Test 2 - Make up

1) Find the 1^{st} order perturbation approximation for the following I.V.P.:

$$x' = 1 - x e^{\varepsilon x} \quad \text{with} \quad x(0) = 1.$$

The 1^{st} order perturbation approximation has the form:

$$p_1(t) = k_0(t) + \varepsilon k_1(t).$$

The L.H.S. of this D.E. is clearly

$$p_1'(t) = k_0'(t) + \varepsilon k_1'(t).$$

The R.H.S. is:

$$1 - p_1(t) e^{\varepsilon p_1(t)} \cong 1 - p_1(t)\left[1 + \varepsilon p_1(t)\right]$$

$$= 1 - (k_0 + \varepsilon k_1) - \varepsilon (k_0 + \varepsilon k_1)^2$$

$$\cong (1 - k_0) + \varepsilon(-k_1 - k_0^2)$$

As a result, we must investigate the following system:

$$k_0' = 1 - k_0 \quad \text{and} \quad k_1' = -k_1 - k_0^2$$

$$k_0(0) = 1 \qquad\qquad k_1(0) = 0.$$

Since $k_0(t) = 1$, is a solution to the 1st D.E.

$k_0(t) = c e^{-t} + 1$ is the general solution.

$1 = c + 1 \Rightarrow c = 0$. Thus $k_0(t) = 1$ is the desired solution.

For the 2nd D.E. we have then that

$$k_1' = -k_1 - 1$$

$k_1(t) = c e^{-t} - 1$ is the general solution

$0 = k_1(0) = c - 1 \Rightarrow c = 1.$

Thus $k_1(t) = e^{-t} - 1$ is the solution. We conclude

$$p_1(t) = k_0(t) + \varepsilon k_1(t) = 1 + \varepsilon(e^{-t} - 1).$$

2) a) Find a fundamental solution matrix for

$\tilde{x}' = A\tilde{x}$ with $\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} -4 & 6 \\ -3 & 2 \end{pmatrix}$.

The eigenvalues of A are given by:

$$\det(A - \lambda I) = 0 \iff (-4-\lambda)(2-\lambda) + 18 = 0$$

$$\iff \lambda^2 + 2\lambda + 10 = 0$$

$$\lambda_\pm = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2}$$

$$= \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

Use Putzer: $\lambda_1 = -1 + 3i \qquad \lambda_2 = -1 - 3i$

$$r_1(t) = e^{\lambda_1 t} = e^{-t} \cdot e^{3it}$$

$$r_2(t) = e^{\lambda_2 t} \int_0^t e^{-\lambda_2 s} \cdot e^{\lambda_1 s}\, ds$$

$$= e^{-t} \cdot e^{-3it} \cdot \int_0^t e^{6is}\, ds$$

$$= e^{-t} \cdot e^{-3it} \cdot \frac{1}{6i}\left(e^{6it} - 1\right)$$

$$= \frac{1}{3} e^{-t} \frac{1}{2i}\left(e^{3it} - e^{-3it}\right) = \frac{1}{3} e^{-t} \sin(3t)$$

In this case,

$$\Phi(t) = r_1(t) P_1 + r_2(t) P_2$$

$$= e^{-t} \cdot e^{3it} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3} e^{-t} \sin(3t) \left(A - \lambda_1 I \right)$$

$$= e^{-t} \left[\begin{pmatrix} \cos(3t) + i\sin(3t) & 0 \\ 0 & \cos(3t) + i\sin(3t) \end{pmatrix} \right.$$

$$\left. + \frac{\sin(3t)}{3} \begin{pmatrix} -3-3i & 6 \\ -3 & 3-3i \end{pmatrix} \right]$$

$$= e^{-t} \begin{pmatrix} \cos(3t) - \sin(3t) & 2\sin(3t) \\ -\sin(3t) & \cos(3t) + \sin(3t) \end{pmatrix}$$

b) The solution of this IVP is

$$\vec{x}(t) = \Phi(t)\, \vec{x}(0)$$

$$= e^{-t} \begin{pmatrix} \cos(3t) - \sin(3t) & 2\sin(3t) \\ -\sin(3t) & \cos(3t) + \sin(3t) \end{pmatrix} \Bigg($$

$$= e^{-t} \begin{pmatrix} 2\cos(3t) - 4\sin(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix}$$

3 a) convert

$$x'' + 4x = \sin(3t)$$

to a 1st order system.

