A variation of constants formula for systems

Last class we learned that linear non-homogeneous systems of the form

$$(*) \quad \tilde{x}' = A(t)\,\tilde{x} + \tilde{g}(t)$$

have general solutions with the form

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$$

where:

- $\tilde{x}_h(t)$ is the general solution of the associated homogeneous system

$$(**) \quad \tilde{x}' = A(t)\,\tilde{x}$$

and

- $\tilde{x}_p(t)$ is any particular solution of $(*)$.

Last class we focused on the solution of (**).

We found that

$$\tilde{X}_h(t) = \Phi(t)\,\tilde{c}$$

where $\Phi(t)$ is a fundamental solution matrix (i.e. a matrix whose columns are L.I. solutions of (**).)

and

$$\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

is a vector of arbitrary coefficients c_1 and c_2.

Today we focus on finding particular solutions of (*) i.e. $\tilde{X}_p(t)$.

There are several methods for this.

Today we will look at variation of constants.

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Recall that the 1^{st} order linear equation

$$(***) \qquad x' = p(t)\,x + q(t)$$

was solved by variation of constants. We found that

$$x_p(t) = e^{P(t)} \int^t e^{-P(s)} q(s)\,ds$$

was a particular solution.

A similar fact holds for systems.

<u>Note</u>: For $(***)$

$$x_h(t) = c \cdot e^{P(t)} = e^{P(t)} \cdot c$$

We will find that the fundamental solution matrix $\Phi(t)$ plays a role similar to

$$e^{P(t)} \; !$$

Variation of constants argues that one should look for a particular solution which looks like the solution of the homogeneous problem with "varying constants".

Suppose

$$\tilde{x}_p(t) = \Phi(t)\,\tilde{c}(t)$$

with $\tilde{c}(t)$ now depending on t.

Can one find a solution of this form? Well try!

LHS. $\tilde{x}_p'(t) = \Phi'(t)\,\tilde{c}(t) + \Phi(t)\,\tilde{c}'(t)$

Use property of fundamental solution matrix Φ $\rightarrow$ $= A(t)\underbrace{\Phi(t)\,\tilde{c}(t)}_{\tilde{x}_p(t)} + \Phi(t)\,\tilde{c}'(t)$

RHS. $A(t)\,\tilde{x}_p(t) + \tilde{g}(t)$

$$= A(t)\,\Phi(t)\,\tilde{c}(t) + \tilde{g}(t)$$

Equality $\Rightarrow$ $\Phi(t)\,\tilde{c}'(t) = \tilde{g}(t)$ (After canceling!)

Thus

Since Φ is invertible

$$\tau'(t) = \Phi(t)^{-1} g(t)$$

and hence

$$\tau(t) = \int^{t} \Phi(s)^{-1} g(s)\,ds.$$

Plugging back into our guess we find that

$$\begin{aligned} x_p(t) &= \Phi(t)\,\tau(t) \\ &= \Phi(t) \int^{t} \Phi(s)^{-1} g(s)\,ds \end{aligned}$$

is a particular solution of (*).

(compare with solution of (***))

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Theorem 5.6 [Variation of constants formula I: general solutions]

Assume the entries in $A(t)$ and $\tilde{g}(t)$ are continuous functions on some interval $\alpha < t < \beta$. Suppose $\Phi(t)$ is a fundamental solution matrix associated to

$$\tilde{x}' = A(t)\,\tilde{x}.$$

Then the general solution of the non-homogeneous system

$$\tilde{x}' = A(t)\,\tilde{x} + \tilde{g}$$

is given by

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$$
$$= \Phi(t)\,\tilde{c} + \Phi(t)\int^t \Phi(s)^{-1}\tilde{g}(s)\,ds.$$

Here $\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is a vector of arbitrary constants.

In fact, by "shifting constants" any anti-derivative of $\Phi(s)^{-1}\tilde{g}(s)$ produces a solution.

