Equation Solving Review

To find the zero(s) of a function $f(x)$ you must solve the equation $f(x) = 0$. Below is a brief review of techniques that are useful when solving equations.

1 Linear Equations

A linear equation is an equation that can be written in the form $ax + b = 0$ where a and b are real numbers. The signature of a linear equation is that the highest exponent on the variable x is one (in more formal language, it is a degree one polynomial equation). Linear equations are solved by simple manipulations of the equation using addition, subtraction, multiplication and division. If you get a result "$0 = 0$" the equation is true for every real number x and if you get a result "$b = 0$" where b is nonzero, then there is no solution. Otherwise, you will obtain exactly one solution.

For more information on solving linear equations, see part 6 of Appendix A on page 389 of Ruud and Shell.

2 Quadratic Equations

A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$ where a, b and c are real numbers and $a \neq 0$. Its signature is that the highest exponent on the variable x is two (in more formal language, it is a degree two polynomial equation). There are three methods for solving quadratic equations.

2.1 Factoring

If you can factor the left hand side of the equation $ax^2 + bx + c = 0$ into a product of two linear factors, then the zeros of each factor are the solutions of the equation. This is a consequence of the **zero product principle**, which says that if a product of two or more quantities is zero, one of the factors must be zero.

WARNING! Do not read more into the zero product principle than it says. First of all, you must have a **product** equal to zero in order to invoke the principle. So if you have a sum of two quantities equal to zeros, the principle does not apply. The other common misuse of this principle is ignoring the fact that the product must equal **zero**. If a product equals something nonzero then the principle does not apply. For example, if you wish to solve $x^2+3x-5 = 0$, you cannot move the five to the other side and factor the x out of what's left and then invoke the zero product principle!

For more on factoring with quadratic expressions, see pages 366 and 367 of Ruud and Shell.

2.2 Square Root Method

If $b = 0$ then you can move the constant term to the other side of the equation and take square roots. Note that when you are solving an equation, if you take a square root you must take the positive **and** negative square roots, since both roots will solve the equation. As an example, solve

$x^2 - 9 = 0$ by moving the 9 over, which gives you $x^2 = 9$, whose solutions are ± 3.

WARNING! This will NOT work if $b \neq 0$.

WARNING! If a and c have the same sign, you will end up with imaginary solutions (can you see why?). Since complex numbers are not involved in our course, we will just say that there are no real solutions when this happens.

2.3 Quadratic Formula

The quadratic formula **ALWAYS** works. Note that you have to have your equation in the form $ax^2 + bx + c = 0$ in order to use this formula. Once you have your equation in this form, the solutions are given by plugging the coefficients a, b and c into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

You should commit this formula to memory as soon as possible. Alternatively, there is a program in the back of the workbook that you can enter into your calculator. It is one of the two programs that you are allowed to use in this course.

2.4 How Many Roots?

Note that there is a square root in the quadratic formula. The quantity under the radical $b^2 - 4ac$ is called the **discriminant**. If the discriminant is negative, the square root in the quadratic formula will give you imaginary numbers, so the quadratic equation will have no real solutions. If the discriminant is zero the quadratic formula will yield exactly one real solution. If the discriminant is positive, you will get two real solutions. You can save yourself some hassles by quickly computing the discriminant for any quadratic equation to make sure it is not negative. If it is, you can simply conclude that there are no real roots and get on with your life. Otherwise, since you have already computed the discriminant, part of the computation with the quadratic formula is already done and you can proceed to finish it off.

2.5 Which Method Should I Use?

The bottom line is as follows. If $b = 0$ use the square root method. Otherwise, if you see a factorization right away, use the factorization (check that it is correct though!). If you do not see a factorization right away, just use the quadratic formula. It ALWAYS works!!

For more information about quadratic equations, see part 7 of Appendix A on page 400 of Ruud and Shell.

3 Polynomial Equations

Polynomials are sums of terms that look like a number multiplied by x raised to a power that is a whole number. Example: $3x^9 + 4x^6 + 23x^2 - 7$ is a polynomial. The largest exponent is called the **degree** of the polynomial. Now you can see that linear expressions are degree one polynomials and quadratic expressions are degree two polynomials. For now, we will solve all equations of degree greater than two by factoring and invoking the zero product principle (recall all the cautions involved with this principle!). Later in the course we will study another method to solve polynomial equations.

For help with factoring, see part 3 of Appendix A on page 364 in Ruud and Shell.

4 Equations with Rational Expressions

The word "rational" should always invoke fractions in your mind. So here we address equations involving fractions. Specifically we will be interested in solving an equation of the form:

$$\frac{f(x)}{g(x)} = 0.$$

To solve such an equation, you want to clear the fractions away and reduce the problem to solving an equation without a fraction. Note that if you multiply both sides of the equation by the denominator $g(x)$, you get $f(x) = 0$ which is an equation without a fraction. Solving the latter equation will finish the problem - with one caveat.

WARNING! This method fails if $g(x) = 0$. As a result, when you finish the problem, go back to the original function and make sure your solutions do not cause the denominator to be zero. Any solutions that cause the denominator to be zero should be deleted from your final solution set.

Note that if you have a sum of several rational expressions in your equation, get all the expressions on one side, express each over a common denominator and perform the addition. This will then give you an equation of the form discussed above.

5 Equations with Radicals

The word "radical" should always invoke roots in your mind. This exposition will treat square roots, but for arbitrary roots the only difference is the power to which you raise both sides of the equation.

If there is a square root in your equation, the best way to undo this square root is to square both sides of the equation and proceed. However, you have to isolate the square root before you can do this (i.e. you have to get the expression with the square root on one side of the equation and everything else on the other side). Here is an example of what can go wrong. Suppose we want to solve $\sqrt{x+3} + x = 0$. If we square both sides without isolating the square root, the left hand side becomes:

$$(\sqrt{x+3} + x)^2 = x + 3 + 2x\sqrt{x+3} + x^2.$$

This doesn't help since we still have a square root. So moving the x to the other side of the equation isolates the square root and allows us to proceed after squaring.

Note that the process of squaring, for some technical reasons, raises the possibility that not all of your solutions will end up working. It is therefore necessary to check your solutions to equations with radicals and toss away anything that doesn't work.

WARNING! When squaring both sides of an equation, you must square the **entire** side. You cannot square term-by-term. For example, if you have $\sqrt{2x-1} = x+3$, when you square both sides, the right side should be $(x+3)^2 = x^2 + 6x + 9$, NOT $x^2 + 9$!

6 Equations with Absolute Values

Note that you cannot do most algebraic manipulations with absolute values because $|a+b| \neq |a|+|b|$. Therefore you cannot solve $|2x+3| = 4$ by moving the three to the other side and dividing by 4. Instead, recall what the absolute value means. This equation tells you that either $2x+3 = 4$ or $2x+3 = -4$. Solving these two equations separately will give you exactly the solutions to the original equation.