MATH 355-002:
TEST 1
MAKE-UP

SPRING 2019

Name	
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Directions: This work is an optional assignment for those who took the first test on Thursday, February 21st. It is due on Tuesday, March 12th at the beginning of class. No late work will be accepted. If you turn this in, I will grade it and your new grade on test 1 will be the average (out of 100%) of the two scores you have received. If you do not turn this in, your grade on test 1 will stay the same.

Where indicated, show work to illustrate you are using the method requested in the particular question.

(1) In the following initial value problems, the number a is a real parameter. Determine the values of a for which our fundamental theorem on existence and uniqueness of solutions applies. Explain your answer.

(a)
$$x' = \ln(a - x) \quad \text{with} \quad x(0) = 0\,.$$

(b)
$$x' = \sqrt{a^2 - x^2} \quad \text{with} \quad x(1) = 2\,.$$

(c)
$$x' = \tan(ax) \quad \text{with} \quad x(0) = \frac{\pi}{2}\,.$$

Test 1 Make up

1 a)

$$x' = \ln(a-x) \qquad x(0) = 0$$

Here $f(t,x) = \ln(a-x)$ does not depend on t.
Note: For any real a, f is only defined when

$$a - x > 0 \quad \text{i.e.} \quad x < a.$$

On this domain f is continuous and

$$\frac{\partial f}{\partial x} = \frac{-1}{a-x} = \frac{1}{x-a}$$

This function is well-defined and continuous whenever $x \neq a$.

In this case, the theorem on existence and uniqueness applies whenever $a > 0$.

b) $x' = \sqrt{a^2 - x^2} \qquad x(1) = 2.$

Here $f(t,x) = \sqrt{a^2 - x^2}$ does not depend on t.
Note: For any real a, f is only defined when

$$0 \leq a^2 - x^2 \quad \text{i.e.} \quad x^2 \leq a^2$$

$$\boxed{-|a| \leq x \leq |a|} \qquad \text{i.e.} \quad |x| \leq |a|$$

1b) continued. f is continuous whenever

$$-|a| \le x \le |a|.$$

Moreover,

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{a^2-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{a^2-x^2}}$$

and this function is continuous whenever

$$-|a| < x < |a|$$

In this case, the theorem on existence and uniqueness applies whenever $|a| > 2$.

1c) $x' = \tan(ax)$

Here $f(t,x) = \tan(ax)$ does not depend on t.

Note: For any real a, f is only defined when

$$ax \neq \frac{\pi}{2} \pm n\pi \qquad \text{for all integers } n \ge 0.$$

Away from these values, f is continuous.

Moreover,

$$\frac{\partial f}{\partial x} = \sec^2(ax) \cdot a$$

This function is continuous on the same domain.

In this case, the theorem on existence and uniqueness applies whenever.

$$a\frac{\pi}{2} \neq \pi/2 \pm n\pi$$

$$(a-1)\pi/2 \neq \pm n\pi$$

$$(a-1)\pi \neq \pm 2n\pi$$

$$a \neq 1 \pm 2n \qquad n \geq 0.$$

(These are odd integers and their negatives...)

(2) Using the variation of constants formula, solve the following initial value problem:

$$x' = \frac{2}{t}x + t^2(1 + e^{-at}) \quad \text{with} \quad x(1) = 1\,.$$

Here $a \neq 0$ is a real parameter.

2) Solve the I.V.P.:

$$x' = \frac{2}{t}x + t^2(1 + e^{-at})$$

with $x(1) = 1$. Here $a \neq 0$.

We know that the general solution to this linear D.E. has the form

$$x(t) = x_h(t) + x_p(t).$$

The general solution to the associated homogeneous equation is

$$x_h(t) = c\,e^{P(t)} \quad \text{where} \quad P(t) = \int^t \frac{2}{s}\,ds$$
$$= 2\ln|t|$$
$$= \ln(t^2).$$

and c is any constant. Thus

$$x_h(t) = c\,e^{\ln(t^2)} = c \cdot t^2.$$

For the particular solution we use variation of constants:

$$x_p(t) = e^{P(t)} \int^t e^{-P(s)} q(s)\,ds$$
$$= t^2 \cdot \int^t \frac{1}{s^2} \cdot s^2 (1 + e^{-as})\,ds$$
$$= t^2\left(t - \frac{1}{a}e^{-at}\right)$$

Note $a \neq 0$

In this case, the general solution is

$$x(t) = x_h(t) + x_p(t)$$

$$= ct^2 + t^2\left(t - \frac{1}{a}e^{-at}\right)$$

$$= t^2\left(c + t - \frac{1}{a}e^{-at}\right)$$

Considering the initial condition, we find that

$$1 = x(1) = c + 1 - \frac{1}{a}e^{-a}$$

$$\Rightarrow c = \frac{1}{a}e^{-a}$$

Thus

$$x(t) = t^2\left(\frac{1}{a}e^{-a} + t - \frac{1}{a}e^{-at}\right)$$

(3) Using the super position principle and the method of undetermined coefficients, find the general solution of the following differential equation:

$$x' = 2x - 3te^{2t} + 5t^3 \,.$$

3) Find the general solution of

$$x' = 2x - 3te^{2t} + 5t^3$$

We use the super position principle.

