Analysis of Ordinary Differential Equations

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Preface

This textbook is for the sole use of students enrolled in the course Mathematics 355 at the University of Arizona. It is available free of charge from the course D2L webpage. You can read this e-version on a tablet or computer, or you may print one paper copy for you own personal use. Do not disseminate it without the author's written permission.

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Preliminaries

Mathematical applications typically involve one or more equations to be solved for unknown quantities. Often applications involve rates of change, and therefore lead to equations containing derivatives. Such equations are called *differential* equations.

A student's first encounter with differential equations is usually in a calculus course where anti-derivatives (or indefinite integrals) are studied. For example, consider the problem of finding the anti-derivative of t^2. This problem can be formulated as follows: find a function $x = x(t)$ whose derivative is t^2, or in other words find a function $x = x(t)$ that satisfies the equation

$$x' = t^2. \tag{0.1}$$

(Here we have used the notation x' for the derivative of x with respect to t. We will also occasionally use the notation dx/dt.) Equation (0.1) is a differential equation for the unknown function $x = x(t)$. Notice what it means to "solve" this equation: find a function $x = x(t)$ that, when substituted into both sides of the equation, makes the left hand side *identically* equal to the right hand side. That is to say, a solution is a function which upon substitution into the equation reduces the equation to a mathematical identity in t. Also notice it is not accurate to speak of *the* solution of this differential equation. This is because it has many solutions, namely $x(t) = t^3/3 + c$ where c is any constant (the so-called "constant of integration").

It is not always as easy to find formulas for solutions of a differential equation as it is for the equation (0.1). For example, consider the differential equation

$$x' = x. \tag{0.2}$$

This equation is fundamentally different from (0.1) because the unknown function x appears on the right hand side. This equation cannot be solved by an anti-differentiation of the right hand side, because the right hand side is not a known function of t. Later we will learn how to solve this equation, but for now notice that $x(t) = e^t$ is a solution, i.e., a substitution of e^t for x into the left and the right hand sides of the equation yields the same result (namely e^t). Similarly, $x(t) = ce^t$ is a solution of this equation for *any* constant c (including $c = 0$). Notice, however, that $x(t) = e^t + c$ is *not* a solution (unless $c = 0$). To see this, we calculate $x'(t) = e^t$ and note that it is not equal to $x(t) = e^t + c$ (unless $c = 0$). This shows that constants of integration do not always appear additively in formulas for solutions of differential equations.

As another example consider the differential equation

$$x' = x^2. \tag{0.3}$$

The function $x(t) = 1/(1-t)$ is a solution of this equation, so long as $t \neq 1$, because the derivative $x' = 1/(1-t)^2$ is identical to x^2 for $t \neq 1$. We say this function is a solution on the interval $-\infty < t < 1$ or on the interval $1 < t < +\infty$ (or on any interval not containing $t = 1$). Similarly, for a constant c, the function $x(t) = 1/(c-t)$ is a solution on any interval that does not contain $t = c$. Notice each solution obtained by assigning a numerical value to c has a different singular point $t = c$ and hence is associated with a different interval of existence. (Incidentally, the constant function $x \equiv 0$ is also a solution which is not included in the formula $x(t) = 1/(c-t)$.)

A solution of a differential equation is associated with an *interval of existence*. The solutions $x(t) = 1/(c-t)$ of equation (0.3) show there is not necessarily a common interval of existence for all solutions of a differential equation. This example also illustrates that the differential equation itself might give little or no clue about the intervals of existence of its solutions.

For differential equations (0.1), (0.2), and (0.3) it is possible, as we have seen, to write down formulas for solutions. For other equations, it is not possible to calculate solution formulas. In the latter case, we must use other methods to study equations and their solutions. In this book we will study some types of equations for which we can derive solution formulas, but we will also study many methods of analysis that do not require solution formulas. These methods are of particular importance since it is not possible to calculate solution formulas for the differential equations that arise in many, if not most, scientific and engineering applications.

The equations (0.1), (0.2), and (0.3) are examples of a general class of ordinary differential equations of the form

$$x' = f(t, x).$$

Here all terms in the equation not involving the derivative have been placed on the right hand side. In general both the independent variable t and the dependent variable x can appear on the right hand side. Letters or symbols representing unspecified numerical constants called *coefficients* or *parameters* might also appear. Here are some further examples:

$$\begin{aligned}
&x' = x^2 + t^2 \\
&x' = -2x \\
&x' = px, \quad \text{where } p \text{ is a constant} \\
&x' = r\left(1 - \frac{x}{K}\right)x, \quad \text{where } r > 0,\ K > 0 \text{ are constants.}
\end{aligned}$$

It is important to recognize those letters and symbols that represent independent variables, those that represent dependent variables, and those that represent coefficients or parameters.

The independent variable is, of course, the variable with respect to which the derivative is being taken. In the above equations we use the letter t for the independent variable; *this will be done throughout the book*. This choice is motivated by the many applications in which the independent variable represents time. (Other letters can, of course, be used.)

On the other hand, throughout the book we use a variety of letters for the *dependent* variable (sometimes referred to as the *state variable*). In applications, a letter suggestive of the meaning of the variable in that application is usually chosen. For example, we will encounter differential equations involving symbols such as x', y', v', N', and P' for the

derivatives of the dependent variables x, y, v, N, and P with respect to t. If it is necessary to emphasize the role of the independent variable t we sometimes write derivatives as

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dN}{dt}, \frac{dP}{dt}.$$

Applications often involve several differential equations for several unknown functions, i.e. a *system of differential equations*. Some examples are

$$\begin{aligned} x' &= y \\ y' &= -\sin x \end{aligned}$$

$$\begin{aligned} x' &= -r_1 x - r_2 y \\ y' &= r_1 x - (r_1 + r_2)\, y \end{aligned}$$

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m} x - \frac{c}{m} y \end{aligned}$$

$$\begin{aligned} x' &= r\left(1 - \frac{x}{K}\right) x - cxy \\ y' &= -dy + xy \end{aligned}$$

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha\left(x^2 - 1\right) y\ . \end{aligned}$$

In each of these examples there are two differential equations for two unknown functions x and y. All other letters represent coefficients (or parameters).

A solution of a system of two equations is a *pair* of functions $x = x(t)$, $y = y(t)$. For example, the pair $x(t) = 2e^{2t}$, $y(t) = -e^{2t}$ is a solution of the system

$$\begin{aligned} x' &= 5x + 6y \\ y' &= x + 4y. \end{aligned}$$

To see this, we note that $x' = 4e^{2t}$ is identical to

$$5x + 6y = 5\left(2e^{2t}\right) + 6\left(-e^{2t}\right) = 4e^{2t}$$

(i.e., the *first* equation is satisfied for all t) and *also* that $y' = -2e^{2t}$ is identical to

$$x + 4y = \left(2e^{2t}\right) + 4\left(-e^{2t}\right) = -2e^{2t}$$

(i.e., the *second* equation is also satisfied for all t). The reader can check that $x(t) = 3e^{7t}$, $y(t) = e^{7t}$ is another solution pair of this same system.

Applications also arise in which higher order derivatives appear in the equation. Here are some examples of higher order differential equations:

$$x'' + x = 0$$

$$mx'' + cx' + kx = a\sin\beta t$$

$$x''' + 3x'' + 3x' + 2x = 0$$

$$\begin{aligned} m_1x'' + (k_1 + k_2)x - k_2y &= 0 \\ m_2y'' - k_2x + k_2y &= 0. \end{aligned}$$

The *order* of a differential equation is that of the highest order derivative appearing in the equation. Thus, the equation $x' = x$ is a *first order* equation. The first two equations above are *second order* and the third equation is *third order*. The last pair of equations constitute a *second order system* of equations.

Solutions of higher order equations must reduce the equation(s) to identities upon substitution. For example, $x(t) = \sin t$ is a solution of the second order equation $x'' + x = 0$ for all t (as is $x(t) = \cos t$). The exponential function $x(t) = e^{-2t}$ is a solution of the third order equation $x''' + 3x'' + 3x' + 2x = 0$ for all t.

Any higher order equation (or system of higher order equations) can be associated with an equivalent system of first order equations. The following example illustrates the most common way to convert a higher order equation to an equivalent first order system. The function $x(t) = \sin t$ is a solution (for all t) of the second order equation

$$x'' + x = 0. \tag{0.4}$$

Define y to be the derivative of x, i.e., $y = x'$. Then $y(t) = \cos t$ and the pair $x(t) = \sin t$, $y(t) = \cos t$ solves the first order system

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \tag{0.5}$$

This shows how a particular solution of the second order equation (0.4) can be used to construct a solution of the first order system (0.5).

More generally, suppose $x = x(t)$ is *any* solution of the second order equation (0.4), i.e., $x''(t) + x(t) = 0$. Define $y = x'(t)$. The calculations

$$x'(t) = y(t)$$

$$y'(t) = x''(t) = -x(t)$$

show the pair $x(t)$, $x'(t)$ solves the system (0.5). This shows that *any* solution of the second order equation (0.4) gives rise to a solution pair for the first order system (0.5). Is the converse true? Can a solution of the first order system (0.5) be used to obtain a solution of the second order equation (0.4)? If so, then we could say that the second order equation (0.4) is "equivalent" to the first order system (0.5) in the sense that solving one is the same as solving the other.

Suppose $x = x(t)$, $y = y(t)$ is a solution pair of the first order system (0.5). Then

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{0.6}$$

We need to show how we can obtain a solution of the second order equation (0.4) from the solution pair of the system. The way to do this is simply to choose the first component x of the solution pair. We can show that the first component $x = x(t)$ satisfies the second order equation by differentiating both sides of the first equation in the system (0.6), to obtain $x''(t) = y'(t)$, and then use the second equation in the system to obtain $x''(t) = -\,x(t)$, or in other words $x'' + x = 0$.

The procedure we used to derive the system (0.5) equivalent to the equation (0.4) is not peculiar to that second order equation. For example, by the same method, we can show that the second order equation

$$x'' + \sin x = 0$$

is equivalent to the first order system

$$\begin{aligned} x' &= y \\ y' &= -\sin x. \end{aligned}$$

In general, we can show (by a similar procedure) that *any* second order differential equation of the general form

$$x'' = f(t, x, x')$$

is equivalent to the first order system

$$\begin{aligned} x' &= y \\ y' &= f(t, x, y). \end{aligned}$$

An extension of the method also applies to equations of order higher than two. For example, we can obtain an equivalent first order system for the third order equation

$$x''' + 3x'' + 3x' + 2x = 0$$

by defining *two* new dependent variables

$$y = x', \quad z = x''.$$

As above, we can show solutions of this equation give rise to solutions of the system

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -2x - 3y - 3z \end{aligned}$$

and vice versa.

A further extension of the method can be used for higher order systems as well. For example, consider the second order system

$$\begin{aligned} x'' + 2x - z &= 0 \\ 2z'' - x' + z &= 0 \end{aligned}$$

for two unknowns x and z. We apply the procedure twice, once on each equation, by defining two new dependent variables

$$y = x', \quad w = z'$$

and obtaining the equivalent first order system of four equations

$$\begin{aligned} x' &= y \\ y' &= -2x + z \\ z' &= w \\ w' &= \tfrac{1}{2}y - \tfrac{1}{2}z. \end{aligned}$$

The ability to convert higher order equations to a first order system is required by many (if not most) computer programs available for the study of differential equations.

One way to classify differential equations is by their order. Another way to classify equations is based on the notion of "linearity". *A differential equation is linear if the dependent variable and all of its derivatives appear linearly.* Thus, in a linear first order equation, both x and x' appear linearly. This means

$$\begin{aligned} x' &= 3x + 1 \\ 2x' - x &= 2 + \sin t \\ x' &= tx + a \\ e^t x' &= \frac{x}{t} + \ln t \end{aligned}$$

are all linear (first order) differential equations. Note that the independent variable plays no role in the definition of linearity. For example, the second equation is linear even though the independent variable t appears in a nonlinear way (in the $\sin t$ term). We can write each of these equations in the form

$$x' = p(t)x + q(t)$$

for appropriate coefficients $p(t)$ and $q(t)$. By definition, an equation is linear if it has this form (or can be rewritten in this form).

The equations

$$\begin{aligned} x' &= x^2 - 1 \\ xx' &= x + t \\ (x')^2 &= tx - 4 \\ x' &= r\left(1 - \frac{x}{K}\right)x \end{aligned}$$

are nonlinear. The first and fourth equations are nonlinear because of the term x^2. The second equation is nonlinear because of the term xx' and the third equation is nonlinear because of the term $(x')^2$ (not because of the term tx).

A second or higher order equation is linear if the dependent variable and all of its derivatives appear linearly in the equation. The second order equations

$$\begin{aligned} x'' + x &= 0 \\ x'' + x' + x &= \sin t \\ x'' + (\sin t)x &= 0 \end{aligned}$$

are linear because x, x' and x'' appear linearly. The equations

$$x'' + \alpha(1-x)x' + x = 0$$
$$x'' + \sin x = 0$$

are nonlinear (the first because of the term xx' and the second because of the term $\sin x$).

