Math 120 - Section 6
Exam #1

Instructors Name: Ryan Smith
Students Name (please print): Exam solutions

These solutions are intended to explain how to solve the problems, so I show more work on some of them than I would expect in a timed exam.

1. The cost of rent at a run-down apartment complex is shown in the table below. Assuming that the rent is growing linearly, write a function to model the cost of rent and use it to estimate the rent in 2008. (Hint: you may wish to make your independent variable the number of years since 1994).

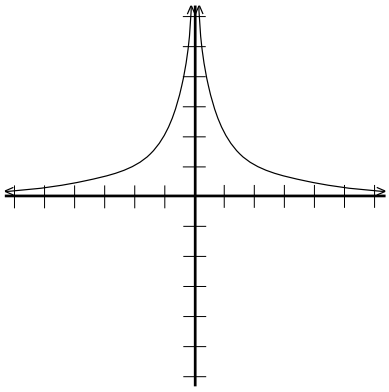
Year	1994	1998	2002	2006
Price	165	181	197	213

First we have to find the slope, $m = \dfrac{213-197}{2006-2002} = \frac{16}{4} = 4$. Since the function measures the number of years since 1994 it is easiest to use the data from 1994 to find the y-intercept: $165 = 4(0) + b$, so $b = 165$. Our function then is $f(x) = 4x + 165$. The year 2008 will be 14 years after 1994, so $f(14) = 221$ will be the rent in 2008.

2. Find the equation of the line which is perpendicular to the line $y = \frac{1}{2}x - 3$ and which intersects it at $(2, -1)$.

In order to be perpendicular the slope must be the negative reciprocal, ie, $m = \dfrac{-1}{\frac{1}{2}} = -2$. Now we simply must find the correct y-intercept for the line to pass through $(2, -1)$. $-1 = -2(2) + b$, so $b = 3$. The equation for the line is $f(x) = -2x + 3$.

3. Let $h(x)$ be the function defined by the graph below:



(a) On what intervals is $h(x)$ increasing? On what intervals is $h(x)$ decreasing?
Increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. (There is no work to show, this problem just asks if you know what increasing and decreasing means).

(b) On what intervals is $h(x)$ concave up? On what intervals is $h(x)$ concave down?
Concave up on $(-\infty, 0) \cup (0, \infty)$, nowhere concave down. The function is undefined at zero, everywhere else the slope is increasing.

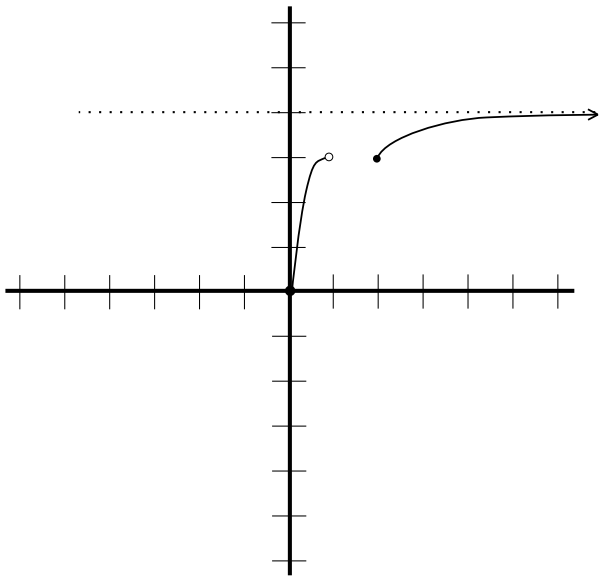
4. Find the intersection(s) of $\Phi(x) = 3x^2 - 5x + 4$ and the line $T(x) = 3x - 1$.

To find the intersection we just set the two functions equal and solve:

$$\begin{aligned} 3x^2 - 5x + 4 &= 3x - 1 \\ 3x^2 - 8x + 5 &= 0 \\ x &= \frac{8 \pm \sqrt{8^2 - 4(3)(5)}}{2(3)} \\ x &= \frac{8 \pm 2}{6} \\ x &= 1, \frac{5}{3} \end{aligned}$$

The intersections are points, so they have an x and a y value. We have $(1, 2)$ and $\left(\frac{5}{3}, 4\right)$.

5. Let $f(x)$ be the function defined by the graph below:



(a) What is the domain of $f(x)$?
The domain is $[0, 1) \cup [2, \infty)$

(b) What is the domain of $f^{-1}(x)$?
This is equivalent to asking what is the range of $f(x)$, since the inverse function flips domain and range. The function achieves every y value from 0 to 4 (but not including it). So the domain of $f^{-1}(x)$ is $[0, 4)$.

