MATH 355-002:
SIMS
TEST 1

SPRING 2019

Name	Key
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Directions: Solve all the problems below. Where indicated, show work to illustrate you are using the method requested in the particular question.

(1) Consider the following initial value problems. In each case, determine whether or not our fundamental theorem on existence and uniqueness of solutions applies. Each part below is worth 2 points. 1 point for answering yes the theorem applies or no it does not; and 1 point for a sentence explaining why.

(a)

$$x' = \ln(x^2 - t^2 + 1) \quad \text{with} \quad x(1) = 2\,.$$

(b)

$$x' = \ln(x^2 - t^2 + 1) \quad \text{with} \quad x(2) = 1\,.$$

(c)

$$x' = \sqrt{x} - 4t \quad \text{with} \quad x(1) = 0\,.$$

(d)

$$x' = \sqrt{x} - 4t \quad \text{with} \quad x(0) = 1\,.$$

(e)

$$x' = \frac{\sin(x)}{\cos(t)} \quad \text{with} \quad x(0) = \frac{\pi}{2}\,.$$

Key to Test 1

1 a) $x' = \ln(x^2 - t^2 + 1)$ with $x(1) = 2$.

The function $f(t,x) = \ln(x^2 - t^2 + 1)$ is well-defined

for $x^2 - t^2 + 1 > 0$. On this domain f is continuous

and
$$\frac{\partial f}{\partial x}(t,x) = \frac{1}{x^2 - t^2 + 1} \cdot (2x) = \frac{2x}{x^2 - t^2 + 1}$$

is well-defined as well. Note: $\frac{\partial f}{\partial x}$ is continuous whenever

$x^2 - t^2 + 1 \neq 0$.

The point $(1,2)$ is in this domain.

$2^2 - (1)^2 + 1 = 4 > 0$.

So, yes the theorem applies.
Both f and $\frac{\partial f}{\partial x}$ are continuous on a rectangle containing
$(1,2)$ in the t,x plane.

b) $x' = \ln(x^2 - t^2 + 1)$ with $x(2) = 1$.

The point $(2,1)$ is not in this domain.

$1^2 - (2)^2 + 1 = -2 < 0$.

No, the theorem does not apply.
f is not defined (hence not continuous) at $(2,1)$.

c) $x' = \sqrt{x} - 4t$ with $x(1) = 0$.

The function $f(t,x) = \sqrt{x} - 4t$ is well-defined (and continuous) for all real t and $x \geq 0$. For all real t and $x > 0$,

$$\frac{\partial f}{\partial x}(t,x) = \frac{1}{2\sqrt{x}}.$$

$\frac{\partial f}{\partial x}$ is continuous for all real t and $x > 0$.

At the point $(1,0)$, $\frac{\partial f}{\partial x}$ is not defined (hence, not continuous). So no, the theorem does not apply.

d) $x' = \sqrt{x} - 4t$ with $x(0) = 1$.

At the point $(0,1)$, both f and $\frac{\partial f}{\partial x}$ are continuous in a rectangle containing $(0,1)$. Thus, yes the theorem applies.

e) $x' = \dfrac{\sin(x)}{\cos(t)}$ with $x(0) = \pi/2$.

The function $f(t,x) = \dfrac{\sin(x)}{\cos(t)}$ is well-defined (and continuous) for all real x and all t except $t = \frac{\pi}{2} \pm n\pi$ for $n = 0,1,2,\ldots$

The function

$$\frac{\partial f}{\partial x}(t,x) = \frac{\cos(x)}{\cos(t)}$$

is well-defined and continuous on the same domain. So yes the theorem does apply at $(0, \pi/2)$.

(2) Using the variation of constants formula, solve the following initial value problem:

$$x' = \sin(at)x + \sin(at) \quad \text{with} \quad x(0) = 1\,.$$

Here $a \neq 0$ is a real parameter.

The general solution of this D.E. has the form

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is the general solution of the homogeneous D.E. and $x_p(t)$ is any particular solution of this D.E.

By variation of constants we know that

$$x_h(t) = c\, e^{P(t)} = c\, e^{\int^t \sin(as)ds} = c\, e^{-\frac{\cos(at)}{a}}$$ for any real number c.

By variation of constants, we also know that

$$x_p(t) = e^{P(t)} \int^t e^{-P(s)} q(s)\,ds = e^{-\frac{\cos(at)}{a}} \int^t e^{\frac{\cos(as)}{a}} \sin(as)\,ds$$

$$\boxed{\begin{aligned} \text{Let } u &= \frac{\cos(as)}{a} \\ du &= -\sin(as)ds \end{aligned}} = e^{-\frac{\cos(at)}{a}}\left(-e^{\frac{\cos(at)}{a}}\right)$$

$$= -1$$

In this case, the general solution is:

$$x(t) = c\, e^{-\frac{\cos(at)}{a}} - 1$$

For the I.V.P., we find that

$$1 = x(0) = c\, e^{-\frac{1}{a}} - 1 \quad \Rightarrow \quad c = 2e^{\frac{1}{a}}$$

$$\Rightarrow \quad x(t) = 2e^{\frac{1}{a}(1-\cos(at))} - 1.$$

(3) Using the super position principle and the method of undetermined coefficients, find the general solution of the following differential equation:

$$x' = -2x + 3ae^{-2t} - 4at^2$$

Here a is a real parameter.

By the super position principle, if

$x_1(t)$ solves $x' = -2x + e^{-2t}$

and $x_2(t)$ solves $x' = -2x + t^2$

then $x(t) = 3a\,x_1(t) - 4a\,x_2(t)$ solves the above D.E.