Systems of equations are linear if (and only if) *all* of the equations are linear in *all* of the *dependent* variables and their derivatives. Thus,

$$x' = y$$
$$y' = -x$$

$$x' = -rx + ry$$
$$y' = rx - 2ry$$

are linear systems and

$$x' = \left(1 - x - \tfrac{1}{2}y\right) x$$
$$y' = \left(1 - \tfrac{1}{2}y - x\right) y$$

$$x' = (x_{in} - x)\, d - \tfrac{1}{\gamma}\tfrac{mx}{a+x} y$$
$$y' = \left(\tfrac{mx}{a+x} - d\right) y$$

are nonlinear systems (because of the terms x^2, xy, and y^2 in the first system and the term $mxy\,(a+x)$ in the second).

0.1 Exercises

What are the orders of the following equations? Explain your answers.

Exercise 0.1 $t^2x' + x^3 = 0$

Exercise 0.2 $3x' - 2x^2 = 0$

Exercise 0.3 $e^t\,(x')^2 + x^3 = 0$

Exercise 0.4 $3\,(x'')^3 - 2x^5\,(x')^2 = 0$

Exercise 0.5 $x'x^3x'' - t^7x^{1/2} = 0$

Exercise 0.6 $x' + t^2x^2 + x''' = 2$

Exercise 0.7 $x' + t^{1/2}x = \ln t$

Exercise 0.8 $x'(x'')^2 - 5t^{1/2}x^3 = 2$

Exercise 0.9 $tx = e^x + (x')^2$

Exercise 0.10 $x'' + a\sin x = 0$

Exercise 0.11 $t^2x' + x^3 = t\cos t$

Exercise 0.12 $x' = p(t)x + q(t)$

Exercise 0.13 $2xx' + x''' + (x'')^3 - x^4 = 0$

Exercise 0.14 $(x''')^2 + (x'')^5 + 3(x')^7 - \sin x = 0$

Which of the following are solutions and which are not solutions of the equation $x' + 3x = 0$? Explain your answers.

Exercise 0.15 e^{-3t}

Exercise 0.16 e^{3t}

Exercise 0.17 $-e^{-3t}$

Exercise 0.18 $3e^{-t}$

Which of the following are solutions and which are not solutions of the equation $x' - 2tx = 0$? Explain your answers.

Exercise 0.19 e^{2t}

Exercise 0.20 $2e^{-2t}$

Exercise 0.21 $-7e^{t^2}$

Exercise 0.22 $1 + e^{t^2}$

Which of the functions below are solutions and which are not solutions of the equation $2x' + 3x^{5/3} = 0$? Explain your answers.

Exercise 0.23 $t^{-3/2}$

Exercise 0.24 $-t$

Exercise 0.25 $(t-1)^{-3/2}$

Exercise 0.26 $(1-t)^{-3/2}$

Exercise 0.27 $t^{3/2}$

Exercise 0.28 $-(t+3)^{-3/2}$

Exercise 0.29 $(t-2)^{-2/3}$

Exercise 0.30 $(t-c)^{-3/2}$ *(where c is any constant)*

Which of the functions below are solutions and which are not solutions of the equation $x'' - 5x' + 6x = 0$? Explain your answers.

Exercise 0.31 e^{-2t}

Exercise 0.32 e^{2t}

Exercise 0.33 e^{3t}

Exercise 0.34 e^{-3t}

Exercise 0.35 $5e^{2t}$

Exercise 0.36 $-7e^{3t}$

Exercise 0.37 $e^{2t} + e^{3t}$

Exercise 0.38 $c_1e^{2t} + c_2e^{3t}$ *for constants* c_1 *and* c_2

In the Exercises 0.39-0.43 determine which of the functions are solutions of the given differential equation and which are not.

Exercise 0.39 *For the equation* $x' + 5x = 0$:
(a) $x = e^{-5t}$ *(b)* $x = 3e^{-5t}$ *(c)* $x = 5e^{-3t}$

Exercise 0.40 *For the equation* $x' = 2x$:
(a) $x = e^{3t}$ *(b)* $x = -3e^{2t}$ *(c)* $x = e^{2t}$

Exercise 0.41 *For the equation* $x' + x^2 = 0$:
(a) $x = t^{-1}$ *(b)* $x = 2t^{-1}$ *(c)* $x = (t-2)^{-1}$

Exercise 0.42 *For the equation* $x' = x + e^t$:
(a) $x = e^t$ *(b)* $x = te^t$ *(c)* $x = e^t + te^t$

Exercise 0.43 *For the equation* $tx'' + x' = 0$.
(a) $x = \ln t$ *(b)* $x = 1$ *(c)* $x = t$

Which of the following are solutions and which are not solutions of the equation $x''' - 4x'' - 4x' + 16x = 0$? Explain your answers.

Exercise 0.44 $x = e^{4t}$

Exercise 0.45 $x = -2e^{4t}$

Exercise 0.46 $x = ce^{4t}$ *where* c *is any constant*

Exercise 0.47 $x = e^{2t}$

Exercise 0.48 $x = e^{-2t}/2$

Exercise 0.49 $x = c_1e^{4t} + c_2e^{2t} + c_3e^{-2t}$ *for any constants* c_1, c_2, c_3

Exercise 0.50 $x = e^{4t}e^{2t}$

Which of the following are solutions and which are not solutions of the equation $x'' + x' - 2x = 0$? Explain your answers.

Exercise 0.51 $x = e^t$

Exercise 0.52 $x = e^{-2t}$

Exercise 0.53 $x = e^te^{-2t}$

Exercise 0.54 $x = e^t + 2e^{-2t}$

Exercise 0.55 *Do* $x = e^{4t}$ *and* $y = -2e^{4t}$ *form a solution pair for the two equations* $x' = 2x - y$, $y' = -6x + y$*?*

Exercise 0.56 *Do* $x = e^{3t}\sin 5t$ *and* $y = e^{3t}\cos 5t$ *form a solution pair for the equations* $x' = 3x + 5y$, $y' = -5x + 3y$?

Which of the following are solution pairs of the system below? Which are not solution pairs? Explain your answers.

$$x' = 4x + 3y$$
$$y' = -2x - y.$$

Exercise 0.57 $x = e^t$, $y = -e^t$

Exercise 0.58 $x = -e^t$, $y = e^t$

Exercise 0.59 $x = e^t$, $y = e^t$

Exercise 0.60 $x = -e^t$, $y = -e^t$

Exercise 0.61 $x = 3e^{2t}$, $y = -2e^{2t}$

Exercise 0.62 $x = e^{2t}$, $y = -e^{2t}$

Exercise 0.63 $x = e^t + 3e^{2t}$, $y = -e^t - 2e^{2t}$

Exercise 0.64 $x = -2e^t + 6e^{2t}$, $y = 2e^t - 4e^{2t}$

Exercise 0.65 $x = c_1e^t + 3c_2e^{2t}$, $y = -c_1e^t - 2c_2e^{2t}$ *for constants* c_1 *and* c_2

Exercise 0.66 *For each function that is a solution in Exercise 0.15-0.18 identify the interval on which it is a solution.*

Exercise 0.67 *For each function that is a solution in Exercise 0.23-0.30 identify the interval on which it is a solution.*

Convert the equations below to equivalent first order systems.

Exercise 0.68 $x'' + x' - 3x = 0$

Exercise 0.69 $x'' - 6x' + 4x = 0$

Exercise 0.70 $3x'' - 6xx' + 12x^2 = 1$

Exercise 0.71 $5x'' + 10x'x = 5e^t$

Exercise 0.72 $2x''' - 6x'' + 4x' + x = -3$

Exercise 0.73 $x''' + 2x'' - x' + x = 1$

Exercise 0.74 $x'' + 2x' + 4x = \cos t$

Exercise 0.75 $2x'' + 3x' + 9x = 0$

Exercise 0.76 $t^2x'' + (x')^2 + \cos x = 0$

Exercise 0.77 $xx'' + (x')^2 + x^{1/2} = e^t$

Exercise 0.78 $x'' = -2x' - x + z$, $z'' = -z' + 2x - z$

Exercise 0.79 $x''' + x'' - 2x' + 7x = t$

Exercise 0.80 *Convert the second order system*

$$\begin{aligned} 2x'' - x' + 2z' + 4x - 8z &= 0 \\ z'' + 2x' - z' - x + 3z &= \sin t \end{aligned}$$

to an equivalent first order system.

Exercise 0.81 *Convert the second order system*

$$\begin{aligned} x'' - 5x' - 6z' + x - z &= 0 \\ 3z'' - 6x' - z' + 12x + 3z &= 21e^{-3t} \end{aligned}$$

to an equivalent first order system.

Which of the following first order equations are linear? If an equation is nonlinear, explain why.

Exercise 0.82 $x' = 2x + 1$

Exercise 0.83 $3x' + 4x = 1/2$

Exercise 0.84 $x' = tx^2 - 1$

Exercise 0.85 $x' = t^2 x - 1$

Exercise 0.86 $t^2 x' = x$

Exercise 0.87 $x' = x \sin t$

Exercise 0.88 $x' = t \sin x$

Exercise 0.89 $x' = e^x$

Which of the following second order equations are linear? If an equation is nonlinear, explain why. (a is a constant.)

Exercise 0.90 $x'' + xx' + x = 0$

Exercise 0.91 $x'' + tx' + x = 0$

Exercise 0.92 $t^2 x'' + tx' + x = 1$

Exercise 0.93 $x^2 x'' + tx' + x = 1$

Exercise 0.94 $x'' + a(1 - x)x = 0$

Exercise 0.95 $x'' + a(1 - t)x = t$

Exercise 0.96 $x'' + e^{-x} t = \sin t$

Exercise 0.97 $x'' + e^{-t} x = \sin t$

Which of the following systems are linear? If a system is nonlinear, explain why.

Exercise 0.98 $\begin{cases} x' = x + y \\ y' = x - y \end{cases}$

Exercise 0.99 $\begin{cases} x' = (1 - x)x - xy \\ y' = -y + xy \end{cases}$

Exercise 0.100 $\begin{cases} x' = x - y \\ y' = xy \end{cases}$

Exercise 0.101 $\begin{cases} x' = ax + by \\ y' = cx + dy \\ \textit{where } a, b, c \textit{ are constants.} \end{cases}$

Which of the following equations (or systems of equations) are linear?

Exercise 0.102 $x' = a(r - x)$ *where* a *and* r *are constants*

Exercise 0.103 $x' = a(r-x)$ *where* $a = a(x)$ *is a decreasing function of* x

Exercise 0.104 $x'' + f(x)x = 0$ *where* $f = f(x)$ *is a function of* x *satisfying* $df(x)/dx > 0$ *(for all* x*).*

Exercise 0.105 $mx'' + cx' + kx = a\sin\beta t$ *where* m, c, k, a *and* β *are positive constants*

Exercise 0.106 $mx'' + c\sin x = 0$ *where* m *and* c *are positive constants*

Exercise 0.107 $\begin{cases} x' + y' = x + y \\ x' - y' = 2x + y \end{cases}$

Exercise 0.108 $\begin{cases} x' = \ln(ty) \\ y' = x \end{cases}$

Exercise 0.109 $\begin{cases} x' = y\sin t \\ y'x' = x + y \end{cases}$

Exercise 0.110 $\begin{cases} x' - 2x = y + \cos t \\ y - e^{2t}x = y' - 1 \end{cases}$

Determine whether the following equations can be rewritten as linear equations or not.

Exercise 0.111 $x' = \ln(2^x)$

Exercise 0.112 $x' = \begin{cases} (x^2-1)/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

Chapter 1

First Order Equations

1.1 Introduction.

In this Chapter we consider first order differential equations of the form

$$x' = f(t, x).$$

A fundamental question concerns the existence of solutions to such an equation. Under what conditions (i.e., for what kind of expressions $f(t, x)$) can we be assured that solutions exist? Another question concerns the number of solutions. We know from calculus that integration problems have infinitely many solutions and, therefore, we anticipate that this is also true for a first order differential equation. On the other hand, in applications there are often requirements (in addition to the differential equation) that serve to select one of solutions. For a first order differential equation the most common requirement is that the solution $x(t)$ equal a specified value x_0 for a specified value of t, that is to say, that $x(t_0) = x_0$ for a given t_0 and x_0. A fundamental mathematical question is whether the resulting *initial value problem*

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a solution. In this chapter we learn conditions which, when placed on $f(t, x)$, guarantee that this initial value problem has one and only one solution (i.e., has a "unique" solution).

As pointed out in the Chapter , we will sometimes in this course use different letters for the unknown (or state variable) x. We will, however, consistently use the letter t for the independent variable.

For specialized equations (i.e., for $f(t, x)$ with special properties) one can calculate formulas for solutions. We study some examples in Chapters 2 and 3. However, for most differential equations it is not possible to find solution formulas. For that reason, this course will not emphasis those equations for which methods for calculating solution formulas are available. One exception, however, will be the case of so-called *linear* differential equations (by which is meant $f(t, x)$ is a linear function of x), which we study in some detail in Chapters 2. Instead, the course will focus on methods of analysis that reveal various properties of solutions (e.g., monotonicity, asymptotic behavior as $t \to +\infty$, and others) or that provide useful approximations to solutions. We will look at many different kinds of solution approximations, including graphic and numeric approximations (for which a computer will

be an important tool), analytic formulas that approximate solutions, and methods based on approximating the differential equation itself.