Let $y = x'$. Then

$$x' = 0x + y$$
$$y' = x'' = -4x + 0y + \sin(3t)$$

$\iff$ $\tilde{x}' = A\tilde{x} + \tilde{g}$ with $\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$

and $\tilde{g}(t) = \begin{pmatrix} 0 \\ \sin(3t) \end{pmatrix}$.

b) Find the general solution.
The eigen values of A are:

$$\det(A - \lambda I) = 0 \iff (-\lambda)(-\lambda) + 4 = 0$$
$$\lambda^2 = -4$$
$$\lambda = \pm 2i$$

$$\Phi(t) = \begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix}$$

$$\det \Phi(t) = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

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Thus

$$\Phi(s)^{-1} = \frac{1}{2}\begin{pmatrix} 2\cos(2s) & -\sin(2s) \\ 2\sin(2s) & \cos(2s) \end{pmatrix}$$

$$\Rightarrow \int^t \Phi(s)^{-1} \vec{g}(s)\,ds = \int^t \frac{1}{2}\begin{pmatrix} 2\cos(2s) & -\sin(2s) \\ 2\sin(2s) & \cos(2s) \end{pmatrix}\begin{pmatrix} 0 \\ \sin(3 \end{pmatrix}$$

$$= \int^t \frac{1}{2}\begin{pmatrix} -\sin(2s)\sin(3s) \\ \cos(2s)\sin(3s) \end{pmatrix} ds$$

$$= \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{9-4}\left(2\cos(2t)\sin(3t) - 3\sin(2t)\cos(3t)\right)\right) \\ \frac{1}{2}\left(\frac{1}{4-9}\left(2\sin(3t)\sin(2t) + 3\cos(2t)\cancel{\sin}\overset{\cos}{}(3t)\right)\right) \end{pmatrix}$$

$$= -\frac{1}{10}\begin{pmatrix} 2\cos(2t)\sin(3t) - 3\sin(2t)\cos(3t) \\ 2\sin(2t)\sin(3t) + 3\cos(2t)\cos(3t) \end{pmatrix}$$

The general solution is then.

$$\vec{x}(t) = \Phi(t)\vec{c} + \Phi(t)\int^t \Phi(s)^{-1}\vec{g}(s)\,ds$$

$$= \begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \frac{-1}{10}\begin{pmatrix} 2\cos(2t)\sin(3t) - 3\sin(2t)\cos t \\ 2\sin(2t)\sin(3t) + 3\cos(2t)\cos \end{pmatrix}$$

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$$\tilde{x}(t) = \begin{pmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ -2c_1 \sin(2t) + 2c_2 \cos(2t) \end{pmatrix}$$

$$-\frac{1}{10}\begin{pmatrix} 2\cos^2(2t)\sin(3t) - 3\cos(2t)\sin(2t)\cos(3t) \\ \quad + 2\sin^2(2t)\sin(3t) + 3\sin(2t)\cos(2t)\cos(3t) \\ -4\sin(2t)\cos(2t)\sin(3t) + 6\sin^2(2t)\cos(3t) \\ \quad + 4\cos(2t)\sin(2t)\sin(3t) + 6\cos^2(2t)\cos(3t) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ -2c_1 \sin(2t) + 2c_2 \sin(2t) \end{pmatrix}$$

$$-\frac{1}{10}\begin{pmatrix} 2\sin(3t) \\ 6\cos(3t) \end{pmatrix}$$

Resolution of the original 2^{nd} order D.E. is then

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{5}\sin(3t)$$

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4) a) Find a normalized solution matrix for:

$$x' = 3x$$
$$y' = -x + 3y$$

The coefficient matrix is $A = \begin{pmatrix} 3 & 0 \\ -1 & 3 \end{pmatrix}$.
The eigenvalues are.

$$\det(A - \lambda I) = 0 \iff (3-\lambda)(3-\lambda) = 0$$

i.e. $\lambda = 3$ is a repeated root.

