Key to Test 2

1 a) Denote by

$$f(t,x,y) = \ln(x+y+1) + 4$$

and

$$g(t,x,y) = \sin(x+y-1) + 2.$$

All the functions $f, g, f_x, f_y, g_x,$ and g_y are continuous in $t, x,$ and y in an open neighborhood of $x(0)=1$ and $y(0)=-1$. For this reason, the fundamental existence and uniqueness result (see Theorem 4.1 page 133) applies.

b) Since $f(t,0,-1) = \ln(0)+4$ is undefined, our fundamental existence and uniqueness result (again Theorem 4.1) does not apply.

c) Denote by

$$f(t,x,y) = 3x + 5y + ax^2$$

and

$$g(t,x,y) = 2x - y + e^t$$

For all values of a, $f, g, f_x, f_y, g_x,$ and g_y are continuous for all values of $t, x,$ and y. In this case, Theorem 4.1 applies to all initial conditions for <u>any</u> choice of a.

d) Theorem 4.1 only guarantees local existence of solutions. Theorem 5.1 (see page 160) guarantees existence of unique solutions on the whole interval (in this case for all t), but this Theorem 5.1 only applies to <u>linear</u> systems. In this case, we only have existence and uniqueness of solutions that are defined for all t in the special case that $a=0$. (When $a \neq 0$, this system is not <u>linear</u>.)

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2 a) Note that

$$A\tilde{x}_1(t) = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} e^{4t} \\ 2e^{4t} \end{pmatrix} = \begin{pmatrix} 4e^{4t} \\ 8e^{4t} \end{pmatrix} = \tilde{x}_1'(t)$$

and similarly

$$A\tilde{x}_2(t) = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} 9e^{-3t} \\ -3e^{-3t} \end{pmatrix} = \tilde{x}_2'(t)$$

and thus both $\tilde{x}_1$ and $\tilde{x}_2$ are solutions of the linear homogeneous system:

$$\tilde{x}' = A\tilde{x}.$$

The solution matrix

$$\Phi(t) = (\tilde{x}_1(t) \quad \hat{x}_2(t)) = \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix}$$ is fundamental

because $\det(\Phi(0)) = \det\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = 7 \neq 0.$

b) In general, the solution of a linear homogeneous system's I.V.P. at $t=0$ is given by (2)

$$\tilde{x}(t) = \Phi(t)\,\Phi(0)^{-1}\,\tilde{x}(0)$$

$$= \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix} \cdot \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} -2e^{4t} + 9e^{-3t} \\ -4e^{4t} - 3e^{-3t} \end{pmatrix}.$$

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3) Let $A = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}$ and $\Phi(t) = \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 3\cos(t) - \sin(t) & \cos(t) + 3\sin(t) \end{pmatrix}$

check that:

$$A\,\Phi(t) = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 3\cos(t) - \sin(t) & \cos(t) + 3\sin(t) \end{pmatrix}$$

$$= \begin{pmatrix} -15\cos(t) + 5(3\cos(t) - \sin(t)) & -15\sin(t) + (5\cos(t) + 15\sin(t)) \\ -10\cos(t) + 3(3\cos(t) - \sin(t)) & -10\sin(t) + 3(\cos(t) + 3\sin(t)) \end{pmatrix}$$

$$= \begin{pmatrix} -5\sin(t) & +5\cos(t) \\ -\cos(t) - 3\sin(t) & 3\cos(t) - \sin(t) \end{pmatrix}$$

$$= \Phi'(t)$$ i.e. $\Phi(t)$ is a solution matrix.

To see that $\Phi(t)$ is fundamental, note that

$$\det(\Phi(0)) = \det\begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} = 5 \neq 0.$$

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b) By Variation of constants, the general solution of this non-homogeneous linear system is:

$$\tilde{x}(t) = \Phi(t)\,\tilde{c} + \Phi(t)\int^{t} \Phi(s)^{-1}\,\tilde{g}(s)\,ds$$

where $\tilde{g}(s) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Note that

$$\det(\Phi(s)) = \det\begin{pmatrix} 5\cos(s) & 5\sin(s) \\ 3\cos(s) - \sin(s) & \cos(s) + 3\sin(s) \end{pmatrix}$$

$$= 5\cos(s)\big(\cos(s) + 3\sin(s)\big) - 5\sin(s)\big(3\cos(s) - \sin(s)\big)$$

$$= 5\cos^2(s) + 15\sin(s)\cos(s) - 15\sin(s)\cos(s) + 5\sin^2(s)$$

$$= 5\left(\sin^2(s) + \cos^2(s)\right) = 5$$

<u>Thus</u>

$$\Phi(s)^{-1} = \frac{1}{5}\begin{pmatrix} \cos(s) + 3\sin(s) & -5\sin(s) \\ \sin(s) - 3\cos(s) & 5\cos(s) \end{pmatrix}$$