Necessary for any methods of analysis is, of course, the knowledge that the differential equation and its initial value problems have solutions.

1.2 The Fundamental Existence and Uniqueness Theorem

We begin with a definition.

Definition 1.1 *A **solution** of a differential equation $x' = f(t,x)$ on an interval $a < t < b$ is a function $x = x(t)$ that is differentiable and that reduces the equation to an identity on the interval, i.e.,*

$$x'(t) \equiv f(t, x(t))$$

for all values of t from the interval.[1] *The interval $a < t < b$ may be the whole real line, in which case we say the function is a solution for all t.*

For the differential equation

$$x' = t^2 \tag{1.1}$$

we have $f(t,x) = t^2$. The function $x(t) = t^3/3 + 1$ is a solution of this equation for all t because $x'(t) = t^2$ equals $f(t, x(t)) = t^2$ for all t.

More generally, the unknown x might appear in $f(t,x)$. For example, for the equation $x' = tx$ we have $f(t,x) = tx$. The function $x(t) = e^{t^2/2}$ is a solution of this equation for all t because $x'(t) = te^{t^2/2}$ and $f(t, x(t)) = tx(t) = te^{t^2/2}$ are identical for all t.

From calculus we know the differential equation (1.1) has infinitely many solutions and the set of all solutions is given by the formula

$$x\,(t) = \frac{1}{3}t^3 + c \tag{1.2}$$

where c is an arbitrary constant. This is an example of a "general solution" of a differential equation.

Definition 1.2 *The collection or set of all solutions of the differential equation $x' = f(t,x)$ is called the **general solution** (or the **solution set**).*

An initial condition $x(t_0) = x_0$ selects a particular solution from the general solution. For example, suppose we require that a solution of the equation (1.1) satisfy the initial condition $x(0) = 1$. From the general solution (1.2) we obtain $x(0) = c$ and therefore this initial condition is satisfied by choosing (and only by choosing) $c = 1$. That is to say, there is a unique solution of the initial value problem

$$x' = t^2, \quad x(0) = 1$$

[1] As a mathematical function $f(t,x)$ has a domain of t and x values. It is assumed, in this definition, that all values of t taken from the interval $a < t < b$ and the corresponding values of $x(t)$ (i.e., the range of the function $x(t)$) lie in the domain of f. Otherwise $f(t, x(t))$ makes no sense.

namely, $x(t) = t^3/3 + 1$.

In an initial value problem the "initial" time need not be $t_0 = 0$. For example, we can use the general solution (1.2) to find the unique solution of the initial value problem

$$x' = t^2, \quad x(2) = -1.$$

From the general solution (1.2) we obtain

$$x(2) = \frac{8}{3} + c.$$

To satisfy the initial condition $x(2) = -1$ we solve

$$\frac{8}{3} + c = -1$$

for $c = -11/3$ and thereby obtain the solution formula

$$x(t) = \frac{1}{3}t^3 - \frac{11}{3}.$$

In fact, we can solve the general initial value problem

$$x' = t^2, \quad x(t_0) = x_0.$$

using the general solution (1.2) by setting

$$x(t_0) = \frac{1}{3}t_0^3 + c$$

equal to the desired initial value x_0 and solving for

$$c = x_0 - \frac{1}{3}t_0^3.$$

This results in the unique solution

$$x(t) = \frac{1}{3}t^3 + x_0 - \frac{1}{3}t_0^3.$$

Example 1.1 *A differential equation for the velocity $v = v(t)$ of a falling object subject to the force of gravity and air resistance is*

$$v' = g - k_0 v \tag{1.3}$$

where in which the unknown dependent (state) variable is $v = v(t)$ and

$$f(t, v) = g - k_0 v.$$

Here the coefficients (or parameters) g and k_0 appearing in the equation are constants (the acceleration due to gravity and the per unit mass coefficient of friction respectively). The differentiable function

$$v(t) = e^{-k_o t} + \frac{g}{k_0}$$

is an example of a solution for all t. To prove this one substitutes this alleged solution into both sides of the differential equation, performs the indicated operations, and see whether or not the results on the right and left are identical. Specifically,

$$v'(t) = -k_0 e^{-k_0 t}$$

and

$$f(t, v(t)) = g - k_0 \left(e^{-k_o t} + g/k_0\right) = -k_0 e^{-k_0 t}$$

are indeed identical for all t.

In a similar fashion, the reader can verify that the function

$$v(t) = ce^{-k_o t} + \frac{g}{k_0} \tag{1.4}$$

is solution for all t for any constant c. Specifically,

$$v'(t) = -k_0 ce^{-k_0 t}$$

and

$$f(t, v(t)) = g - k_0 \left(ce^{-k_o t} + \frac{g}{k_0}\right) = -k_0 ce^{-k_0 t}$$

are identical for all t.

Verifying that a function, even one with an arbitrary constant c as in the Example above, does not prove that the formula is the general solution. The problem is: how does one know whether or not there are other solutions, ones not represented by the formula? This requires further analysis. (In Chapter 2 it is shown that the formula (1.4) is in fact the general solution of the differential equation (1.3).)

Example 1.2 *The solution of the differential equation (1.3) that satisfies the initial condition $v(0) = 0$ (which describes an object that is initially dropped) is found from the solution formula (1.4) by solving*

$$x(0) = c + \frac{g}{k_0} = 0$$

for

$$c = -\frac{g}{k_0}.$$

This yields the following solution formula

$$x(t) = -\frac{g}{k_0} e^{-k_o t} + \frac{g}{k_0}.$$

Verifying that a solution formula satisfies an initial value problem does not, in and of itself, does not prove that it is the only solution of the initial value problem. An important question is: when is the solution of an initial value problem unique?

In applications solutions are not always defined for all t. Here is an example.

Example 1.3 *An equation describing the growth of the world's human population $x(t)$ in billions as a function of time t (in years) is*

$$x' = kx^2 \tag{1.5}$$

where the coefficient $k > 0$ is a positive constant estimated from data. The function

$$x(t) = \frac{1}{1-kt} \tag{1.6}$$

is defined and differentiable on both of the intervals $t < 1/k$ and $t > 1/k$. (The denominator vanishes at $t = 1/pk$.) To prove that this formula is a solution (on either one of these intervals) we substitute it into both sides of the differential equation, perform the indicated operations, and check that the right side is identical to the left side on the interval. Specifically,

$$x'(t) = k\frac{1}{(1-kt)^2}$$

and

$$f(t, x(t)) = k\left(\frac{1}{1-kt}\right)^2 = k\frac{1}{(1-kt)^2}$$

are identically equal for all $t < 1/k$ and for all $t > 1/k$.

The solution (1.6) satisfies the initial value problem

$$x' = kx^2, \quad x(0) = 1$$

on the interval $t < 1/k$. (The reason it does not satisfy this initial value problem on the interval $t > 1/k$ is that the initial value $t_0 = 0$ for t does not belong to this interval.) Is it the only solution? And do other initial value problems $x(t_0) = x_0$ have solutions?

An initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{1.7}$$

does not necessarily have a solution and, when it does, it does not always have just one solution (i.e. a "unique" solution). Examples appear below. It all depends on $f(t, x)$ and on t_0 and x_0. Only if $f(t, x)$ has some special properties will (1.7) have a unique solution.

Theorem 1.1 provides some easily usable criteria for the existence and uniqueness of a solution to the initial value problem (1.7). One of the criteria involves the derivative of $f(t, x)$ with respect to x, which we denote by

$$\frac{df(t, x)}{dx}.$$

Those readers who have had a multi-variable course know this as the "partial derivative" of f with respect to x. In multi-variable calculus it is denoted instead by

$$\frac{\partial f(t, x)}{\partial x}.$$

Theorem 1.1 ***(Fundamental Existence and Uniqueness Theorem) Suppose***
(a) $f(t,x)$ ***and*** $df(t,x)/dx$ ***are continuous in*** t ***and continuous in*** x ***on intervals*** $a<t<b$ ***and*** $c<x<d$***,***
(b) the initial conditions lie in these intervals: $a<t_0<b, \quad c<x_0<d.$
Then the initial value problem (1.7) has a solution on an interval $\alpha<t<\beta$ ***containing*** t_0***. Moreover, there is no other solution of the initial value problem on this interval.***

Remark 1. This theorem is called a "local" existence and uniqueness theorem because it guarantees existence and uniqueness only on *some* interval around the initial time t_0. The theorem gives no information about the size of this interval, which might be infinite or finite (even quite small).

Remark 2. Sometimes we refer to the criteria in Theorem 1.1 this way: $f(t,x)$ is "*continuously differentiable in* x at $t=t_0$ and $x=x_0$".

Remark 3. To apply Theorem 1.1 one can, of course, compute the derivative $df(t,x)/dx$ in order to investigate its continuity at the initial point. One can, however, often avoid this calculation by relying on known theorems from calculus about continuous and differentiable functions. For example, recall that polynomials in x are continuous at all values of x and polynomials in t are continuous at all values of t. Moreover, derivatives of polynomials are polynomials. So, any differential equation (1.7) in which $f(t,x)$ is a polynomial in t and x (for example, if $f(t,x)=x^2+t^2$) will satisfy the criteria of Theorem 1.1 for any initial value problem. One can also make use of theorems from calculus about products, quotients and composites of continuously differentiable functions. For example, in this way we see that

$$f(t,x)=\frac{1}{1+x^2}\sin\left(t^2x^2\right)$$

satisfies the criteria of Theorem 1.1 for any initial value problem. (This is because sine is continuously differentiable for all values of is arguments and because of the chain and quotient results.)

Remark 4. Do not make the mistake of deducing something from Theorem 1.1 when the criteria of continuously differentiability of $f(t,x)$ fail to hold for an initial value problem. *When the assumptions of a theorem do not hold, all one can say is that the theorem is not applicable and nothing at all can be deduced from it.*

Remark 5. If $f(t,x)$ does not depend on t, then the continuity requirements with respect to t that are needed to apply Theorem 1.1 are satisfied.

Example 1.4 *For the initial value problem*

$$x'=tx, \quad x(0)=\frac{1}{2}.$$

the function

$$f(t,x)=tx$$

and its derivative

$$\frac{df(t,x)}{dx} = t,$$

are continuous for all x and t (and therefore, certainly for x near $x_0 = 1/2$ and t near $t_0 = 0$). (Also see Remark 3.) Therefore, by Theorem 1.1 this initial value problem has a unique solution on some interval $a < t < \beta$ containing $t_0 = 0$.

From the formula

$$x(t) = \frac{1}{2}e^{t^2/2}$$

for the solution (check this!) we see that the solution is in fact defined for all t, i.e., on the interval $-\infty < t < +\infty$. This fact not obtainable from Theorem 1.1.)

Example 1.5 *An initial value problem describing the growth of a population in a periodically fluctuating environment is*

$$x' = rx\left(1 - \frac{x}{K + a\sin t}\right), \quad x(0) = x_0$$

where x_0 is the initial population size and the coefficients r, K and a are positive constants (with $a < K$). Since the denominator never vanishes the function

$$f(t,x) = r\left(x - \frac{x^2}{K + a\sin t}\right)$$

and its derivative with respect to x

$$\frac{df(t,x)}{dx} = r\left(1 - 2\frac{x}{K + a\sin t}\right)$$

are continuous for all x and t. Therefore, the initial value problem has a unique solution on an interval containing $t_0 = 0$. No algebraic formula is available for the general solution of this equation, nor for the solution of initial value problems.

If one or both of the conditions on $f(t,x)$ in the existence and uniqueness Theorem 1.1 fail to hold, then one can draw no conclusions from this theorem (Remark 4). In particular, in this circumstance one *cannot* conclude that there is no solution. For example, for the initial value problem

$$x' = 3x^{2/3}, \quad x(0) = 0 \tag{1.8}$$

the function

$$f(t,x) = 3x^{2/3}$$

fails to satisfy the conditions in Theorem 1.1 because the derivative

$$\frac{df(t,x)}{dx} = 2x^{-1/3},$$

is not continuous at the initial $x_0 = 0$ of interest (it is not even defined there). Therefore, nothing is deducible from Theorem 1.1. In particular, one cannot deduce Theorem 1.1 that

the initial value problem (1.8) has no solution. In fact, it does have a solution! Here's a formula for a solution on the interval $-\infty < t < +\infty$ (check it!):

$$x(t) = t^3.$$

(For an example of an initial value problem that has no solution see Exercise 1.42.)

The initial value problem (1.8) also serves to illustrate another point, namely, that initial value problems might not have unique solutions (i.e., have more than one solution). Here are two solutions to the initial value problem (1.8)[2]:

$$x(t) = t^3 \text{ and } x(t) = 0.$$

Note that this does not contradict Theorem 1.1 because, as we've shown, the theorem does not apply to this initial value problem.

Remark 6. Here's an interesting, non-obvious fact (which we make no attempt to prove). If an initial value problem (1.7) has two solutions, then it has infinitely many solutions. Thus, an initial value problem (1.7) has either 0, 1 or infinitely many solutions.