Use Putzer.

$$r_1(t) = e^{3t}$$

$$r_2(t) = e^{3t} \int_0^t e^{-3s} \cdot e^{3s}\, ds = t e^{3t}$$

$$\Phi(t) = e^{3t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t e^{3t} \left(\begin{pmatrix} 3 & 0 \\ -1 & 3 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} e^{3t} & 0 \\ -te^{3t} & e^{3t} \end{pmatrix}$$

b) Solve the IVP. with $\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\vec{x}(t) = \Phi(t)\, \vec{x}(0)$$

$$= \begin{pmatrix} e^{3t} & 0 \\ -te^{3t} & e^{3t} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t} \\ -e^{3t} - te^{3t} \end{pmatrix}$$

5a) $$A_1 = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix}$$

<u>Eigenvalues</u>: $\det(A_1 - \lambda I) = 0 \iff (-\lambda)(5-\lambda) + 4 = 0$

$$\iff \lambda^2 - 5\lambda + 4 = 0$$

$$\iff (\lambda - 4)(\lambda - 1) = 0$$

$$\lambda = 1, 4$$

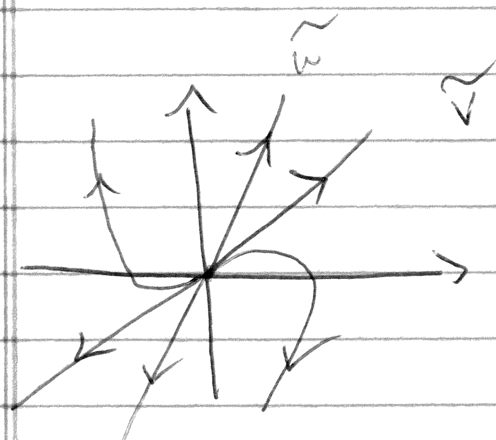
$\lambda = 1$

$$(A_1 - I)\vec{v} = 0 \iff \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = 4$

$(A_1 - 4I)\tilde{w} = 0 \iff \begin{pmatrix} -4 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



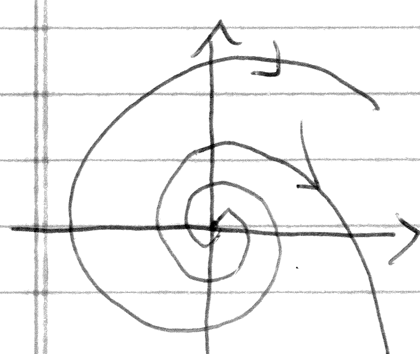
$\tilde{w} = \begin{pmatrix} 1 \\ +4 \end{pmatrix}$

$\boxed{\text{unstable origin}}$

b) $A_2 = \begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix}$

<u>Eigenvalues</u> $\det(A_2 - \lambda I) = 0 \iff (3-\lambda)(-1-\lambda) + 8$

$\iff \lambda^2 - 2\lambda + 5 = 0$



spiral

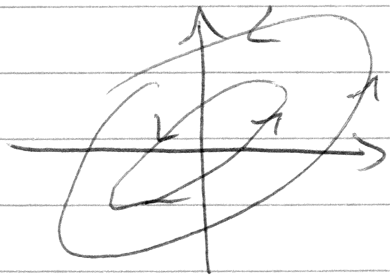
$\lambda_{\pm} = \dfrac{2 \pm \sqrt{4 - 4(1)(5)}}{2}$

$= \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

$\boxed{\text{unstable origin}}$

c) $A_3 = \begin{pmatrix} 9 & -17 \\ 5 & -9 \end{pmatrix}$

Eigenvalues $\det(A_3 - \lambda I) = 0 \iff (9-\lambda)(-9-\lambda) + 17(5)$



$\iff \lambda^2 + 4 = 0$

$\lambda = \pm 2i$

center. Lyapunov stable, but not a local attractor.

d) $A_4 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

Eigenvalues $\det(A_4 - \lambda I) = 0 \iff (-\lambda)(-2-\lambda) + 1 = 0$

$\iff \lambda^2 + 2\lambda + 1 = 0$

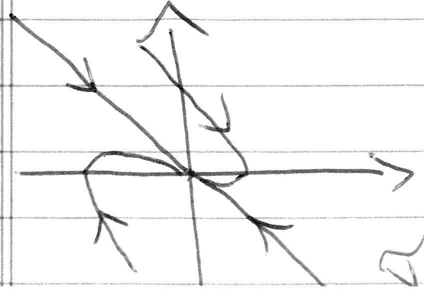
$(\lambda+1)^2 = 0$

$\lambda_\pm = -1$ double root.