When solving initial value problems it is often useful to "anchor" the indefinite integral.

i.e. consider $\int_{t_0}^{t} \Phi(s)^{-1} \underset{\sim}{g}(s)\,ds$

We conclude that the solution

$$\tilde{x}(t) = \Phi(t)\tilde{c} + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1} \underset{\sim}{g}(s)\,ds$$

of $(*)$ which satisfies the initial value

$\tilde{x}(t_0) = \tilde{x}_0$ is the one for which

$$\tilde{x}_0 = \tilde{x}(t_0) = \Phi(t_0)\tilde{c} + \cancel{\Phi(t_0)} \cdot 0$$

$$\Rightarrow \tilde{c} = \Phi(t_0)^{-1}\tilde{x}_0.$$

This observation is the content of the next result.

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Theorem 5.7 [Variation of Constants formula: Initial value problems].

Assume the entries in $A(t)$ and $\tilde{g}(t)$ are continuous functions on $\alpha < t < \beta$. Suppose $\Phi(t)$ is a fundamental solution matrix associated to

$$\tilde{x}' = A(t)\tilde{x}.$$

Then the solution of the I.V.P. with

$$\tilde{x}' = A(t)\tilde{x} + \tilde{g}, \quad \tilde{x}(t_0) = \tilde{x}_0$$

is given by

$$\tilde{x}(t) = \Phi(t)\Phi(t_0)^{-1}\tilde{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}\tilde{g}(s)\,ds.$$

Examples:

Last class we looked at 2 basic examples.

Example 1

$$x' = -2x + 2y$$
$$y' = 2x - 5y$$

which we re-wrote as

$$\tilde{x}' = A\tilde{x} \quad \text{where } \tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{and} \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

We found that

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}$$

is a fundamental solution matrix.

In this case

$$\tilde{x}_h(t) = \Phi(t)\tilde{c} = \begin{pmatrix} 2c_1e^{-t} + c_2e^{-6t} \\ c_1e^{-t} - 2c_2e^{-6t} \end{pmatrix}$$

is the general solution.

Example 2

We considered the 2nd order equation

$$x'' + x = 0.$$

By introducing $y = x'$, we re-write this as:

$$x' = y \qquad \text{i.e.} \quad \tilde{x}' = A\tilde{x}$$
$$y' = -x$$

with $\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We found that

$$\Phi(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

is a fundamental solution matrix and hence

$$\tilde{x}_h(t) = \Phi(t)\,\tilde{c}$$
$$= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= \begin{pmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{pmatrix}$$

is the general solution.

Example 1 continued

Find the general solution of

$$x' = -2x + 2y + e^{-2t}$$

$$y' = 2x - 5y$$

i.e.

$$\tilde{x}' = A\tilde{x} + \tilde{\xi}$$

where

$$\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

and $$\tilde{\xi} = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$$

We know from before that

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}$$

is a fundamental solution matrix for

$$\tilde{x}' = A\tilde{x}.$$

By what we have learned, the general solution is then

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$$

$$= \Phi(t)\,\tilde{c} + \Phi(t)\int^t \Phi(s)^{-1}\tilde{q}(s)\,ds$$

Note that

$$\det(\Phi(t)) = 2e^{-t}(-2e^{-6t}) - (e^{-t})(e^{-6t})$$

$$= -4e^{-7t} - e^{-7t} = -5e^{-7t}$$

$$\Rightarrow \quad \Phi(t)^{-1} = \frac{1}{\det(\Phi(t))}\begin{pmatrix} -2e^{-6t} & -e^{-6t} \\ -e^{-t} & 2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5}e^{t} & \frac{1}{5}e^{t} \\ \frac{1}{5}e^{6t} & -\frac{2}{5}e^{6t} \end{pmatrix}$$

$$\Rightarrow \quad \Phi(t)^{-1}\tilde{q}(t) = \begin{pmatrix} \frac{2}{5}e^{t} & \frac{1}{5}e^{t} \\ \frac{1}{5}e^{6t} & -\frac{2}{5}e^{6t} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5}e^{-t} \\ \frac{1}{5}e^{4t} \end{pmatrix}$$