The Fundamental Existence and Uniqueness Theorem 1.1 provides criteria under which an initial value problem has a solution on *some* interval containing the initial point $t = t_0$. The *maximal interval* of the solution is the largest interval containing t_0 on which it solves the differential equation. *Theorem 1.1 gives no information about the maximal interval of a solution.* In fact, without a solution formula it is usually difficult to determine the maximal interval (directly from $f(t, x)$ and the initial condition). For example, the function $f(t, x)$ can satisfy the criteria of Theorem 1.1 *for all values of* t *and* x and yet solutions of initial problems might *not* be defined for all t! Here is an example.

Example 1.6 *Consider the initial value problem*

$$x' = 2tx^2, \quad x(0) = 1.$$

By Remark 3 above (applied to the function $f(t, x) = 2tx^2$*) we know that Theorem 1.1 applies to any initial value problem for this differential equation. Theorem 1.1 implies there exists a unique solution on some interval containing* $t_0 = 0$*. Here's a solution formula (check it!):*

$$x(t) = \frac{1}{1 - t^2}.$$

This formula shows that the maximal existence interval of this solution is $-1 < t < 1$*. See Figure 1.1.*

[2] Any constant number (such as 0) can be used to define a constant function. Such a function has a zero derivative, of course, and it has a horizontal straight line graph.

The importance of the interval of existence of a solution can sometimes be overlooked, with erroneous results. Here is an example.

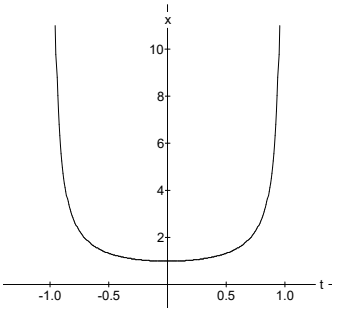


Figure 1.1

Example 1.7 *Some popular computer programs that find solution formulas of initial value problems give the* $x(t) = \sin t$ *for the solution of the initial value problem*

$$x' = \sqrt{1 - x^2}, \quad x(0) = 0$$

without indicating the solution interval. The (unfortunate) implication of the computer generated answer (since $\sin t$ *is defined for all* t*) is that* $x(t) = \sin t$ *is a solution for all* t*. However, this is false. Recall the definition of a solution. Substituting* $\sin t$ *into both sides of the differential equation we obtain*

$$x'(t) = \cos t \quad \text{and} \quad f(t, x(t)) = \sqrt{1 - \sin^2 t} = \sqrt{\cos^2 t}.$$

Recalling that for a real number z

$$\sqrt{z^2} = |z|$$

we see that these two expressions for $x'(t)$ *and* $f(t, x(t))$ *are identical only on intervals for which* $\cos t \geq 0$*. The largest such interval that contains the initial condition* $t_0 = 0$ *is the interval* $-\pi/2 < t < \pi/2$*. On this interval* $x(t) = \sin t$ *is a solution formula for the initial value problem. However,* $x(t) = \sin t$ *is* ***not*** *a solution on any larger interval containing* $t_0 = 0$ *and certainly not for all* t*. (NOTE: it turns out, perhaps unexpectedly, that the interval* $-\pi/2 < t < \pi/2$ *is* ***not*** *the maximal interval of the solution of the initial value problem! See Exercise 1.43.)*

1.3 Approximation of Solutions

Formulas for solutions of differential equations are not in general available. For this reason we need other methods for studying equations and their solutions. For some applications it is sufficient to obtain approximations to solutions. For example, roughly sketched graphs of solutions are sometimes adequate. In other applications, more accurate graphs or even numerical approximations are necessary. One can also obtain algebraic formulas for approximations to solutions. In this section we study some graphical and numerical approximation methods. Analytic approximation methods are studied in Chapter 3. We begin with a procedure for making sketches of solution graphs.

1.3.1 Slope Fields

From algebra and calculus we learn that graphs are a useful way to study functions. The derivative of a function is the slope of its graph. A differential equation therefore tells us something about the slopes of the graphs of its solutions.

Specifically, if the graph of a solution $x = x(t)$ of

$$x' = f(t, x) \tag{1.9}$$

passes through a point (t, x), then the slope $x'(t)$ of its graph at this point equals $f(t, x(t))$. In other words, in the (t, x)-plane each point (t, x) in the domain of f is associated with a slope equal to the number $f(t, x)$.

For example, in the (t, x)-plane the graph of a solution of $x' = t^2 + x^2$ that passes through the point $(t, x) = (1, 1)$ necessarily has slope $1^2 + 1^2 = 2$ at this point. Similarly, the solution whose graph passes through the point $(-2, 1/3)$ must have slope $(-2)^2 + (1/3)^2 = 37/9$ at this point.

The association of a slope $f(t, x)$ with each point (t, x) in the (t, x)-plane defines the *slope field* of the differential equation (1.9). Solutions of differential equation must "fit" its slope field. This means at each point on a solution's graph the slope (of the tangent) must equal the slope associated with that point.

One way to obtain a picture of a slope field is to draw, through each of several points in the (t, x)-plane, a short straight line segment that has the slope associated with that point. By drawing such line segments through a sufficient number of points in the plane, we can get a good approximation to the overall slope field and hence the graphs of solutions.

Rather than randomly choosing points in the plane, it is better to proceed in a systematic manner. We discuss two ways to do this: the "grid" and the "isocline" methods. The grid method is particularly well suited for computer use. The isocline method is sometimes a convenient way to obtain a sketch of the slope field by hand.

The Grid Method

One way to approximate a slope field is to draw a short line segment with the appropriate slope at points lying on a rectangular grid in the (t, x)-plane. This *grid method* can be done by hand; however, most computer programs that "solve" differential equations will also draw slope fields using this "grid" method and display the results graphically.

When sketching a slope field by the grid method, one must chose a grid fine enough so that the essential features of the slope field are apparent, but coarse enough so as not to be visually cluttered. It usually takes a several attempts to find a suitable gird size. Sample slope fields for several differential equations, drawn using the grid method, appear in Figure 1.2.

One can sketch the solution graph of an initial value problem $x(t_0) = x_0$ by drawing a curve that both fits the slope field and passes through the point (t_0, x_0). Such a sketch can often suggest important properties of solutions. For example, the slope field and solution sketched in Figure 1.3 suggest that the solution is monotonically increasing without bound as $t \to +\infty$ and that the x-axis is a horizontal asymptote as $t \to -\infty$.

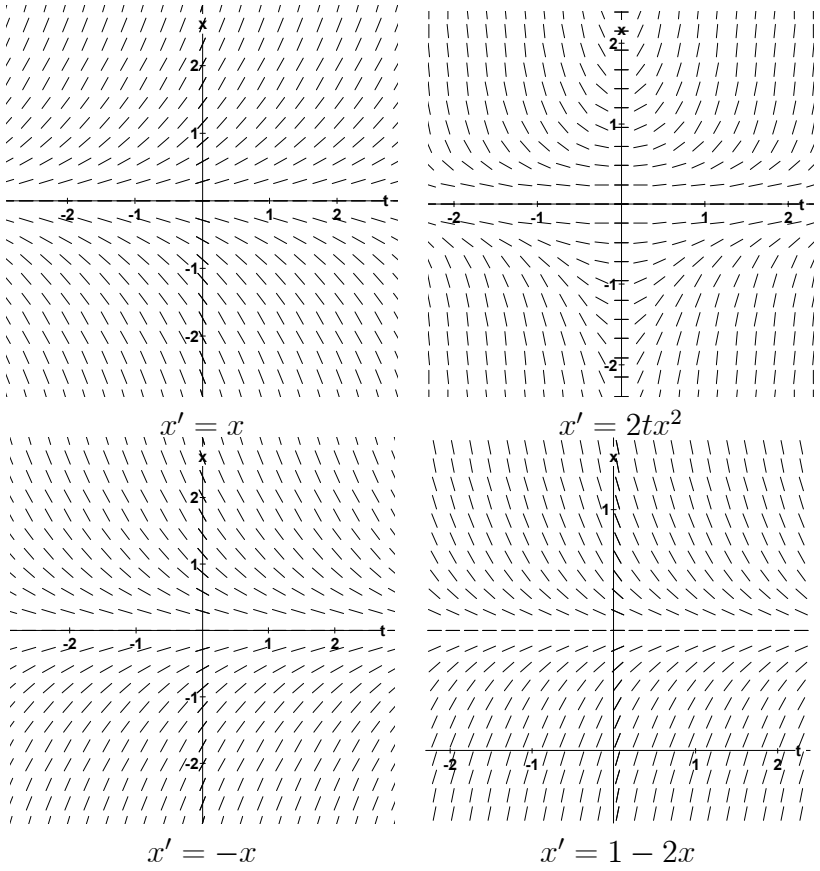


Figure 1.2. Slope fields are shown for four different differential equations.

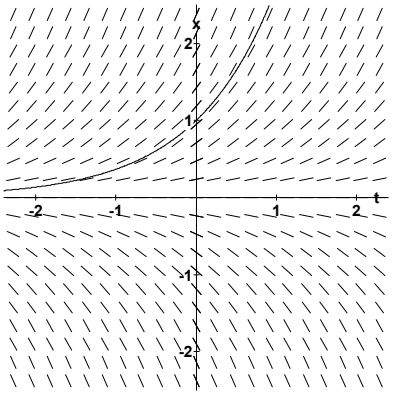


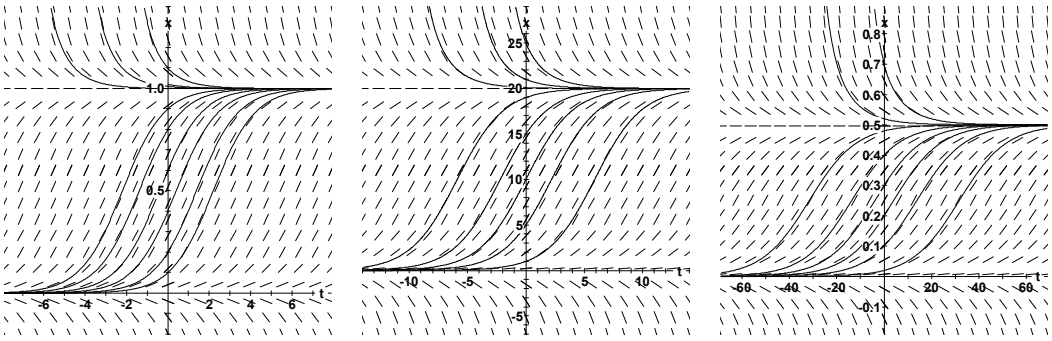
Figure 1.3 The slope field for $x' = x$ and the solution satisfying the initial condition $x(0) = 1$.

The next example shows how a slope field can yield important properties of solutions.

Example 1.8 *Figure 1.4 shows the slope fields of the logistic equation*

$$x' = rx\left(1 - \frac{x}{K}\right)$$

for several choices of the parameters r and K. These slope fields, together with the sample solution graphs, suggest that solutions with positive initial conditions $x(0) = x_0 < K$ tend monotonically to a horizontal asymptote at $x = K$ as $t \to +\infty$. This important fact about the logistic equation will be proved in Chapter 3. Note that $x(t) = K$ is a solution.



(a) $r = 1$, $K = 1$ (b) $r = 0.5$, $K = 20$ (c) $r = 0.1$, $K = 0.5$

Figure 1.4. Selected slope fields and solutions for the logistic equation $x' = r\,(1 - x/K)\,x$.

The Isocline Method

In Figure 2.3 it is interesting to note that the points lying on a horizontal straight line appear to be associated with the same slope. The reason for this is that

$$f(t, x) = rx(1 - x/K),$$

and hence the slope at a point (t, x), does not depend on t. This observation in fact applies to any equation whose right hand side f does not depend on the independent variable t, i.e. to any so-called *autonomous* differential equation (Chapter 3).

A curve all of whose points are associated with the same slopes in the slope field of a differential equation is called an *isocline*. ("iso" means "same" and "cline" means "slope" .) The isoclines of an autonomous equation $x' = f(x)$ are horizontal straight lines. Points on a horizontal line $x = a$ are associated with slope $f(a)$. This fact can be a useful aid in sketching the slope field of an autonomous equation. Figure 1.5 shows a sketch of the slope field for the equation $x' = x(1-x)$ obtained using this isocline method.

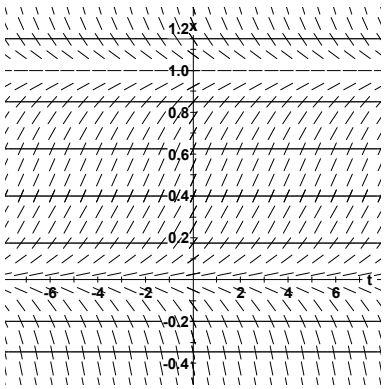


Figure 1.5

The concept of an isocline is not restricted to autonomous equations. For any equation $x' = f(t, x)$ we can find isoclines by determining those points in the plane that are associated with a common slope m. These points satisfy the *isocline equation*

$$f(t, x) = m.$$

The graph of this equation is, in general, a curve in the plane called the *isocline* associated with slope m.

For non-autonomous equations isoclines are not necessarily horizontal lines. If they can be conveniently graphed, isoclines can be used to sketch slope fields for non-autonomous equations in the same way they were used for autonomous equations. On an isocline we draw several short line segments each having the slope associated with that isocline. Doing this for a collection of isoclines we obtain a sketch of the slope field. The following example illustrates the method.