In this case,

$$\int^t \Phi(s)^{-1} \tilde{g}(s)\,ds = \int^t \frac{1}{5}\begin{pmatrix} \cos(s)+3\sin(s) & -5\sin(s) \\ \sin(s)-3\cos(s) & 5\cos(s) \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} ds$$

$$= \int^t \frac{1}{5}\begin{pmatrix} -\cos(s)-8\sin(s) \\ -\sin(s)+8\cos(s) \end{pmatrix} ds$$

$$= \frac{1}{5}\begin{pmatrix} -\sin(t)+8\cos(t) \\ \cos(t)+8\sin(t) \end{pmatrix}$$

and so

$$\tilde{x}(t) = \Phi(t)\tilde{c} + \Phi(t)\int^t \Phi(s)^{-1}\tilde{g}(s)\,ds$$

$$= \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 3\cos(t)-\sin(t) & \cos(t)+3\sin(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$+ \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 3\cos(t)-\sin(t) & \cos(t)+3\sin(t) \end{pmatrix} \frac{1}{5}\begin{pmatrix} -\sin(t)+8\cos(t) \\ \cos(t)+8\sin(t) \end{pmatrix}$$

$$= \begin{pmatrix} 5c_2\sin(t) + 5c_1\cos(t) \\ (3c_2-c_1)\sin(t) + (c_2+3c_1)\cos(t) \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

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4) The coefficient matrix $A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$ has eigenvalues:

a) $\det(A-\lambda I)=0 \iff \lambda^2-2\lambda+2=0 \iff \lambda_\pm = 1\pm i$

$\lambda_1 = 1+i$ and $\lambda_2 = 1-i$.

By Putzer: Take $\lambda_1 = 1+i$

$r_1(t) = e^{(1+i)t}$

$$r_2(t) = e^{(1-i)t}\int_0^t e^{-(1-i)s}\cdot e^{(1+i)s}\,ds$$

$$= e^t\cdot e^{-it}\int_0^t e^{2is}\,ds = e^t\cdot e^{-it}\cdot\frac{1}{2i}(e^{2it}-1)$$

$$= e^t\cdot\frac{1}{2i}(e^{it}-e^{-it})$$

$$= e^t\sin(t)$$

In this case,

$$\Phi(t) = r_1(t)P_1 + r_2(t)P_2$$

$$= e^{(1+i)t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{t}\sin(t)\left(\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} - (1+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= e^{t}\left(\begin{pmatrix} \cos(t)+i\sin(t) & 0 \\ 0 & \cos(t)+i\sin(t) \end{pmatrix} + \sin(t)\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \right)$$

$$= e^{t}\begin{pmatrix} \cos(t)+\sin(t) & \sin(t) \\ -2\sin(t) & \cos(t)-\sin(t) \end{pmatrix}$$

b) The solution of the initial value problem is

$$\vec{x}(t) = \Phi(t)\,\Phi(0)^{-1}\,\vec{x}(0)$$

$$= e^{t}\begin{pmatrix} \cos(t)+\sin(t) & \sin(t) \\ -2\sin(t) & \cos(t)-\sin(t) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos(t) \\ -e^{t}(\sin(t)+\cos(t)) \end{pmatrix}$$

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5) a) $x' = y$ $\quad \Longleftrightarrow \quad \tilde{x}' = A\tilde{x}$ with

$y' = 6x + y$ $\qquad A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$

b) $\det(A - \lambda I) = 0 \Longleftrightarrow \lambda^2 - \lambda - 6 = 0$

$\Longleftrightarrow \lambda_{\pm} = 3, -2$

$\lambda = 3$ $\quad (A - \lambda_1 I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\tilde{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\qquad -3v_1 + v_2 = 0$

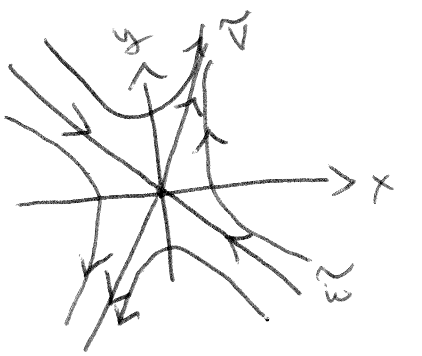
$6v_1 - 2v_2 = 0$

$\lambda_2 = -2$ $\quad (A - \lambda_2 I)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$

$\tilde{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\qquad 2w_1 + w_2 = 0$

$6w_1 + 3w_2 = 0$

c)



d) The origin is unstable.