Math 110 - Section 3
Exam #3 - Sample study problems

Very important: These problems do not outline everything that will or will not be on the exam! They should be similar, however the goal of the exam is to test understanding - not regurgitation.

1. Solve $(\ln x)^2 + \ln(x^5) = -4$
   First thing, to deal with like terms we can take down the exponent on the $\ln(x^5)$, so the equation becomes
   $$(\ln x)^2 + 5\ln x = -4$$
   This is starting to look like a candidate for a u substitution, let $u = \ln x$.
   $$u^2 + 5u = -4$$
   We know how to solve this
   $$\begin{aligned} u^2 + 5u &= -4 \\ u^2 + 5u + 4 &= 0 \\ (u+1)(u+4) &= 0 \\ u &= -1, -4 \end{aligned}$$
   Changing our variables back we have the equations $\ln x = -1$ and $\ln x = -4$. We can rewrite these via the Golden Rule of logarithms: $e^{-1} = x$ and $e^{-4} = x$. So our solutions are $x = \frac{1}{e}, \frac{1}{e^4}$.

2. What is the domain of $f(x) = 2^{x-5}$? What is the domain of $g(x) = \log_2(x-5)$?

   $f(x)$ will be defined everywhere since it's an exponential function (the -5 will shift the graph to the right, but it won't make it undefined anywhere). $g(x)$ however is another story. It doesn't matter what base the logarithm is, what matters is when the inside is greater than zero. $x - 5 > 0 \Longrightarrow x > 5$, so the domain of $g(x)$ is $(5, \infty)$

3. Solve $e^{x^3-6x^2+9x} = 1$

   All of the x's are in the exponent, so we can take the natural log of both sides:
   $$\begin{aligned} \ln(e^{x^3-6x^2+9x}) &= \ln 1 \\ x^3 - 6x^2 + 9x &= 0 \\ x(x^2 - 6x + 9) &= 0 \\ x(x-3)^2 &= 0 \\ x &= 0, 3 \end{aligned}$$

4. To the lasting dismay of the Australian authorities, the Arizona cactus is taking over the out-back. Every 11 years the number of cactus double; if there are an estimated 90,000 cactus right now, how many will there be in 20 years? How many years will it take until there are 200,000 cactus (solve with algebra)?

   This is a doubling problem, so we want to use the equation $A = P(2^{\frac{t}{d}})$ where t is the number of years, d is the doubling time, and P is the initial number of cactus.

   If we want to know how many cactus there will be in 20 years we're going to substitute 20 in for t:

$$\begin{aligned} A &= 90,000(2^{\frac{20}{11}}) \\ A &= 317,372 \text{ cactus} \end{aligned}$$

   For the second part we're told how many cactus there will be and now t is the unknown:

$$\begin{aligned} 200,000 &= 90,000(2^{\frac{t}{11}}) \\ \frac{20}{9} &= 2^{\frac{t}{11}} \\ \log\left(\frac{20}{9}\right) &= \log\left(2^{\frac{t}{11}}\right) \\ \log\left(\frac{20}{9}\right) &= \frac{t}{11}\log(2) \\ 11\frac{\log\left(\frac{20}{9}\right)}{\log(2)} &= t \\ t &= 12.67 \text{ years} \end{aligned}$$

5. If $f(x) = 3\ln(x) - \ln(x^2) + \pi$, find $f^{-1}(x)$

   First we flip the x and y, then we solve for y by combining the logs together

$$\begin{aligned} x &= 3\ln(y) - \ln(y^2) + \pi \\ x - \pi &= \ln(y^3) - \ln(y^2) \\ x - \pi &= \ln\left(\frac{y^3}{y^2}\right) \\ x - \pi &= \ln(y) \\ e^{x-\pi} &= y \end{aligned}$$

   So $f^{-1}(x) = e^{x-\pi}$

6. A bank is setting up a loan for \$1400. If they are charging 4% and compounding quarterly, how long will they have to wait until to make \$323.50 in interest?