Example 1.9 *What are the isoclines associated with the equation*

$$x' = t^2 + x^2 \ ?$$

Suppose we find the isocline associated with slope $m = 1$. *The equation for this isocline is* $t^2 + x^2 = 1$ *which we recognize as the equation the circle with radius* 1 *and center at the origin* $(0, 0)$. *Drawing this circle and placing on it several short line segments with slope* 1, *we obtain part of the slope field. This procedure can be repeated using other slopes* m. *Points associated with slope* $m = 2$ *lie on the circle of radius* $\sqrt{2}$ *while points associated with slope* $m = 0.25$ *lie on the circle of radius* $\sqrt{0.25}$ *and so on. The typical isocline equation* $t^2 + x^2 = m$ *yields the circle of radius* $\sqrt{m}$, *provided* $m > 0$. *A "degenerate" isocline is obtained for slope* $m = 0$, *namely the single point* $(0, 0)$. *There are no isoclines associated with negative slopes* $m < 0$. *See Figure 1.6(a).*

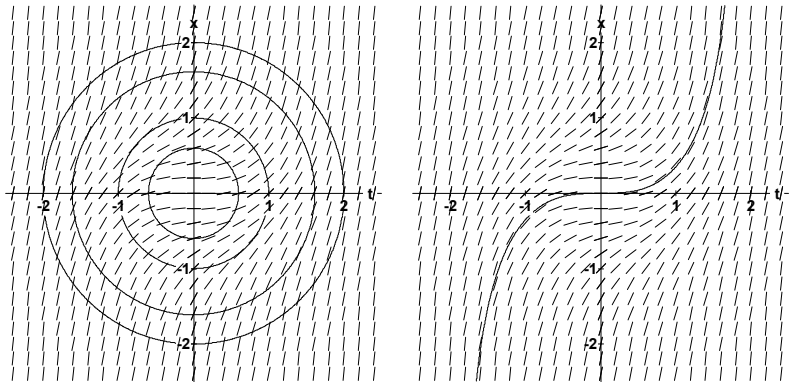


Figure 1.6. (a) Selected circular isoclines of $x' = t^2 + x^2$. (b) The solution satisfying the initial condition $x(0) = 0$.

Isoclines are not necessarily easy to identify or graph. Their usefulness for slope field sketching depends on the right hand side $f(t, x)$ of the differential equation. If we can easily identify and graph isoclines, then this method for drawing slope fields is convenient. Otherwise it is not.

Caution: A common mistake is to confuse isoclines with the solution graphs. Isoclines are *not* graphs of solutions. For example, compare the solution graph in Figure 1.6(b) to the circular isoclines in Figure 1.6(a).

1.3.2 Numeric and Graphic Approximations

Slope fields provide approximate graphs of solutions of differential equations. However, it is often desirable to have a more accurate approximation to a solution and its graph than can be obtained from a slope field. Another way to obtain an *approximate graph of a solution* on an interval $t_0 \le t \le T$ is to calculate numerical approximations x_i to the solution $x(t_i)$ at $t = t_i$ where

$$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T.$$

and, in the (t, x)-plane, connect the points (t_0, x_0), (t_1, x_1),..., (t_n, x_n) by straight line segments. See Figure 1.7.

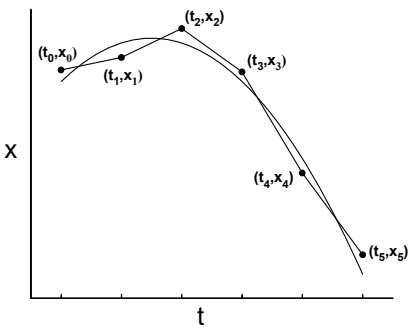


Figure 1.7

We want to obtain the approximations $x_i \approx x(t_i)$ in such a way that if the number of points t_i increases (and the distances between them tend to zero) then the approximations x_i become more accurate and the approximate ("broken line") graphs approach the (smooth) graph of the solution $x = x(t)$.

The Euler Method

In this section we study a basic method for numerically approximating the values of the solution $x(t)$ of the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{1.10}$$

at specified values of $t > t_0$. The method, called the *Euler Algorithm*, is a basic method that serves as an introduction to the numerical approximation of solutions of differential equations. It is, however, rarely used for other than pedagogical reasons because it "converges" too slowly. Sec. 1.3.2 gives some methods that converge more quickly (and hence are more commonly used). Nonetheless Euler's Algorithm, by providing a basis for understanding how solutions are numerically approximated, is a good starting point for the study of more efficient (and hence complicated) algorithms.

Consider the problem of approximating the solution $x = x(t)$ of (1.10) at $t = t_1 > t_0$. Since $x(t)$ is a solution, we can integrate both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_0$ to $t = t_1$ to obtain

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(t, x(t))dt$$

or, using the initial condition,

$$x(t_1) = x_0 + \int_{t_0}^{t_1} f(t, x(t))dt. \tag{1.11}$$

The right hand side of this equation does not give a formula for $x(t_1)$ because it involves the unknown solution $x(t)$. However, we can use (1.11) to approximate $x(t_1)$ by making an approximation to the integral on the right hand side. For example, we can use integration approximation methods studied in calculus, such as the rectangle rule, the trapezoid rule, or Simpson's rule.

The Euler Algorithm is obtained by using the (left hand) rectangle rule to approximate the integral :

$$\int_{t_0}^{t_1} f(t, x(t))dt \approx (t_1 - t_0)f(t_0, x(t_0)).$$

Defining the first step size by $s_0 = t_1 - t_0$ and recalling the initial condition $x(t_0) = x_0$ we have

$$\int_{t_0}^{t_1} f(t, x(t))dt \approx s_0 f(t_0, x_0)$$

and consequently from (1.11) we have the approximation

$$x(t_1) \approx x_0 + s_0 f(t_0, x_0).$$

Denote this approximation by x_1; that is, we define x_1 by

$$x_1 = x_0 + s_0 f(t_0, x_0).$$

To obtain an approximation x_2 to the solution value $x(t_2)$ at the next point t_2 we proceed in a similar manner. Integrate both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_1$ to $t = t_2$. Using the Fundamental Theorem of Calculus, the (left hand) rectangle rule to approximate the integral and the approximation $x_1 \approx x(t_1)$, we obtain

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} f(t, x(t))dt \approx x_1 + (t_2 - t_1)f(t_1, x_1)$$

We denote this approximation to the solution at $t = t_2$ by

$$x_2 = x_1 + s_1 f(t_1, x_1), \quad s_1 = t_2 - t_1.$$

In calculating the approximation x_2 we introduced *two* sources of error. First, there is the error made in using the rectangle rule to approximate the integral (called the **"**truncation error") and, secondly, there is the error in using the approximation x_1 to $x(t_1)$. Together these errors account for the **"**accumulation error**"** at the point $t = t_2$.

If this procedure is repeated we obtain the following formulas

$$\begin{aligned} x_0 &= x_0 \\ x_{i+1} &= x_i + s_i f(t_i, x_i), \qquad s_i = t_{i+1} - t_i, \quad i = 0, 1, 2, \cdots \end{aligned}$$

of the Euler Algorithm. The number x_i is an approximation to the solution $x = x(t)$ of the initial value problem (1.10) at the point $t = t_i$. Usually equally spaced points are chosen, in which case $s_i = s$ for all i and the algorithm reduces to

$$\begin{aligned} x_0 &= x_0 \\ x_{i+1} &= x_i + sf(t_i, x_i) \quad \text{for} \quad i = 0, 1, 2, \cdots, n. \end{aligned} \tag{1.12}$$

The common distance s is called the *step size* of the algorithm**.**

The formulas (1.12) are recursive. That is to say, one utilizes the same formula sequentially to calculate the approximations at each of the points t_1, $t_2, \ldots t_n$, using at each step the approximation made at the previous step. This makes the method ideally suited for programming on a computer or calculator.

The accuracy of the integral approximation obtained by the rectangle rule increases if the step size s decreases. For this reason we expect the accuracy of the approximations obtained from the Euler Algorithm (1.12) to increase if the step size s decreases. There is a cost for this increased accuracy, however, because decreasing the step size s will increase the number n of steps necessary to get from the initial condition t_0 to the end point T. This means more repetitions of the algorithm (1.12) are required, and consequently more arithmetic work is necessary to reach the end point $t_n = T$. (This also means more round off errors!)

Example 1.10 *In this example we use the Euler Algorithm (1.12) to approximate the solution $x = x(t)$ of the initial value problem*

$$x' = x, \quad x(0) = 1$$

at $T = 1$ using step size $s = 0.2$. The Euler algorithm (1.12) for this problem is

$$x_{i+1} = x_i + sx_i \quad \text{for} \quad i = 0, 1, 2, \cdots$$

with $x_0 = 1$. Using step size $s = 0.2$ we need to calculate approximations at the five points $t = 0.2$, 0.4, 0.6, 0.8, 1.0. The calculations are

$$\begin{aligned}
x_1 &= x_0 + sx_0 = 1 + 0.2 \times 1 = 1.2 \\
x_2 &= x_1 + sx_1 = 1.2 + 0.2 \times 1.2 = 1.44 \\
x_3 &= x_2 + sx_2 = 1.44 + 0.2 \times 1.44 = 1.728 \\
x_4 &= x_3 + sx_3 = 1.728 + 0.2 \times 1.728 = 2.0736 \\
x_5 &= x_4 + sx_4 = 2.0736 + 0.2 \times 2.0736 = 2.48832.
\end{aligned}$$

The Euler Algorithm with step size $s = 0.2$ yields the approximation $x(1) \approx x_5 = 2.48832$.

How good is the approximation x_5 in the previous example? More generally, how accurate are the approximations (1.12) of the Euler Algorithm? Can we estimate the size of the error and if not how can we have any confidence in the numerical approximations obtained from the formulas (1.12)?

An accurate estimate of the error resulting from approximation methods such as the Euler Algorithm is usually not possible. However, we expect the numerical approximations will get more accurate as the step size s decreases and that they will tend to the exact solution in the limit as $s \to 0$. This turns out to be true for the Euler Algorithm, on the solution's interval of existence, under the assumptions of the Fundamental Existence and Uniqueness Theorem 1.1.

One useful way to study the accuracy of the Euler Algorithm (and of other algorithms as well) is to consider the *rate* at which the approximations converge to the exact solution. The Euler Algorithm is said to be "first order" or of "order 1". What this means is that the magnitude of the error at $t = T$ is no larger than constant multiple of the first power of

s. That is to say, there exists a constant $c > 0$ such that $| x(T) - x_n | \leq cs$. This inequality guarantees the Euler approximations converge to the value of the solution at least as fast as s decreases to 0. Thus, roughly speaking, if the step size s is halved, then in general we expect the error to be (at least) halved. If the step size is decreased by a factor of 1/10, then in general we expect the error to decrease by a factor of 1/10 and so on. (For an example, see Table 1.2 below.) We summarize this by saying that the Euler Algorithm is "$O(s)$" (pronounced "Oh of s").

We can gain confidence in the accuracy of numerical approximations by observing their changes as the step size s decreases. This is commonly done by decreasing s by a fixed fraction. For example, if s is decreased by one half several times, we expect the error to be cut in half each time. Since the approximations at a fixed t approach the solution value $x(t)$, the leading digits in the resulting sequence of approximations should eventually "stabilize" (i.e., remain unchanged as s decreases further). As a practical matter we accept these digits as correct. However, none of these digits may be accurate, since we cannot be sure that they will remain unchanged if the step size s decreases further.

Example 1.11 *In this example we repeat Example 1.10 by halving the step size s six consecutive times and observe the resulting change in the approximation to* $x(1)$*. The number of calculations necessary to perform the approximation increases as s decreases. For example, the algorithm (1.12) must be used* 320 *times for the step size* $s = 0.003125$*.*

We use a computer to perform the calculations and the results appear in Table 1.1. We expect the approximation $x(1) \approx 2.714047$ *obtained from the smallest step size* $s = 0.003125$ *to be the most accurate, but how many of these digits are correct? We know the sequence of approximations converges to the exact value of the solution at* $T = 1$*. Since only two digits appear to have stabilized in Table 1.1, we accept only the two digit approximation* 2.7 *as accurate.*

Step size s	Approximation to $x(1)$
0.200000	2.488320
0.100000	2.593742
0.050000	2.653298
0.025000	2.685064
0.012500	2.701485
0.006250	2.709836
0.003125	2.714047

Table 1.1. The Euler Algorithm approximations to the solution at $t = 1$ of the initial value problem $x' = x$, $x(0) = 1$ obtained by repeatedly halving the step size.

There is a formula for the solution of the initial value problem in Examples 1.10 and 1.11, namely $x(t) = e^t$. Therefore, the exact value of the solution at $t = 1$ is $x(1) = e$ (recall $e \approx 2.718282$). Using this formula we can investigate how accurate the approximations in Table 1.1 really are.

The percent error of each approximation is given in Table 1.2. Notice the percent error decreases by a factor of (approximately) 1/2 at each consecutive step. This is what we expect,

since the step size s decreases by a factor of $1/2$ at each step and the Euler Algorithm is $O(s)$.