Not proportional to identity.

Eigenvector

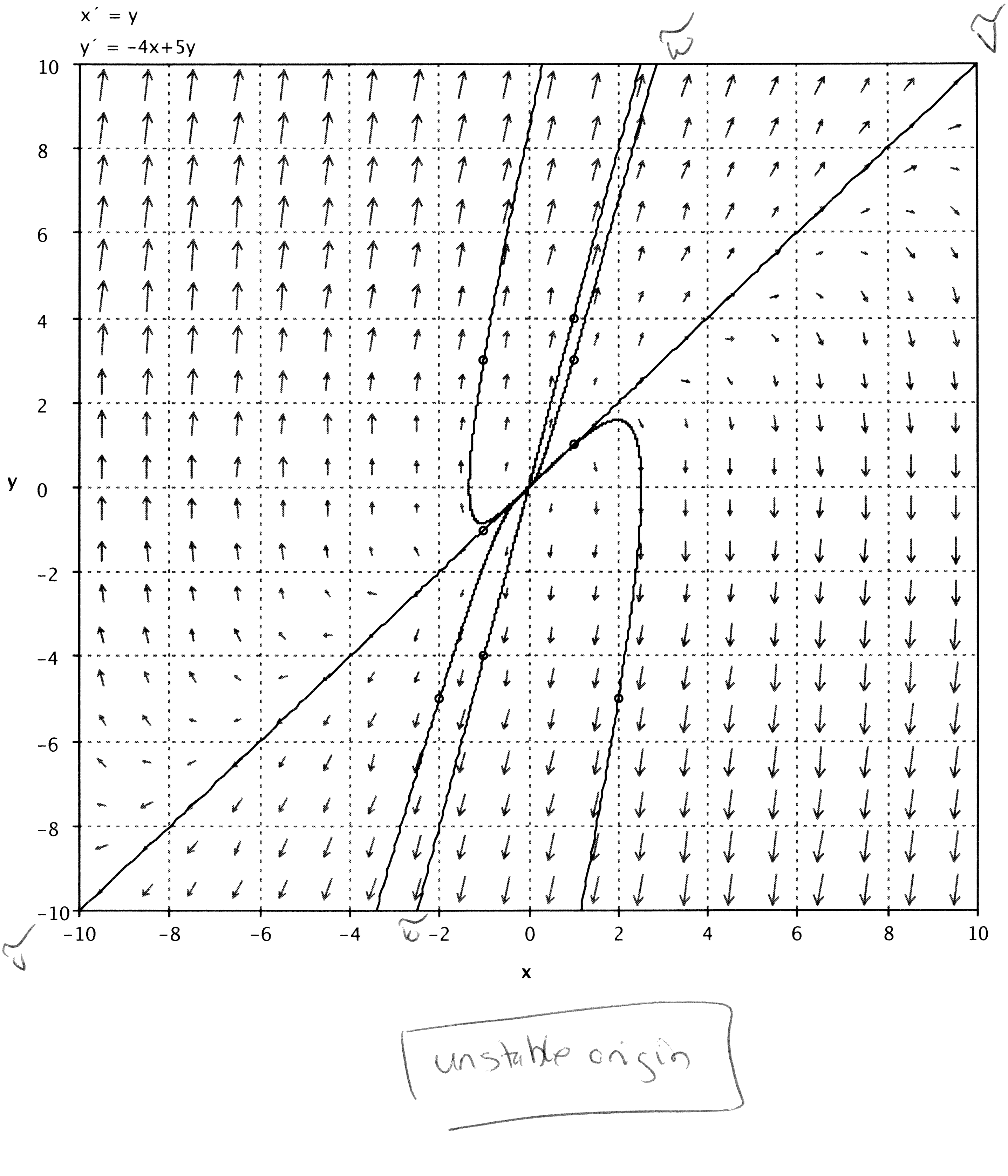
$\vec{v}$ $(A_4 + I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \iff \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$



$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

stable, improper node.