The \$323.50 will be the amount they've gained, so if x is the unknown length of time

$$\begin{aligned}
1400 + 323.50 &= 1400\left(1+\frac{0.04}{4}\right)^{4x}\\
\frac{1723.50}{1400} &= \left(1+\frac{0.04}{4}\right)^{4x}\\
\log\frac{1723.50}{1400} &= \log\left(1+\frac{0.04}{4}\right)^{4x}\\
\log\frac{1723.50}{1400} &= 4x\log\left(1+\frac{0.04}{4}\right)\\
\frac{\log\frac{1723.50}{1400}}{\log(1+\frac{0.04}{4})} &= 4x\\
x &= 5.22 \text{ years}
\end{aligned}$$

7. Simplify the expression: $\ln\left(\dfrac{81x^5(z^2+100)}{e^4 5^x x^5}\right)$

We just have to apply the same rules over logarithms over and over,

$$\begin{aligned}
\ln\left(\frac{81x^5(z^2+100)}{e^4 5^x x^4}\right) &= \ln\left(81x^5(z^2+100)\right) - \ln\left(e^4 5^x x^4\right)\\
&= \ln(81)+\ln(x^5)+\ln(z^2+100)-\ln(e^4)-\ln(5^x)-\ln(x^4)\\
&= \ln(3^3)+\ln(x^5)+\ln((z+10)^2)-\ln(e^4)-\ln(5^x)-\ln(x^4)\\
&= 3\ln(3)+5\ln(x)+2\ln(z+10)-4-x\ln(5)-4\ln(x)\\
&= 3\ln(3)+\ln(x)+2\ln(z+10)-4-x\ln(5)
\end{aligned}$$

8. Consider the functions defined by the tables:

x	f(x)
-3	-1
0	1
1	0
2	7
3	24

x	g(x)
-1	8
0	5
1	3
2	2
6	-2

Use the tables to compute $f^{-1}\left(g\left(f(2)-f^{-1}(0)\right)+g^{-1}(3)\right)$

For taking the inverse we want to start with the value of the function and then find the x. Working from the inside out,

$$\begin{aligned}
&= f^{-1}\left(g\left(7-1\right)+g^{-1}(3)\right)\\
&= f^{-1}\left(g\left(6\right)+g^{-1}(3)\right)\\
&= f^{-1}\left(-2+g^{-1}(3)\right)\\
&= f^{-1}\left(-2+1\right)\\
&= f^{-1}\left(-1\right)\\
&= -3
\end{aligned}$$

9. Solve $\frac{1}{2}(e^x - e^{-x}) = 4$. (This is a challenging problem)

   There is a trick to this problem, we need to get it into a good form to use the u substitution trick. We can do this by multiplying both sides by e^x since we need to get rid of the negative.

$$\begin{aligned}
\frac{1}{2}(e^x - e^{-x}) &= 4\\
e^x(e^x - e^{-x}) &= 8e^x\\
e^{2x} - e^0 &= 8e^x\\
(e^x)^2 - 8e^x - 1 &= 0
\end{aligned}$$

   Now the equation is in a form we can factor. Let $u = e^x$ for the purpose of factoring,

$$\begin{aligned}
u^2 - 8u - 1 = 0&\\
u &= \frac{-(-8) \pm \sqrt{8^2 - 4(-1)(1)}}{2(1)}\\
u &= 4 \pm \sqrt{17}
\end{aligned}$$

   Now we substitute back,

$$\begin{aligned}
e^x &= 4 \pm \sqrt{17}\\
x &= \ln\left(4 \pm \sqrt{17}\right)
\end{aligned}$$

   The negative solution won't work with the logarithm, so we have just one solution:

$$x = \ln\left(4 + \sqrt{17}\right)$$

10. Simplify the expression $b^{\log_{\frac{1}{b}}(2x) - \log_{\frac{1}{b}}(z^2)}$. (This is a challenging problem)

First we need to combine the logarithms if we're going to have any hope of simplifying things,

$$b^{\log_{\frac{1}{b}}(2x)-\log_{\frac{1}{b}}(z^2)} = b^{\log_{\frac{1}{b}}(\frac{2x}{z^2})}$$

What do we do with the odd logarithm? Let's go back to the Golden Rule of logs:

$$\log_{\frac{1}{b}}(a) = c \Longleftrightarrow \left(\frac{1}{b}\right)^c = a$$

If we remember that $\frac{1}{b} = b^{-1}$, we can combine the exponents:

$$\begin{aligned}(b^{-1})^c &= a\\ b^{-c} &= a\\ \log_b a &= -c\end{aligned}$$

So we can rewrite $\log_{\frac{1}{b}}(x)$ as $-\log_b x$.

$$\begin{aligned}b^{\log_{\frac{1}{b}}(\frac{2x}{z^2})} &= b^{-\log_b\left(\frac{2x}{z^2}\right)}\\ &= b^{\log_b\left(\frac{z^2}{2x}\right)}\\ &= \frac{z^2}{2x}\end{aligned}$$

11. **Note:** Be sure to also review section 3.3 on asymptotes of rational functions, asymptotes of exponential and logarithmic functions, and everything else in sections 3.3 and chapter 4.