Step size s	Approximation to $x(1)$	% Error
0.200000	2.488320	8.4598
0.100000	2.593742	4.5816
0.050000	2.653298	2.3906
0.025000	2.685064	1.2220
0.012500	2.701485	0.6179
0.006250	2.709836	0.3107
0.003125	2.714047	0.1558

Table 1.2. The percent errors of the approximations in Table 1.1.

We approximate the graph of solution of the initial value problem $x' = x$, $x(0) = 1$ by connecting the points (t_i, x_i) with straight line segments. This is done in Figure 1.8 for decreasing step sizes on the interval $0 \le t \le 5$. The convergence, as s decreases, of these approximate graphs to the graph of the solution $x = e^t$ is apparent.

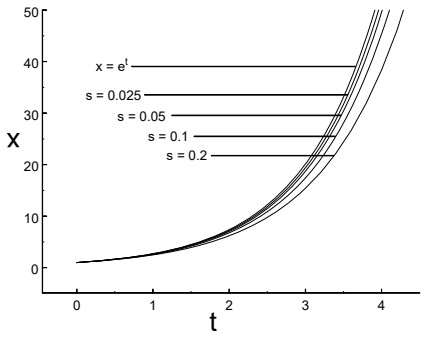


Figure 1.8

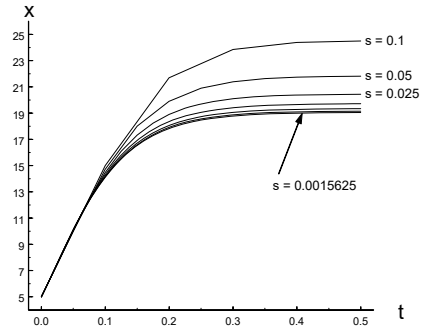


Figure 1.9

One should not accept a graphical approximation to a solution obtained from a single step size s alone (e.g., the default step size in a computer program). Instead, before accepting a graphical approximation, one should decrease the step size until little change occurs in two consecutive graphical approximations.

Example 1.12 *The equation $x' = ae^{-bt}x$ describes the growth of a tumor where $x = x(t)$ is a measure of its size (e.g., weight or number of cells) and t is time. Figure 1.9 shows approximate graphs of the solution of the initial value problem with $x_0 = 5$ and parameter values $a = 20$ and $b = 15$. These graphs result from the Euler Algorithm using a decreasing sequence of step sizes starting with $s = 0.1$. Little change occurs in the graphs for the last two steps sizes $s = 0.003125$ and 0.0015625 and therefore we accept the final graph as an accurate approximation. All of the graphs indicate that the tumor size x approaches a maximal size as $t \to +\infty$. However, the inaccurate graphs obtained from the larger steps sizes considerably over estimate the maximal size of the tumor.*

The convergence rate $O(s)$ of the Euler Algorithm is sometimes too slow for practical purposes. In Table 1.2 only two digits of accuracy for $x(1)$ are obtained with a step size $s = 0.003125$. To obtain more accuracy a smaller step size is needed. However, there are more intermediate steps with each decrease in step size and it takes longer to perform all of the necessary calculations. Furthermore, other sources of error, such as round-off errors at each step, might eventually prevent increased accuracy if the number of steps (and hence calculations) becomes too large.

Table 1.3 shows an example that dramatically illustrates the slow convergence of the Euler Algorithm. In this example no accurate digits are found with a step size as small as $s = 0.000391$.

Step size s	Approximation to $x(1)$
0.100000	5.862897
0.050000	8.905711
0.025000	13.766320
0.012500	21.242856
0.006250	31.967263
0.003125	45.709606
0.001563	60.736659
0.000781	74.330963
0.000391	84.517375

Table 1.3. Euler Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.[3]

Fortunately, practical algorithms with faster rates of convergent are available. In the following section we discuss algorithms of orders two and four. An algorithm has *order of convergence* p (or more succinctly is of *order* p), written $O(s^p)$, if the accumulative error is bounded in magnitude by a constant multiple of s^p, i.e., if

$$| x(T) - x_n | \leq cs^p.$$

To see the advantage of a convergence rate of order greater than $p = 1$ consider an algorithm of order $p = 2$, for which the error satisfies

$$| x(T) - x_n | \leq cs^2.$$

We can expect the error to decrease by a factor of $(1/2)^2 = 1/4$ if the step size s is decreased by a factor of $1/2$, or by a factor of $(1/10)^2 = 1/100$ if the step size s is decreased by a factor of $1/10$, and so on. For an algorithm of order 4 the error decreases even faster, e.g., by a factor of $(1/10)^4 = 1/10000$ if the step size is decreased by a factor of $1/10$.

[3]The interval of existence for the solution is $-\infty < t < 100/99 \approx 1.0101$. It is interesting to note that the Euler Algorithm will calculate "approximations" at t values outside of this interval. For example, with step size $s = 0.1$, eleven repetitions of the algorithm produce the number $x_{11} = 9.30025$. However, this number cannot be taken as an approximation to the solution at $t = 1.1$ because the solution is not defined at this value of $t > 100/99$.

The Modified Euler Method

In deriving the Euler Algorithm we used the Rectangle Rule to approximate the integral

$$\int_{t_i}^{t_{i+1}} f(t, x(t))dt.$$

More accurate approximations to this integral lead to algorithms that converge faster than the Euler Algorithm. For example, we could use the Trapezoid Rule. Integrating both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_i$ to $t = t_{i+1}$ we obtain

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(t, x(t))dt.$$

From the Trapezoid Rule approximation

$$\int_{t_i}^{t_{i+1}} f(t, x(t))dt \approx \frac{s_i}{2} \left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x(t_i)) \right]$$

we get

$$x(t_{i+1}) \approx x(t_i) + \frac{s_i}{2} \left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x(t_i)) \right].$$

Assuming that we already have an approximation $x(t_i) \approx x_i$ to the solution at the point $t = t_i$, we can write

$$x(t_{i+1}) \approx x_i + \frac{s_i}{2} \left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x_i) \right].$$

Unfortunately we cannot use the right hand side to calculate an approximation x_{i+1} to $x(t_{i+1})$ because it involves $x(t_{i+1})$. This is an example of what is called an *implicit* algorithm because the equation

$$x_{i+1} = x_i + \frac{s_i}{2} \left[f(t_{i+1}, x_{i+1}) + f(t_i, x_i) \right]$$

is not explicitly solved for the approximation x_{i+1}. (The Euler Algorithm is an example of an *explicit* algorithm.) To find the approximation x_{i+1}, we have to solve this equation. To do this at each step results in a highly complicated algorithm. One way to deal with this difficulty is to perform another approximation. For example, we can use the Euler approximation for the x_{i+1} on the right hand side. Thus, at each step we use the formulas

$$\begin{aligned} x^*_{i+1} &= x_i + s_i f(t_i, x_i) \\ x_{i+1} &= x_i + \frac{s_i}{2} \left[f(t_{i+1}, x^*_{i+1}) + f(t_i, x_i) \right], \quad i = 0, 1, 2, \cdots \end{aligned}$$

to calculate the approximation x_{i+1}. This algorithm is called *Modified Euler Algorithm* . It is an example of a "predictor-corrector" algorithm. At each step the Euler approximation x^*_{i+1} is the prediction and x_{i+1} is the correction.

If equal step sizes $s_i = s$ are used, the Modified Euler Algorithm becomes

$$\begin{aligned} x^*_{i+1} &= x_i + s f(t_i, x_i) \\ x_{i+1} &= x_i + \frac{s}{2} \left[f(t_{i+1}, x^*_{i+1}) + f(t_i, x_i) \right], \quad i = 0, 1, 2, \cdots . \end{aligned} \tag{1.13}$$

The initial condition x_0 starts the algorithm. It turns out that Modified Euler Algorithm of order $O(s^2)$.

Compare the results in Table 1.4 with those in Table 1.2. Note that the error in Table 1.4 decreases approximately by a factor of 1/4 as the steps size is decreased by a factor of 1/2. Modified Euler Algorithm is a popular procedure; for example, it is often used with programmable hand calculators.

Step size s	Approximation to $x(1)$	% Error
0.200000	2.702708	0.5729
0.100000	2.714081	0.1545
0.050000	2.717191	0.0401
0.025000	2.718004	0.0102
0.012500	2.718212	0.0026
0.006250	2.718264	0.0007

Table 1.4. The Modified Euler Algorithm approximations to the solution of the initial value problem $x' = x$, $x(0) = 1$, at $t = 1$ obtained by repeatedly halving the step size.

We saw in Table 1.3 an example of an initial value problem for which the Euler Algorithm converges too slowly to be practical. Table 1.5 shows the results of applying Modified Euler Method to the same initial value problem. The estimates obtained from the two numerical algorithms differ considerably. At each step size Modified Euler Algorithm provides a more accurate approximation to $x(1) = 99$ than does the Euler Algorithm.

Step size s	Approximation to $x(1)$
0.100000	19.346653
0.050000	33.073325
0.025000	52.217973
0.012500	72.662362
0.006250	87.787581
0.003125	95.273334
0.001563	97.719807
0.000781	98.719804
0.000391	98.928245

Table 1.5. Modified Euler Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.

The Fourth Order Runge-Kutta Algorithm

Higher order algorithms necessarily involve more complicated formulas at each step. A widely used class of algorithms is the class of so-called Runge-Kutta algorithms. Runge-Kutta algorithms are available for any order of convergence. A popular algorithm is the

fourth order Runge-Kutta Algorithm (RK-4) :

$$x_{i+1} = x_i + s\frac{L_1+2L_2+2L_3+L_4}{6} \text{ for } i = 0, 1, 2, \cdots$$

where

$$\begin{aligned} L_1 &= f(t_i, x_i) \\ L_2 &= f\left(t_i + \frac{s}{2}, x_i + \frac{s}{2}L_1\right) \\ L_3 &= f\left(t_i + \frac{s}{2}, x_i + \frac{s}{2}L_2\right) \\ L_4 &= f(t_i + s, x_i + sL_3) \end{aligned}$$

At each step one must calculate, in order, the four numbers L_1, L_2, L_3, and L_4 before calculating x_{i+1}. Table 1.6 shows the results applying this algorithm to the same initial value problem in Table 1.3 and 2.5. This faster converging algorithm provides an accurate approximation to $x(1) = 99$.

Step size s	Approximation to $x(1)$
0.100000	53.355933
0.050000	75.881773
0.025000	91.639594
0.012500	97.671604
0.006250	98.856123
0.003125	98.988718
0.001563	98.999238
0.000781	98.999951
0.000391	98.999997

Table 1.6. Fourth order Runge-Kutta Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.

1.4 Chapter Summary

A solution $x = x(t)$ of the differential equation $x' = f(t, x)$ is a differentiable function for which $x'(t) = f(t, x(t))$ holds for all t on an interval. In general a differential equation has infinitely many solutions. The general solution is the set of all solutions. We need an additional requirement in order to specify a unique solution. For a given point (t_0, x_0), the initial condition $x(t_0) = x_0$ is such a requirement. Theorem 1.1 gives conditions under which an initial value problem $x' = f(t, x)$, $x(t_0) = x_0$ has one and only one solution. Specifically, if $f(t, x)$ and its derivative $df(t, x)/dx$ with respect to x are both continuous for t near t_0 and x near x_0, then there is one and only one solution. Although formulas for the solution cannot always be calculated, many kinds of approximation methods are available. The slope

field associated with the differential equation helps in to sketching a graph of the solutions. A computers is useful for plotting the slope fields by the grid method; this method associates the slope $f(t, x)$ with each point (t, x) on from a chosen grid of points in the (t, x)-plane. Also useful for sketching slope fields are isoclines, which are curves in the (t, x)-plane made up of those points associated with a common slope. Numerical approximations to solution values $x(t)$ yield more accurate graphs of the solution. If x_1, x_2, ..., x_n approximate the solution values $x(t_1)$, $x(t_2)$, ..., $x(t_n)$ for $t_1 < t_2 < \cdots < t_n$, then by connecting the points (t_i, x_i) with straight line segments we construct an approximate (broken line segment) solution graph. Usually equally spaced points t_i are chosen and the common distance between them is the step size s of the method. If the approximations converge to the solution values as s tends to 0, then the broken line graph tends to the solution graph as s tends to 0. The Euler Algorithm is one method for calculating such approximations. It is based on the left hand rectangle rule for approximating an integral. Under the conditions on $f(t, x)$ in Theorem 1.1 the Euler approximations converge to the solution values as the step size s decreases to 0. The Euler Algorithm is of order 1, which means the errors tend to 0 at the same rate that s tends to 0. Faster converging algorithms are available. Modified Euler Algorithm is of order 2, which means the error tends to 0 at the same rate that s^2 tends to 0. A fourth order method called the Runge-Kutta Algorithm is commonly used.

1.5 Exercises

Find a formula for the general solution of the following differential equations. (These are simply integration or anti-differentiation problems from calculus reformulated as differential equations.)

Exercise 1.1 $x' = 1 + t^2$

Exercise 1.2 $x' = \cos \pi t$

Exercise 1.3 $x' = e^{2t}$

Exercise 1.4 $x' = te^{-t}$

Find a formula for the unique solutions of the following initial value problems. (These are simply definite integral problems from calculus reformulated as differential equations.)