By the method of undetermined coefficients,

$x_1(t) = Ate^{-2t}$ should solve the 1st D.E.

- $x_1'(t) = Ae^{-2t} - 2Ate^{-2t}$ $\Rightarrow$ $\boxed{A = 1}$
- $-2x_1(t) + e^{-2t} = -2Ate^{-2t} + e^{-2t}$

Similarly,

$x_2(t) = At^2 + Bt + C$ should solve the 2nd D.E.

- $x_2'(t) = 2At + B$
- $-2x_2(t) + t^2 = (1-2A)t^2 - 2Bt - 2C$

$\Rightarrow$ $1 - 2A = 0$, $2A = -2B$, $B = -2C$

$A = 1/2$, $B = -1/2$, $C = \frac{1}{4}$

$\Rightarrow$ $x_p(t) = 3ate^{-2t} - 4a\left(\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}\right)$

is a particular solution of this D.E.

$x(t) = ce^{-2t} + x_p(t)$

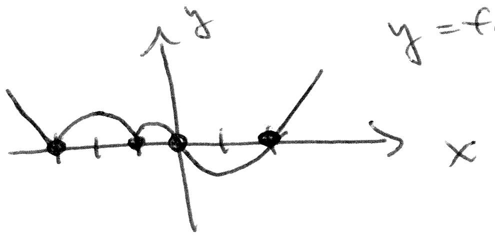
(4) Consider the following differential equation:

$$x' = x(x+1)^2(x-2)^3(x+3)^4$$

Sketch the phase line portrait.
Find and classify all equilibria.
Find the linearization at any hyperbolic equilibrium.

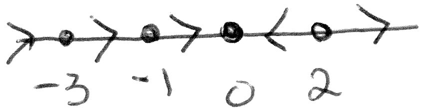
$x' = x(x+1)^2(x-2)^3(x+3)^4$

$f(x) = x(x+1)^2(x-2)^3(x+3)^4$



This is a degree 10 polynomial with a positive coefficient.
$\Rightarrow$ ∪ leading behavior.

Even roots "bounce"
odd roots "cross-through".



P.L.P.

$x = -3$ is a shunt
$x = -1$ is a shunt
$x = 0$ is an attractor
$x = 2$ is a repeller

By the product rule:

$$\frac{d}{dx} f(x) = (x+1)^2(x-2)^3(x+3)^4 + 2x(x+1)(x-2)^3(x+3)^4 + 3x(x+1)^2(x-2)^2(x+3)^4 + 4x(x+1)^2(x-2)^3(x+3)^3$$

$\Rightarrow \frac{d}{dx} f(-3) = 0 + 0 + 0 + 0 = 0$

$\Rightarrow \frac{d}{dx} f(-1) = 0 + 0 + 0 + 0 = 0$

$\Rightarrow \frac{d}{dx} f(0) = (1)^2(-2)^3(3)^4 + 0 + 0 + 0 = -8 \cdot 81 = -648$

$\Rightarrow \frac{d}{dx} f(2) = 0 + 0 + 0 + 0 = 0.$

The only hyperbolic equilibrium is at $x = 0$. The linearization is:

$u' = -648u$

It is an attractor.

(5) Consider the following differential equation:

$$x' = x^3(x^2 - p)$$

Here p is a parameter.

Sketch a bifurcation diagram for this differential equation which indicates the type of equilibria in each branch of the diagram. Also indicate all bifurcation values and classify the type of each bifurcation.

$x' = x^3(x^2 - p)$

$f(x,p) = x^3(x^2 - p)$

Equilibria

$x^3(x^2 - p) = 0$

$x^3 = 0$

$\Rightarrow x = 0$ is an equilibrium for all p.

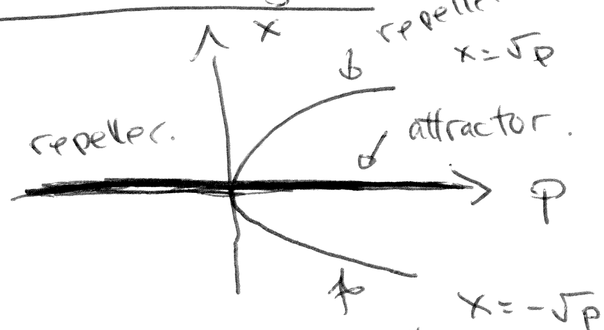
$x^2 - p = 0$

$\Rightarrow$ No equilibria if $p < 0$.

$\Rightarrow$ one equilibrium ($x=0$) if $p = 0$

$\Rightarrow$ $x = \pm\sqrt{p}$ are both equilibria if $p > 0$.

Bifurcation diagram



Bifurcation value:

$p = 0$ is a pitchfork bifurcation.

$\frac{df}{dx}(x,p) = 3x^2(x^2 - p) + 2x^4$

$\frac{df}{dx}(0,p) = 0$ No information.

$\frac{df}{dx}(\pm\sqrt{p},p) = 3p(\cancel{p-p}) + 2p^2 > 0$ if $p>0$

Both repellers.

If $p<0$, then $f(x,p) < 0$ for $x<0$
and $f(x,p) > 0$ for $x > 0$

Thus $x = 0$ is a repeller

If $p>0$, $f(x,p) = x^3(x+\sqrt{p})(x-\sqrt{p}) > 0$
for $-\sqrt{p} < x < 0$
$f(x,p) = x^3(x+\sqrt{p})(x-\sqrt{p}) < 0$

$\Rightarrow x = 0$ is an attractor