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$$\int^t \Phi(s)^{-1} \tilde{g}(s)\,ds = \int^t \begin{pmatrix} \frac{2}{5}e^{-s} \\ \frac{1}{5}e^{4s} \end{pmatrix} ds$$

$$= \begin{pmatrix} -\frac{2}{5}e^{-t} \\ \frac{1}{20}e^{4t} \end{pmatrix}$$

$$\Rightarrow \quad \tilde{x}_p(t) = \Phi(t) \int^t \Phi(s)^{-1} \tilde{g}(s)\,ds$$

$$= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \begin{pmatrix} -\frac{2}{5}e^{-t} \\ \frac{1}{20}e^{4t} \end{pmatrix}$$

$$-\frac{4}{5} = \frac{-16}{20} \qquad = \begin{pmatrix} -\frac{4}{5}e^{-2t} + \frac{1}{20}e^{-2t} \\ -\frac{2}{5}e^{-2t} - \frac{1}{10}e^{-2t} \end{pmatrix}$$

$$-\frac{4}{5} + \frac{1}{20} = -\frac{15}{20}$$

$$= -\frac{3}{4} \qquad = \begin{pmatrix} -\frac{3}{4}e^{-2t} \\ -\frac{1}{2}e^{-2t} \end{pmatrix}$$

$$\Rightarrow \quad \tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$$

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Example 2 (cont.)

Find the general solution of

$$\tilde{x}' = A\tilde{x} + \tilde{g}$$

when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tilde{g} = \begin{pmatrix} 0 \\ \cos(2t) \end{pmatrix}$

Recall we know that

$$\Phi(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

is a fundamental solution matrix.

Since

$$\det(\Phi(t)) = \cos(t)^2 + \sin(t)^2 = \underline{1}$$

we have that

$$\Phi(t)^{-1} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

$$\Rightarrow \quad \Phi(t)^{-1}\tilde{g}(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 \\ \cos(2t) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(t)\cos(2t) \\ \cos(t)\cos(2t) \end{pmatrix}$$

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$$\int^t \Phi(s)^{-1}\vec{g}(s)\,ds = \begin{pmatrix} -\int^t \sin(s)\cos(2s)\,ds \\ +\int^t \cos(s)\cos(2s)\,ds \end{pmatrix}$$

$\boxed{\text{int table!}}$ (12) (11)

$$= \begin{pmatrix} -\frac{1}{3}\left(2\sin(t)\sin(2t) + \cos(t)\cos(2t)\right) \\ \frac{1}{3}\left(2\cos(t)\sin(2t) - \sin(t)\cos(2t)\right) \end{pmatrix}$$

$$\Rightarrow \Phi(t)\int^t \Phi(s)^{-1}\vec{g}(s)\,ds$$

$$= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} -\frac{1}{3}\left(2\sin(t)\sin(2t) + \cos(t)\cos(2t)\right) \\ \frac{1}{3}\left(2\cos(t)\sin(2t) - \sin(t)\cos(2t)\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}\left(2\sin(t)\cos(t)\sin(2t) + \cos^2(t)\cos(2t)\right) \\ \quad +\frac{1}{3}\left(2\sin(t)\cos(t)\sin(2t) - \sin^2(t)\cos(2t)\right) \\ +\frac{1}{3}\left(2\sin^2(t)\sin(2t) + \sin(t)\cos(t)\cos(2t)\right) \\ \quad +\frac{1}{3}\left(2\cos(t)^2\sin(2t) - \sin(t)\cos(t)\cos(2t)\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}\cos(2t) \\ +\frac{2}{3}\sin(2t) \end{pmatrix}$$

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Thus the general solution is:

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$$

$$= \Phi(t)\tilde{c} + \Phi(t)\int^t \Phi(s)^{-1}\tilde{g}(s)\,ds$$

$$= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}\cos(2t) \\ \frac{2}{3}\sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} c_1\cos(t) + c_2\sin(t) - \frac{1}{3}\cos(2t) \\ -c_1\sin(t) + c_2\cos(t) + \frac{2}{3}\sin(2t) \end{pmatrix}$$

$\Rightarrow$ $x(t) = c_1\cos(t) + c_2\sin(t) - \frac{1}{3}\cos(2t)$

is the general solution of

$x'' + x = \cos(2t)$.