Exercise 1.5 $x' = t^2$, $x(1) = 2$

Exercise 1.6 $x' = e^{-3t}$, $x(0) = 1$

Exercise 1.7 $x' = te^{-t}$, $x(0) = 1$

Exercise 1.8 $x' = \sin 3t$, $x(\pi/6) = 0$

For which initial value problems can the Fundamental Existence and Uniqueness Theorem 1.1 be applied and for which can it not be applied? Explain your answer. In each case, what do you conclude about the initial value problem?

Exercise 1.9 $x' = t^2 + x^2$, $x(0) = 0$

Exercise 1.10 $x' = t^2 x^{-2}$, $x(0) = 0$

Exercise 1.11 $x' = \tan x$, $x(\pi/2) = 0$

Exercise 1.12 $x' = \tan x$, $x(0) = 0$

Exercise 1.13 $x' = \tan x$, $x(0) = \pi/2$

Exercise 1.14 $x' = \ln(tx)$, $x(1) = 2$

Exercise 1.15 $x' = 1/\sin x$, $x(0) = \pi/2$

Exercise 1.16 $x' = (t - x)^{-1}$, $x(-1) = 2$

Exercise 1.17 $x' = |x|$, $x(0) = 10$

Exercise 1.18 $x' = |x|$, $x(10) = 0$

For what values of the constant a can the Fundamental Existence and Uniqueness Theorem 1.1 be applied to the four initial value problems below? Explain your answer. What do you conclude from this theorem for such values of a? What do you conclude from this theorem for other values of a?

Exercise 1.19 $x' = \ln(a - x)$, $x(0) = 0$

Exercise 1.20 $x' = \tan ax$, $x(0) = \pi/2$

Exercise 1.21 $x' = \sqrt{a^2 - x^2}$, $x(1) = 2$

Exercise 1.22 $x' = (a - x)^{-1}$, $x(1) = 2$

For which t_0 and x_0 does the Fundamental Existence and Uniqueness Theorem 1.1 apply to the initial value problem $x' = f(t, x)$, $x(t_0) = x_0$, with the four functions $f(t, x)$ below? Explain your answer. What do you conclude from this theorem for such initial points? What do you conclude from this theorem for other initial points?

Exercise 1.23 $f(t, x) = \ln(t^2 + x^2)$

Exercise 1.24 $f(t, x) = t^2 x^{-1}$

Exercise 1.25 $f(t, x) = \tan bx$, $b =$ *constant*

Exercise 1.26 $f(t, x) = \sqrt{t^2 + x^2 - b^2}$, $0 < b =$ *constant.*

Exercise 1.27 $f(t, x) = t^{1/3} + x^{2/3}$

Exercise 1.28 $f(t, x) = t^{1/3} + x^{4/3}$

Exercise 1.29 $f(t,x) = (1-e^{2x-1})^{4/3}$

Exercise 1.30 $f(t,x) = (1-e^{2x-1})^{2/3}$

Exercise 1.31 $f(t,x) = \ln|x-t|$

Exercise 1.32 $f(t,x) = \sqrt{9-x^2-t^2}$

Exercise 1.33 $f(t,x) = |x|$

Exercise 1.34 $f(t,x) = t^{1/3}x$

Exercise 1.35 *Does existence and uniqueness Theorem 1.1 apply to the initial value problem* $x' = \sqrt{1-x}$, $x(1) = 0$*? Explain your answer. What do you conclude?*

Exercise 1.36 *Does the existence and uniqueness Theorem 1.1 apply to the initial value problem* $x' = (4-x^2)^{-1}$, $x(2) = 0$. *Explain your answer. What do you conclude?*

Explain why Theorem 1.1 does not apply to the two initial value problems below. What do you conclude?

Exercise 1.37 $x' = (x+t)^{1/3}$, $x(0) = 0$

Exercise 1.38 $x' = (\sin(x+t))^{1/3}$, $x(0) = 0$

Exercise 1.39 *Apply the existence and uniqueness Theorem 1.1 to the initial value problem* $x' = \sqrt{1-x^2}$, $x(0) = 0$ *in Example 1.7. What do you conclude?*

Exercise 1.40 *Let* $f(t,x)$ *be a polynomial in* t *and* x. *Prove that any initial value problem* $x' = f(t,x)$, $x(t_0) = x_0$ *has a unique solution on an interval containing* t_0.

Exercise 1.41 *Let* $p(z,w)$ *be a polynomial in* z *and* w *and let* $f(t,x) = p(\sin t, \sin x)$. *Prove that any initial value problem* $x' = f(t,x)$, $x(t_0) = x_0$, *has a unique solution on an interval containing* t_0.

Exercise 1.42 *Consider the initial value problem* $x' = f(t,x)$, $x(0) = 0$ *where*

$$f(t,x) = \begin{cases} 1, & \text{for } t \geq 0 \text{ and all } x \\ -1, & \text{for } t < 0 \text{ and all } x. \end{cases}$$

(a) Show the existence and uniqueness Theorem 1.1 does not apply. What do you conclude?

(b) Show this initial value problem does not have a solution on any interval containing $t_0 = 0$.

Exercise 1.43 *Consider the equation* $x' = \sqrt{1-x^2}$.

(a) Show the constant functions $x(t) = 1$ *and* $x(t) = -1$ *are solutions for all* t.

(b) Show the function

$$x(t) = \begin{cases} 1 & \text{for } t \geq \pi/2 \\ \sin t & -\pi/2 < t < \pi/2 \\ -1 & \text{for } t \leq -\pi/2 \end{cases}$$

is a solution for all t. *Thus, the maximal interval for the solution of the initial value problem* $x(0) = 0$ *is the whole real line.*

(c) The solution $x(t) = 1$ *and the solution in (b) both satisfy the same initial value problem* $x(\pi/2) = 1$ *for all* t. *Why does this not contradict Theorem 1.1?*

The following exercises provide practice in using a computer to obtain slope fields and numerically approximated graphs of solutions to initial value problems. Use a computer to obtain the slope fields for the differential equations in the exercises below and then have the computer draw a graph of the solution to each initial value problem. Observe how the solution graphs relate to the slope field.

Exercise 1.44 $x' = 1 - x$ *with initial conditions* $x(0) = 3$, $x(0) = 0$, *and* $x(-1) = 2$

Exercise 1.45 $x' = 2 - 3x$ *with initial conditions* $x(0) = 1$, $x(0) = 2/3$,*and* $x(0) = -1$

Exercise 1.46 $x' = 1 - x^2$ *with initial conditions* $x(0) = -1$, $x(-2) = 1$, $x(0) = 0$, *and* $x(1) = 1.2$

Exercise 1.47 $x' = x\left(1 - x\left(2 + \cos t\right)^{-1}\right)$ *with initial conditions* $x(0) = 0$, $x(0) = 1$, $x(0) = -0.1$, *and* $x(-2) = 2$

Exercise 1.48 $x' = x\cos t$ *with initial conditions* $x(0) = 0$, $x(1) = 4$, $x(0) = 1$, *and* $x(0) = -1$

Exercise 1.49 $x' = -x/2 + \sin t$ *with initial conditions* $x(0) = 0$, $x(0) = 1$, $x(-2) = -1$, *and* $x(0) = -1$

Exercise 1.50 $x' = x\sin x$ *with initial conditions* $x(0) = 0$, $x(0) = \pi/2$, $x(0) = -3$, *and* $x(0) = 4$

Exercise 1.51 $x' = \left(1 + t^2 + x^2\right)^{-1/2}$ *with initial conditions* $x(0) = 0$ *and* $x(-1) = -1.5$

Exercise 1.52 $x' = (1 - x)\, x \sin^2 t$ *with initial conditions* $x(0) = -0.25$, $x(0) = 2$, *and* $x(-2) = 0.5$

Exercise 1.53 $x' = \left(1 - x^2\right)(\sin t - x)$ *with initial conditions* $x(0) = 0$, $x(0) = -0.5$, $x(1) = 1.5$, *and* $x(0) = 1$

Exercise 1.54 $x' = x(1 - x)(x + 1)$ *with initial conditions* $x(0) = 0.5$, $x(0) = -0.5$, $x(0) = 1.5$, *and* $x(0) = -1.5$

Exercise 1.55 $x' = t^2 + 4x^2$ *with initial conditions* $x(0) = 0$ *and* $x(0.5) = 0.5$

Exercise 1.56 $x' = -t/x$ *with initial conditions* $x(0) = 1$ *and* $x(1) = -1$

Exercise 1.57 $x' = (t^2 - x^2)(t^2 + x^2)^{-1}$ *with initial conditions* $x(0) = 1$ *and* $x(-1) = -1$

Exercise 1.58 $x' = \ln(t^2 + x^2)$ *with initial conditions* $x(1) = 0$ *and* $x(0) = 0.1$

Exercise 1.59 *Consider the differential equation in Example 1.5 :*

$$x' = rx\left(1 - \frac{x}{K + a\sin t}\right).$$

(a) Use a computer to sketch the slope fields of the equation in the window $0 \le t \le 20$, $0 \le x \le 10$ *for each of the cases below.*

$$\textit{(i)} \quad r = 1, \quad K = 2, \quad a = 1$$

$$\textit{(ii)} \quad r = 1, \quad K = 5, \quad a = 1$$

$$\textit{(iii)} \quad r = 0.5, \quad K = 5, \quad a = 2$$
$$\textit{(iv)} \quad r = 0.5, \quad K = 5, \quad a = 4$$

(b) For each case in (a), have the computer draw a graph of the solution satisfying the initial condition $x(0) = 1$.

(c) What do all the solutions graphed in (b) seem to have in common (if anything)?

Use a computer to obtain the slope field for the equations below. Do this for a selection of values for the constant a. How are the slope fields for $a > 1$ different from those for $a < 1$?

Exercise 1.60 $x' = -a + 2x - x^2$

Exercise 1.61 $x' = x(x - a)(1 - x)$

Describe (geometrically) the isoclines for the four differential equations below and sketch some sample isoclines and use them to obtain a sketch of the slope field.

Exercise 1.62 $x' = 1 - x$

Exercise 1.63 $x' = 4 - 2x$

Exercise 1.64 $x' = (1 + t^2 + x^2)^{-1/2}$

Exercise 1.65 $x' = -x + \sin t$

Exercise 1.66 *Find the isocline equation for the differential equations in Exercises 1.55-1.56 and graph several typical isoclines. Use your results to sketch the slope field of the equation.*

Find first order differential equations whose isoclines are as described below, if possible. Here m denotes the slope in the field slope. If there is no such equation, explain why.

Exercise 1.67 *The family of lines $x = t + m$ where m allowed to be any constant.*

Exercise 1.68 *The family of parabolas $x = t^2 + m$ where m is allowed to be any constant.*

Exercise 1.69 *The family of lines $x = t + 1/m$ where m is allowed to be any nonzero constant.*

Exercise 1.70 *The family of parabolas $x = t^2 + \frac{1}{m}$ where m is allowed to be any nonzero constant.*

Exercise 1.71 *The family of ellipses $2x^2 + 3t^2 = m^{1/3}$ where m is allowed to be any positive constant.*

Exercise 1.72 *The family of circles $x^2 + t^2 = 1 - 2m^2$ where m is allowed to be any positive constant satisfying $0 < c < 1/\sqrt{2}$.*

Exercise 1.73 *(a) Use the fourth order Runge-Kutta method to approximate the solution of $x' = x$, $x(0) = 1$ at $t = 1$. Start with step size $s = 0.2$ and calculate a sequence of approximations by repeated step size halving.*

(b) Use the solution formula $x = e^t$ to calculate percent errors. Do the errors decrease at the expected rate?

(c) Compare the results in (a) and (b) with those of the Euler and Modified Euler Algorithms in Tables 1.2 and 1.4.

Exercise 1.74 *Let $x = x(t)$ denote the solution of the initial value problem $x' = x^3$, $x(0) = 0.6$. It turns out that $x(1) \approx 1.1338934190$.*

(a) Use the Euler Algorithm to obtain an approximation to $x(1)$ with step size $s = 0.1$. How many correct significant digits does this approximation have?

(b) Obtain Euler approximations by repeatedly halving the step size (starting at $s = 0.1$). At which step size s is the Euler approximation first correct to 2 decimal places? To 3 decimal places?

(c) Compute the absolute error at each step size, starting from $s = 0.1$ and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?

Exercise 1.75 *Repeat Exercise 1.74 using the Modified Euler Algorithm.*

Exercise 1.76 *Repeat Exercise 1.74 using the fourth order Runge-Kutta Algorithm.*

Exercise 1.77 *Repeat Exercise 1.74 using any other algorithm available on your computer.*

Exercise 1.78 *Let $x = x(t)$ denote the solution of the initial value problem $x' = e^x$, $x(0) = 0$. It turns out that $x(0.8) \approx 1.6094379124$.*

(a) Use the Euler Algorithm to obtain an approximation to $x(0.8)$ with step size $s = 0.1$. How many correct significant digits does this approximation have?

(b) Obtain Euler approximations by repeatedly halving the step size. At which step size s is the Euler approximation first correct to 2 decimal places? To 3 decimal places?