First, solve for a particular solution of

$x' = 2x + te^{2t}$ $\{te^{2t}, e^{2t}\}$ ← multiply by t

Let $x_1(t) = Ate^{2t} + Bt^2e^{2t}$

$x_1'(t) = Ae^{2t} + 2Ate^{2t} + 2Bte^{2t} + 2Bt^2e^{2t}$

$2x_1(t) + te^{2t} = 2Ate^{2t} + 2Bt^2e^{2t} + te^{2t}$

$\Rightarrow$ $A = 0$ | $2A + 2B = 2A + 1$ | $2B = 2B$

$2B = 1$

$B = 1/2$.

$x_1(t) = \frac{1}{2}t^2e^{2t}$ ✓

Next, we solve.

$$x' = 2x + t^3 \qquad \{t^3, t^2, t, 1\}$$

let $x_2(t) = At^3 + Bt^2 + Ct + D$

$$x_2'(t) = 3At^2 + 2Bt + C$$

$$2x_2(t) + t^3 = 2At^3 + 2Bt^2 + 2Ct + 2D + t^3$$

$\Rightarrow$ $0 = 2A + 1$	$3A = 2B$	$2B = 2C$	$C = 2D$
$A = -1/2$	$B = -\frac{3}{4}$	$C = -3/4$	$D = -\frac{3}{8}$

$$x_2(t) = -\frac{1}{2}t^3 - \frac{3}{4}t^2 - \frac{3}{4}t - \frac{3}{8} \quad \checkmark$$

Thus (by Super position principle)

$$x_p(t) = -3x_1(t) + 5x_2(t)$$

is a particular solution of this D.E. Hence

$$x(t) = x_h(t) + x_p(t)$$
$$= Ce^{2t} + -3\left(\frac{1}{2}t^2e^{2t}\right) + 5\left(-\frac{1}{2}t^3 - \frac{3}{4}t^2 - \frac{3}{4}t - \frac{3}{8}\right)$$

is the general solution.

(4) Let $0 < a < b$ be real parameters. Consider the following differential equation:

$$x' = (x+a)(x-a)^3(x-b)^4(x+b)^3$$

Sketch the phase line portrait.
Find and classify all equilibria.
Find the linearization at any hyperbolic equilibrium.

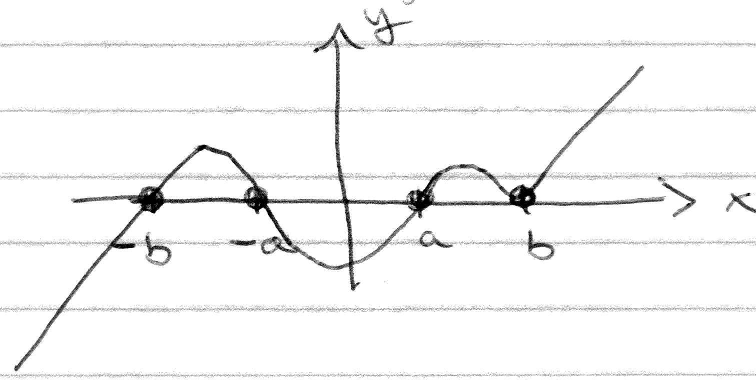
4) Let $0 < a < b$. Consider

$$x' = (x+a)(x-a)^3(x-b)^4(x+b)^3$$

- Sketch the phase line portrait.

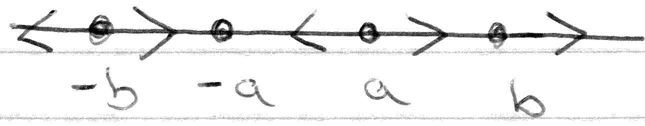
Let $f(x) = (x+a)(x-a)^3(x-b)^4(x+b)^3$

Observe that f is a degree 11 polynomial.
It has 4 roots, namely $-b, -a, a, b$.
In this case, its graph is



rough sketch of $y = f(x)$

Thus the phase line portrait is



- Classify equilibria:

$x = -b$ repeller
$x = -a$ attractor
$x = a$ repeller
$x = b$ shunt

- Find the linearization at any hyperbolic equilibrium.

$$\frac{d}{dx}f(x) = (x-a)^3(x-b)^4(x+b)^3$$
$$+ (x+a)\,3(x-a)^2(x-b)^4(x+b)^3$$
$$+ (x+a)(x-a)^3\,4(x-b)^3(x+b)^3$$
$$+ (x+a)(x-a)^3(x-b)^4\,3(x+b)^2$$

$\Rightarrow$ $\frac{d}{dx}f(-b) = 0+0+0+0$ $\quad -b$ is non-hyperbolic

$\frac{d}{dx}f(-a) = (-2a)^3(a+b)^4(b-a)^3 + 0+0+0 \neq 0$ $\quad -a$ is hyperbolic

$\frac{d}{dx}f(a) = 0+0+0+0$ $\quad a$ is non-hyperbolic

$\frac{d}{dx}f(b) = 0+0+0+0$ $\quad b$ is non-hyperbolic

Since $\frac{d}{dx}f(-a) = -8a^3(a+b)^4(b-a)^3 < 0$, the linearization at $x=-a$ is:

$$u' = -8a^3(a+b)^4(b-a)^3\,u$$

(5) Consider the following differential equation:

$$x' = (x^2 - 4)(p - x^2)$$

Here p is a parameter.