6. Let $f(x) = x^2 - 2x + 5$. Find $f\left(\frac{1}{x+1}\right)$ and simplify your answer.

First we just substitute, and then we clean up the algebra some. (If the substitution seems confusing just ask yourself what $f\left(\frac{1}{\mu+1}\right)$ would be. There is nothing special about the letter x.)

$$f\left(\frac{1}{x+1}\right) = \left(\frac{1}{x+1}\right)^2 - 2\left(\frac{1}{x+1}\right) + 5 = \frac{1}{(x+1)^2} - \frac{2(x+1)}{(x+1)^2} + \frac{5(x+1)^2}{(x+1)^2}$$
$$= \frac{5x^2 + 8x + 4}{(x+1)^2}$$

7. (a) Does $x^{-3} + z^{-2} = 4$ define z as a function of x? If so, solve for z and write it as a function.

In order to answer this we first try to solve for z and then determine if that would be a function.

$$\begin{aligned} x^{-3} + z^{-2} &= 4 \\ z^{-2} &= 4 - x^3 \\ \frac{1}{z^2} &= 4 - \frac{1}{x^3} \\ z^2 &= \frac{1}{4 - \frac{1}{x^3}} \\ z &= \pm\sqrt{\frac{1}{4 - \frac{1}{x^3}}} \end{aligned}$$

But then for a particular x we could get two different z, so z is not a function of x.

(b) Does $x^{-3} + z^{-2} = 4$ define x as a function of z? If so, solve for x and write it as a function.

This is similar to the previous part of the problem except that we solve for x instead of z:

$$\begin{aligned} x^3 &= \frac{1}{4 - \frac{1}{z^2}} \\ x &= \sqrt[3]{\frac{1}{4 - \frac{1}{z^2}}} \end{aligned}$$

The cube root has no problem with $\pm$, so for a given z there is at most one x. Written as a function (and simplified a little):

$$x(z) = \sqrt[3]{\frac{z^2}{4z^2 - 1}}$$

8. Let $g(x) = \sqrt{\dfrac{x-1}{x+2}}$, find $g^{-1}(5)$.

In order to find $g^{-1}(5)$ we set the output of the function equal to 5 and solve

`for` x:

$$\begin{aligned} 5 &= \sqrt{\frac{x-1}{x+2}} \\ 25 &= \frac{x-1}{x+2} \\ 25x + 50 &= x - 1 \\ 24x &= -51 \\ x &= \frac{-51}{24} = -\frac{17}{8} \end{aligned}$$

9. A new aquarium tank is being filled with water and the engineers have modeled the pressure in the tank as a function of the height of the water. The height of the water is measured in inches and the pressure is measured in pounds per square inch.

   (a) Without knowing an explicit formula for $P(h)$, find $P(0)$ and explain your answer.
   $P(0)$ represents the pressure with 0 inches of water in the tank. With no water there should be no pressure, so $P(0) = 0$. [1]

   (b) What would $P(102)$ correspond to in this model?
   $P(102)$ would correspond to the pressure when the tank is filled with 102 inches of water.

   (c) What would $P^{-1}(244)$ correspond to in this model?
   $P^{-1}(244)$ would correspond to the height of the water when the pressure is 244 pounds per square inch.

10. Find the domain of $P(x) = \dfrac{\sqrt{4-x}}{x+1}$ and write your answer in interval notation.

    The function has a square root and the possibility of dividing by zero, so we have to check for both. If $x + 1 = 0$ then $x = -1$ and $P(x)$ would be undefined. In order for the square root to be defined we must have $4-x \geq 0$, so $4 \geq x$. So the domain is $(-\infty, 1) \cup (-1, 4]$.

[1] One person with eye for scientific correctness pointed out that there would still be 1 atmosphere of pressure. This isn't really the point here, but this would work alright for an answer as long there is also a clear explanation to go along with it.

11. True or False: If the statement is true briefly explain why, if it is false provide a counter example.

(a) If $f(x)$ is a function then $f(a+b) = f(a) + f(b)$.

False. Just let $f(x) = 1$, then $f(a+b) = 1$ for every a and b, but $f(a) + f(b) = 1 + 1 = 2$.

(b) $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

True. $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$

(c) If $h(x)$ is a quadratic function then the range of $h(x)$ cannot be all real numbers.

True. The graph of a quadratic function is a parabola, so there will always a be minimum (or maximum) y value it reaches.

(d) If the point (c, d) lies on the graph of a function $T(\alpha)$ then $(\frac{1}{c}, \frac{1}{d})$ lies on the graph of $T^{-1}(\alpha)$.

False. If (c, d) is lies on the graph of $f(x)$ then (d, c) lies on the graph of f^{-1}).

(e) If $g(x)$ is concave up on the interval $[a, b]$ then it is increasing on $[a, b]$.

False. The slope of a function can be positive but decreasing (just think of the graph of $\sqrt{x}$.

12. If $G(y)$ is the functioned defined by the table of values below, determine whether $G(y)$ is linear. If it is linear write it as $G(y) = my + b$; if it is not linear explain why not. **Make sure to show your work and explain what you are doing.**

y	2	4	8	16
G(y)	15	30	45	60

Not a linear function since the slope is not constant. The slope from 2 to 4 is $\frac{30-15}{4-2} = \frac{15}{2}$, but the slope from 4 to 8 is $\frac{45-30}{8-4} = \frac{15}{4}$.