlab? Well, I would first set up your step size in your grid, $h$. Then I would set up the vector of interior locations $x_i$. I would use this to calculate the $f$ values on the right hand side, then modify $\mathbf{b}$ so that it contains the proper boundary values. For programming $A$, notice the distinct pattern. There is a nice little trick we can use to set the diagonal and off diagonals of a matrix. Try:

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>> A = diag(-2*ones(4,1)) + diag(ones(4-1,1),1) +diag(ones(4-1,1),-1)
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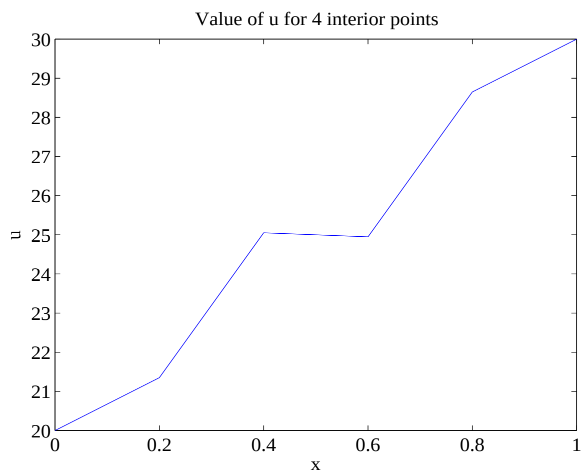
Once we have $A$ and $b$, we solve the system using Gaussian Elimination (the \ command) and plot the solution. We get (for 4 interior points):




***Assigned Problem:*** Approximate the solution to

$$\begin{aligned} u''(x) &= 100\sin(4\pi x), \quad x \in (0,1) \\ u(0) &= 20 \\ u(1) &= 30 \end{aligned}$$

using a mesh with 100 interior points. Do NOT print out your matrix $A$, right hand side $\mathbf{b}$ or solution $\mathbf{u}$. Instead plot your solution. Try to write the program as a function, where you input any $N$ and the function displays a plot of the solution for that given $N$.