## 1. Vector Spaces

For clarity, here is the definition of a field.

**Note:** A non-empty set $\mathbb{F}$ is said to have a binary operation $\Box$ if: to each $a, b \in \mathbb{F}$, there is a unique $c \in \mathbb{F}$ with $a \Box b = c$.

**Definition 1.1.** *Let $\mathbb{F}$ be a non-empty set. Then, $\mathbb{F}$ is a field if: There are two binary operations on $\mathbb{F}$; one denoted by $+$ and one denoted by $*$, and two distinguished elements $0, 1 \in \mathbb{F}$ for which:*
*i) For all $x, y, z \in \mathbb{F}$:*
$x + y \in \mathbb{F}$
$x + y = y + x$
$(x + y) + z = x + (y + z)$
$0 + x = x$
*ii) For all $x, y, z \in \mathbb{F}$:*
$x * y \in \mathbb{F}$
$x * y = y * x$
$(x * y) * z = x * (y * z)$
$1 * x = x$
*iii) For all $x, y, z \in \mathbb{F}$:*
$x * (y + z) = x * y + x * z$
*iv) For all $x \in \mathbb{F}$, there exists a unique $y \in \mathbb{F}$ for which $x + y = 0$.*
*v) For all $x \in \mathbb{F}$ with $x \neq 0$, there exists a unique $z \in \mathbb{F}$ for which $x * z = 1$.*

It is customary to write $x * y = xy$ for elements of a field, and I will often do so, I assume that you have seen a proof that $\mathbb{R}$ equipped with the usual addition and multiplication is a field. We proved that $\mathbb{C}$ is a field in class. For most of what we discuss in class the fields we consider will be $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

The following definition will be crucial in this course.

**Definition 1.2.** *Let $\mathbb{F}$ be a field. A* **vector space** *$V$* **over** *$\mathbb{F}$ is a non-empty set $V$ (the elements of $V$ are called* **vectors***) over a field $\mathbb{F}$ (the elements of $\mathbb{F}$ are called* **scalars***) equipped with two operations:*
*i) To each pair $u, v \in V$, there exists a unique element $u + v \in V$. This operation is called* **vector addition***.*
*ii) To each $u \in V$ and $\alpha \in \mathbb{F}$, there exists a unique element $\alpha u \in V$. This operation is called* **scalar multiplication***.*
*which satisfy the following relations:*
*For all $\alpha, \beta \in \mathbb{F}$ and all $u, v, w \in V$,*

(1) $u + (v + w) = (u + v) + w$ *and* $u + v = v + u$
(2) *There is a vector $0 \in V$ (called the* **additive identity***) such that $u + 0 = u$ for all $u \in V$*
(3) *For each vector $u \in V$, there is a vector $-u \in V$ (called the* **additive inverse** *of $u$) such that $u + (-u) = 0$*
(4) $\alpha(u + v) = \alpha u + \alpha v$
(5) $(\alpha + \beta)u = \alpha u + \beta u$

(6) $(\alpha\beta)u = \alpha(\beta u)$
(7) $1u = u$ *for all* $u \in V$