A1

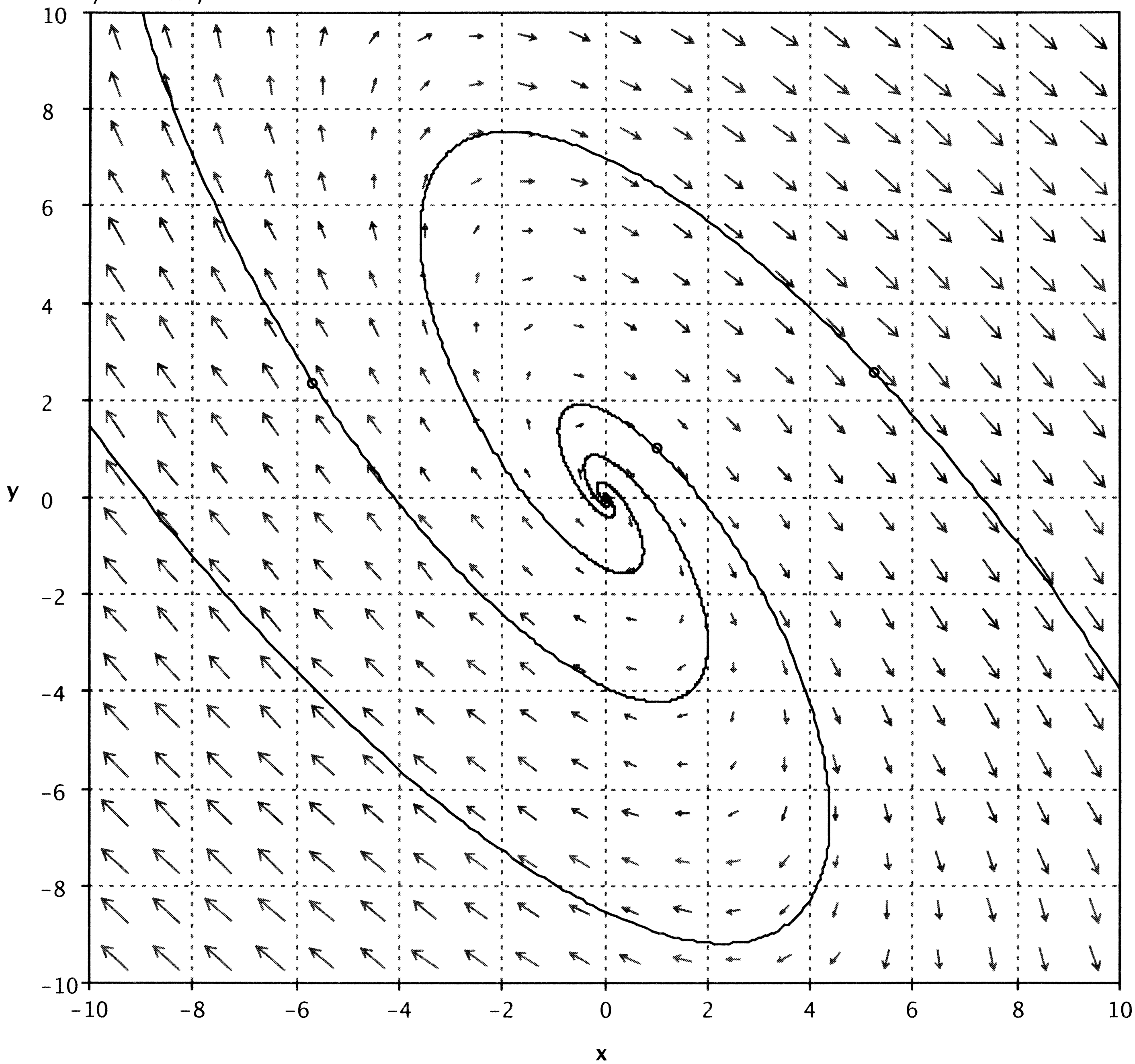


$\boxed{\text{unstable origin}}$

A2

