MATH 464
ON SEMANTICS AND SET THEORY

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The following may be useful in translating probabilistic language into set theoretic language and vice versa.

In general, probability theory concerns making precise statements about experiments with random outcomes. Loosely speaking, one is interested in an experiment which produces outcomes. Typically, one knows all possible outcomes of an experiment, but it is not possible to predict exactly which outcome will occur with any given experiment.

Here is an example. Consider a fair coin. A simple experiment one can perform is to flip this coin. The results of this experiment (i.e. the outcomes) are clear: either the coin will turn up heads or tails. However, before you flip the coin, you do not know which outcome will occur. It is possible, though, to calculate precise likelihoods that heads or tails occur. This is the notion of probability we want to understand.

For any given experiment, we will let Ω denote the set of all possible outcomes. In this case, Ω is called the sample space corresponding to this experiment.

We will often denote by ω a possible outcome of this experiment. We write $\omega \in \Omega$ to mean that ω is a possible outcome in the sample space Ω. Other lower case letters may also be used to denote outcomes.

For a given experiment, there will usually be a collection of questions we would like to ask. They are all of the form: "How likely is this particular collection of outcomes?". Mathematically, we denote this collection of questions as a set of subsets of Ω. We will learn in class that a "good" collection of questions is called a σ-field. Often we will denote by $\mathcal{F}$ a σ-field of subsets of Ω.

In the language of probability theory, any set $A \in \mathcal{F}$ is called an event. The statement "*The event A occurred.*" means that:
an experiment was done,
the outcome ω was obtained, and
$\omega \in A$, i.e. this outcome is in the set A.

To answer the questions of interest, we define a probability measure. A probability measure P is a real valued function with domain $\mathcal{F}$. We will learn the precise definition in class. The important thing to know is that: to each event $A \in \mathcal{F}$ the probability measure P assigns a value $P(A)$ to A and $P(A)$ is called the probability, think likelihood, that the event A occurs.

To make writing easier, we introduce the notion of a probability space. For any probability measure P on $\mathcal{F}$, the triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Below we describe some basic set theory/notation:

Let $A, B \in \mathcal{F}$ be events.

i) If all outcomes in A are also in B, we write $A \subset B$. In this case, we may say that A is part of B, or that A is contained in (or included in) B. In particular, this means that A implies B, i.e. if A occurs, then B occurs.

ii) The common part of two events, A and B, is called their intersection. This is the set of outcomes that are in both A and B, and this intersection is denoted by $A \cap B$. It is also an event, i.e. $A \cap B \in \mathcal{F}$. In words, it occurs whenever both A and B occur.

iii) The event consisting of all outcomes that are either in A or in B is called the union of A and B. This union is denoted by $A \cup B$. It is also an event, i.e. $A \cup B \in \mathcal{F}$. In words, it occurs whenever A occurs, or B occurs, or both occur.

iv) If A is an event, then the rest of Ω is called the complement of A. It is denoted by A^c and it equals $A^c = \Omega \setminus A$. In words, if the event A occurs, then the event A^c does not occur; and vice versa. Said differently, if A^c occurs, then no outcome in A occurs.

v) The event that A occurs but B does not, is denoted by $A \setminus B$. It is defined to be the set $A \setminus B := A \cap B^c$, and as such, it is an event i.e. $A \setminus B \in \mathcal{F}$.

vi) The event that either A occurs or B occurs, but not both is called the symmetric difference of A and B. It is denoted by $A \Delta B$. It is defined to be the set $A \Delta B := (A \setminus B) \cup (B \setminus A)$, and as such, it is an event.

vii) If two events A and B have no common part, i.e. $A \cap B = \emptyset$, then A and B are said to be disjoint or mutually exclusive. They cannot occur simultaneously.

Some further basics of Set Theory

De Morgan's Laws:

Let A and B be sets. There is a simple way to understand the compliments of unions and intersections:

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c\end{aligned}$$

These facts are easy to prove; just show reverse containment.

These facts also generalize readily to countably many sets. Let A_n be a set for all $n \geq 1$. Then

$$\begin{aligned}\left(\bigcup_{n=1}^{\infty} A_n\right)^c &= \bigcap_{n=1}^{\infty} A_n^c \\ \left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \bigcup_{n=1}^{\infty} A_n^c\end{aligned}$$

Distribution Laws for Sets:

Let A, B, and C be sets. We often want to construct new sets from these sets using intersections and unions. There are useful distribution laws associated with these constructions.

$$\begin{aligned}A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)\end{aligned}$$

Again, these facts are easy to prove; just show reverse containment.

These facts also generalize readily to countably many sets. Let A be a set and for all $n \geq 1$, let B_n be a set as well. Then

$$\begin{aligned}A \cap \left(\bigcup_{n=1}^{\infty} B_n\right) &= \bigcup_{n=1}^{\infty} (A \cap B_n) \\ A \cup \left(\bigcap_{n=1}^{\infty} B_n\right) &= \bigcap_{n=1}^{\infty} (A \cup B_n)\end{aligned}$$

Any of these facts make good homework/test questions.