Sketch a bifurcation diagram for this differential equation which indicates the type of equilibria in each branch of the diagram. Also indicate all bifurcation values and classify the type of each bifurcation.

5) Consider

$$x' = (x^2-4)(p-x^2) \quad \text{with } p \text{ a parameter.}$$

Sketch a bifurcation diagram.

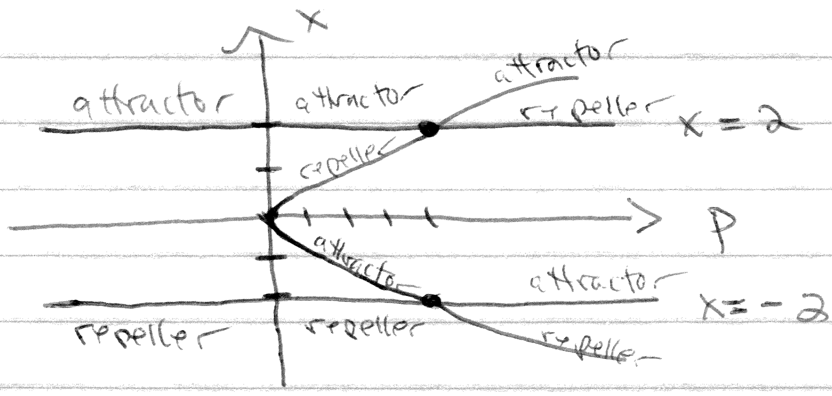
Let $f(x,p) = (x^2-4)(p-x^2)$

Then $f(x,p) = 0 \iff x^2-4 = 0$ or $p-x^2=0$

Thus $x = \pm 2$ are equilibria for all p.

If $p<0$, $p-x^2=0$ has no solutions.
If $p\geq 0$, $p-x^2=0$ i.e. $x^2=p$ has solutions $x = \pm\sqrt{p}$.

The Bifurcation diagram is:

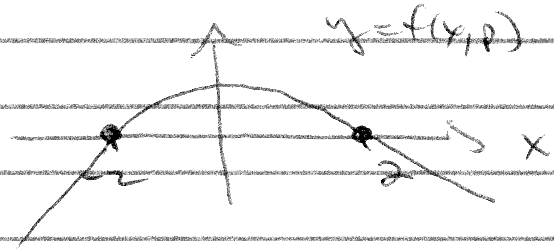


$p=0$ and $p=4$ are bifurcation values.

↑ blue sky ↑ transcritical

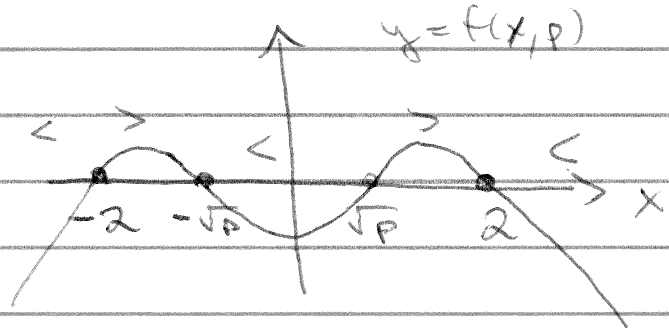
If $p<0$, then $(p-x^2)<0$ for all x.

Thus

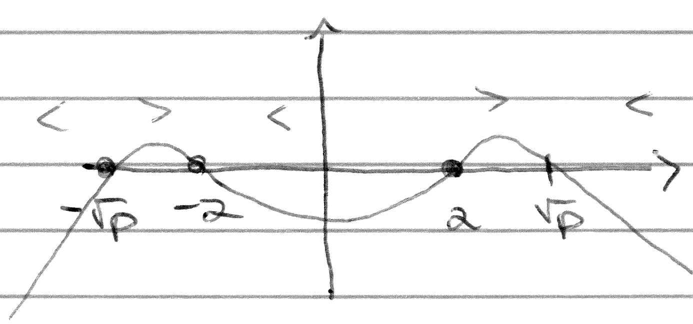


Thus $x=-2$ is a repeller and $x=+2$ is an attractor.

If $p>0$, then $f(x,p) = (x+2)(x-2)(\sqrt{p}-x)(\sqrt{p}+x)$
$= -(x+2)(x-2)(x-\sqrt{p})(x-(-\sqrt{p}))$



if $\sqrt{p}<2$
ie. $p<4$



if $2<\sqrt{p}$
ie $p>4$