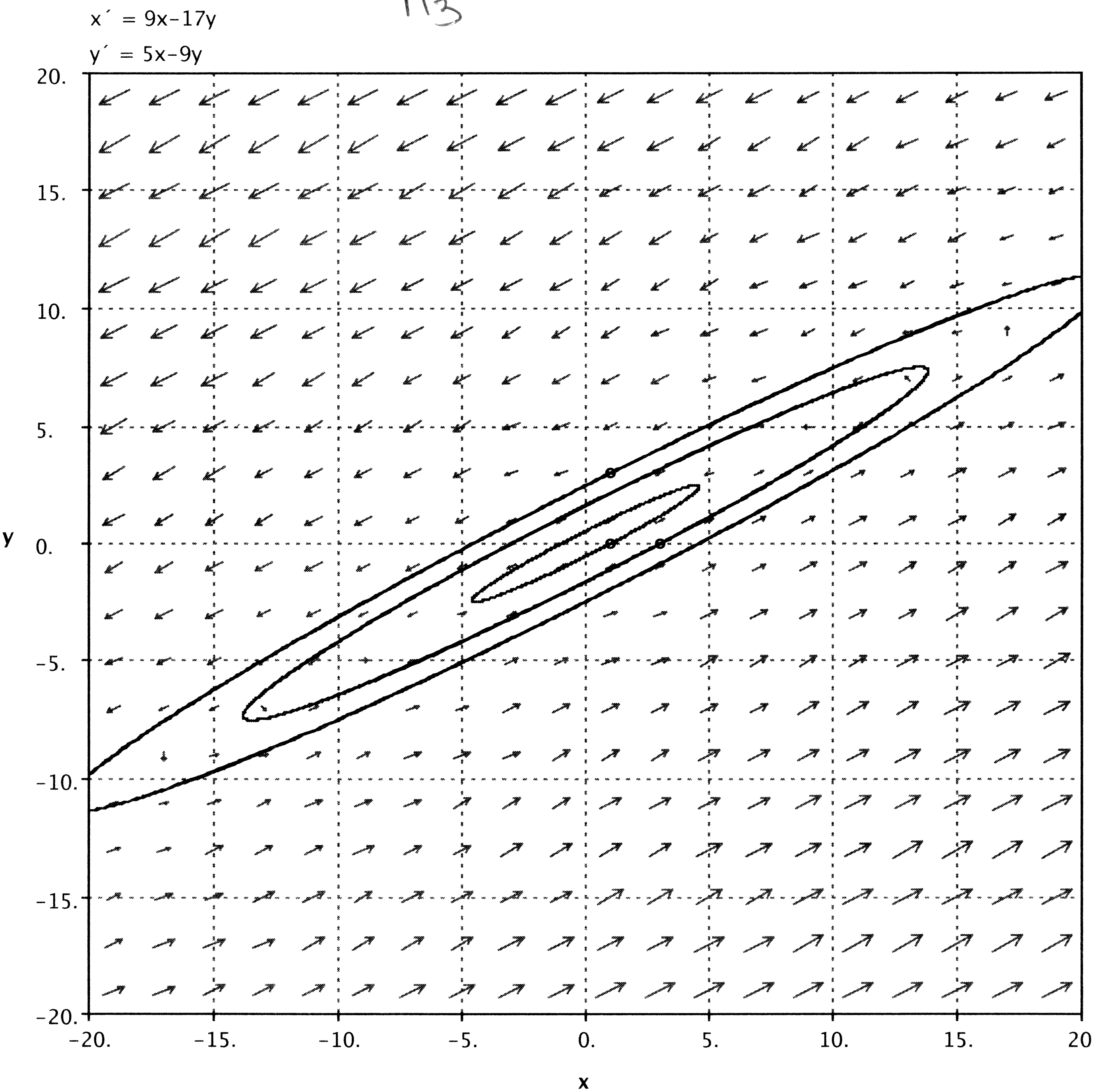
$x' = 3x+2y$

$y' = -4x-y$



unstable origin

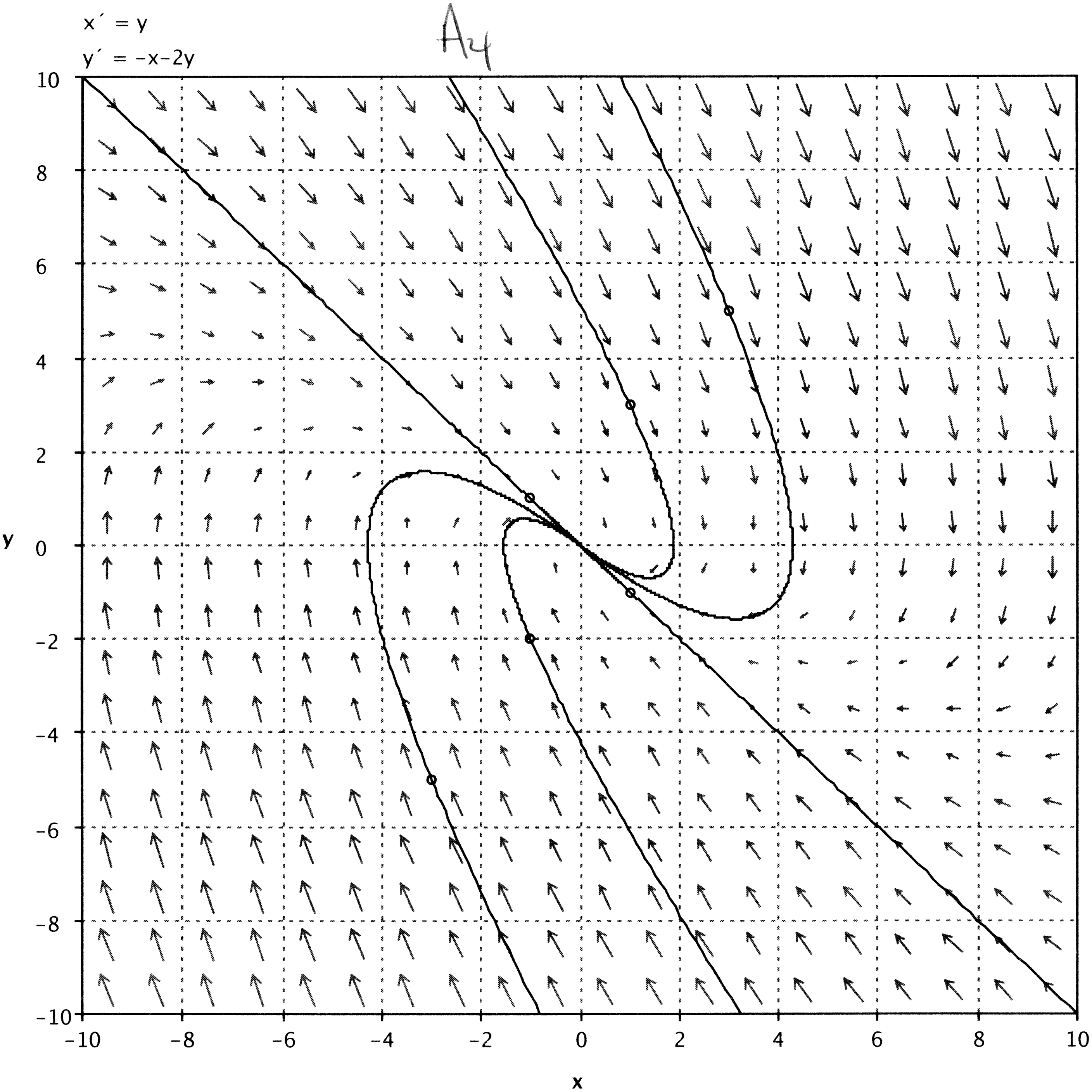
A_3



Center

$\boxed{\text{Lyapunov stable}}$ (not a local attractor)

A4



Stable improper node.