(c) Compute the absolute error at each step size, starting from $s = 0.1$ and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?

Exercise 1.79 *Repeat Exercise 1.78 using the Modified Euler Algorithm.*

Exercise 1.80 *Repeat Exercise 1.78 using the fourth order Runge-Kutta Algorithm.*

Exercise 1.81 *Repeat Exercise 1.78 using any other algorithm available on your computer.*

Exercise 1.82 *Approximate the solution of the initial value problem $x' = t^2 + x^2$, $x(0) = 0$ at $T = 0.5$ using the Euler Algorithm, Modified Euler Algorithm, and the Runge-Kutta Algorithm. Start with step size $s = 0.1$ and repeat by halving the step size four times. What are the accurate digits obtained from each algorithm? What is the best approximation obtained from all methods?*

Exercise 1.83 *Use a computer obtain an accurate graphical solution of the initial value problem $x' = t^2 + x^2$, $x(0) = 0$ on the interval from $t = 0$ to $T = 1$ using the Euler Algorithm. Repeatedly halve the step size s starting with $s = 0.1$. What step size did you stop with and why?*

Exercise 1.84 *Repeat Exercise 1.83 using the Modified Euler Algorithm.*

Exercise 1.85 *Repeat Exercise 1.83 using the fourth order Runge-Kutta Algorithm.*

Exercise 1.86 *Repeat Exercise 1.83 using any other algorithm available on your computer.*

Exercise 1.87 *Use a computer obtain an accurate graphical solution of the initial value problem $x' = x^3/(x - t)$, $x(0) = 1$ on the interval from $t = 0$ to $T = 1$ using the Euler Algorithm. Use a window size of $-20 < x < 20$. Repeatedly decrease the step size s by a factor of one tenth, starting with $s = 0.1$. What step size did you stop with and why?*

Exercise 1.88 *Repeat Exercise 1.87 using the Modified Euler Algorithm.*

Exercise 1.89 *Repeat Exercise 1.87 using the fourth order Runge-Kutta Algorithm.*

Exercise 1.90 *Repeat Exercise 1.87 using any other algorithm available on your computer.*

Exercise 1.91 *(a) Use any algorithm you wish to obtain a graphical solution of the initial value problem $x' = 500\cos(200t)$, $x(0) = 0$. Start with step size $s = 0.1$ and decrease until the graph has stabilized. What do you conclude about the solution?*

(b) Obtain a formula for the solution and use it to explain the graphical solution.

Exercise 1.92 *Consider the initial value problem* $x' = x^3e^{-t}$, $x(0) = 1$. *Apply the Euler Algorithm to approximate the solution at* $T = 0.6$.

(a) Start with step size $s = 0.1$ *and halve it four times. Which digits in the resulting approximations do you think are accurate? Explain your answer.*

(b) Halve the step size four more times. Now which digits in the resulting approximations do you think are accurate? Explain your answer.

Exercise 1.93 *Consider the initial value problem* $x' = x^3e^{-t}$, $x(0) = 1$. *Apply Modified Euler Algorithm to approximate the solution at* $T = 0.6$.

(a) Start with step size $s = 0.1$ *and halve it four times. Which digits in the resulting four approximations do you think are accurate? Explain your answer.*

(b) Halve the step size four more times. Now which digits in the resulting four approximations do you think are accurate? Explain your answer.

Exercise 1.94 *Consider the initial value problem* $x' = x^3e^{-t}$, $x(0) = 1$. *Apply the Runge-Kutta algorithm to approximate the solution at* $T = 0.6$. *(See Exercise 1.73.)*

(a) Start with step size $s = 0.1$ *and halve it four times. Which digits in the resulting four approximations do you think are accurate? Explain your answer.*

(b) Halve the step size four more times. Now which digits in the resulting four approximations do you think are accurate? Explain your answer.

Exercise 1.95 *Use the formula* $x(t) = (2e^{-t} - 1)^{-1/2}$ *for the solution of the initial value problem in Exercises 1.92, 1.93, and 1.94 to calculate the error and the per cent error of the approximations in these exercises for step size* $s = 0.00625$. *Round all numbers to* 6 *significant digits.*

Exercise 1.96 *Use the Euler Algorithm and a computer program to obtain an accurate graph of the solution of the initial value problem* $x' = 1.5x^3 \sin 10t$, $x(0) = 1$ *on the interval from* $t = 0$ *to* $T = 1$. *Use a window size of* $-2 < x < 2$. *Repeatedly halve the step size* s *starting with* $s = 0.2$. *At what step size did you stop and why?*

Exercise 1.97 *Repeat Exercise 1.96 using the Modified Euler Algorithm.*

Exercise 1.98 *Repeat Exercise 1.96 using the fourth order Runge-Kutta Algorithm.*

Exercise 1.99 *Suppose the decay rate of a radioactive isotope is* $r = -0.35$ *per year. The differential equation for the amount* $x(t)$ *at time* t *is* $x' = -0.35x$.

(a) Use a computer to study the graphs of solutions with many different initial conditions $x_0 > 0$ *and formulate a conjecture about the length of time it takes a sample amount of the isotope to decay to one half of its initial amount.*

(b) Use the solution formula $x(t) = x_0e^{-0.35t}$ *to verify or disprove your conjecture.*

Exercise 1.100 *Let* $x = x(t)$ *be the dollars in an investment account which is compounded continuously at a rate of* 4.5%.

(a) Perform numerical experiments on the model equations $x' = 0.045x$, $x(0) = x_0$ *to formulate a conjecture about how long will it take for the initial investment of* x_0 *dollars to triple.*

(b) Use the solution formula $x(t) = x_0e^{0.045t}$ *to calculate a formula for the exact tripling time and compare it to your conjecture in (a).*

Exercise 1.101 *Suppose a population has a per capita death rate $d > 0$ and a per capita birth rate that is proportional to population size x (with constant of proportionality denoted by $a > 0$).*

(a) Use the "inflow-outflow" rule x' = birth rate − death rate to write down a model differential equation for the population size $x = x(t)$.

(b) Perform numerical experiments and formulate a conjecture about the fate of the population. (Hint: choose a pair of model parameter values, such as $a = 1$ and $d = 1$, and compute solution graphs for many initial population sizes $x(0) = x_0$. Then repeat for other values for a and d.)

(c) Use the solution formula

$$x(t) = \frac{dx_0}{x_0 a + e^{dt}\,(d - x_0 a)}$$

to verify or disprove your conjectures in (b).

Chapter 2

Linear First Order Equations

2.1 Introduction

There is no method that will succeed in calculating a formula for the general solution of every first order differential equation $x' = f(t, x)$. We can find solution formulas only for certain kinds of equations, i.e., for equations with specialized right hand sides $f(t, x)$. Nonetheless, for several reasons it is important to learn solution methods for some specialized types equations, even though they are necessarily limited in scope. First of all, some types of equations arise often enough in applications that formulas for their general solutions are useful. Secondly, certain types of equations serve as approximations to more complicated equations. Thirdly, a study of various types of equations increases one's general understanding of differential equations. In this chapter we study one of the most important special types of differential equation, namely, linear differential equations.

A first order equation

$$x' = f(t, x)$$

is linear if the right hand side $f(t, x)$ is linear as a function of x, i.e., if

$$f(t, x) = p(t)x + q(t).$$

Notice it is irrelevant how the *independent* variable t appears. What matters is that the *dependent* variable x, and its derivative x', appear linearly (i.e., to the first power and the first power only).

Definition 2.1 *A **linear differential equation of first order** has the form*

$$x' = p(t)x + q(t) \tag{2.1}$$

*where the **coefficients** $p(t)$ and $q(t)$ are defined on an interval $a < t < b$.*

If the coefficients $p(t)$ and $q(t)$ are continuous on an interval $a < t < b$, then

$$f(t, x) = p(t)x + q(t)$$

and its partial derivative

$$\frac{df(t, x)}{dx} = p(t)$$

are both continuous for t on the interval $a < t < b$ and for all x. Thus, the Fundamental Existence and Uniqueness Theorem 1.1 in Chapter 1 applies to the initial value problem

$$x' = p(t)x + q(t), \quad x(t_0) = x_0 \tag{2.2}$$

for any initial time t_0 from the interval $a < t < b$ and for any initial condition x_0. As a result, we know there exists a unique solution of this initial value problem on some interval containing t_0. As we will see (Section 2.2), it turns out that a stronger existence and uniqueness result holds for linear equations, namely that the solution exists on the *whole* interval $a < t < b$.

It is important to distinguish between linear equations in which the term $q(t)$ is present and those in which it is not.

Definition 2.2 *The linear equation*

$$x' = p(t)x \tag{2.3}$$

is ***homogeneous****. If the term $q(t)$ is not identically equal to 0, the linear equation*

$$x' = p(t)x + q(t) \tag{2.4}$$

is ***nonhomogeneous****. The term $q(t)$ is called the* ***nonhomogeneous term*** *(or* ***forcing function****).*

In many applications a quantity changes at a rate proportional to the amount present at each moment of time. For example, this is true for radioactive sample as it decays over time. Another example is the account balance of a continuously compounded investment. Yet another example is the growth of a bacterial culture in an environment of abundant resources. In all of these cases the rate of change x' is proportional to x, i.e., $x' = px$ for a constant of proportionality p. This is an example of a homogeneous, linear differential equation. Because the coefficient p is constant, the equation is also called *autonomous*. An *autonomous* linear equation is one in which both p and q are constants (so that the independent variable t does not appear in $f = px + q$). Otherwise, the equation is *non-autonomous* . Below is a list of linear equation that arise in applications that we will encounter in examples and exercises.

$$\begin{array}{ll} \text{(a)} & x' = -rx \\ \text{(b)} & x' = g - cx \\ \text{(c)} & x' = ae^{-bt}x \\ \text{(d)} & x' = a\left(b_{av} + \alpha \sin\left(\frac{2\pi}{T}t\right) - x\right) \\ \text{(e)} & x' = c_{in}r_{in} - r_{out}\frac{1}{rt+V_0}x, \text{ where } r = r_{in} - r_{out}. \end{array} \tag{2.5}$$

All coefficients are positive constants. Equation (2.5a) is homogeneous and autonomous since $q(t) = 0$ and $p(t) = p$ are both constants. Since $p(t) = -c$ and $q\,(t) = g$ are constants, equation (2.5b) is also autonomous. Equation (2.5c) is homogeneous and non-autonomous since $p(t) = ae^{-bt}$ and $q\,(t) = 0$. In equation (2.5d)

$$p(t) = -a \quad \text{and} \quad q\,(t) = a\left(b_{av} + \alpha \sin\left(2\pi t/T\right)\right)$$

and therefore this equation is nonhomogeneous and non-autonomous. Equation (2.5e) is also nonhomogeneous and non-autonomous.

2.2 Solution Formulas for Linear Equations

We learn in this section how to calculate formulas for solutions of linear equations and their initial value problems. We begin with homogeneous linear equations.

2.2.1 Homogeneous Linear Equations

A particularly simple example of an homogeneous linear equation is

$$x' = x \tag{2.6}$$

(in which $p(t) \equiv 1$). In recalling the derivatives of basic mathematical functions learned in calculus, the reader will likely remember a function that equals its own derivative, namely, e^t. In fact, the same is true of any constant multiple ce^t of e^t. It is straightforward to check that ce^t satisfies the definition of a solution on the interval $-\infty < t < +\infty$ for any value of c (cf. Definition 1.1). But are there any other solutions to the differential equation (2.6)? To answer that question, consider *any* solution $x(t)$ of the differential equation (2.6) on an interval $a < t < b$. The claim is that this solution must be a constant multiple of e^t. Here's how to see that this is true. Pick an t_0 in the interval $a < t_0 < b$ and choose x_0 to be $x(t_0)$. Then $x(t)$ is a solution of the initial value problem

$$\begin{aligned} x' &= x \\ x(t_0) &= x_0. \end{aligned}$$

However, you can check that $x_0 e^{-t_0} e^t$ is *also a solution* of this same initial value problem. By the Fundamental Existence and Uniqueness Theorem, there is *only one solution* to this initial value problem, and it must then follow that $x(t)$ and $x_0 e^{-t_0} e^t$ are identical, i.e. that $x(t) = x_0 e^{-t_0} e^t$ which is a multiple of e^t! We conclude that the general solution is

$$x(t) = ce^t, \quad c = \text{ an arbitrary constant.} \tag{2.7}$$

Consider now the general homogeneous linear equation

$$x' = p(t)\, x \tag{2.8}$$

where

$$p(t) \text{ is continuous on } a < t < b.$$

There are numerous ways to derive a formula for the general solution of this differential equation. Here we will use a method that is based on a general procedure that we will use many times throughout the course. This procedure is called the *Method of Undetermined Coefficients* (sometimes the *Method of Judicious Guessing*).

Since we will use this method, or variants of it, in several contexts in this course, it is worthwhile pausing for a moment to learn the general idea and steps involved. There are three basic steps to the procedure.

THE METHOD OF UNDETERMINED COEFFICIENTS

Step 1. Make a guess for a solution, one containing undetermined coefficients or terms.

Step 2. Substitute the guess into both sides of the differential equation, perform the required operations on both sides, and equate the results.

Step 3. The resulting equation should allow one to calculate the undetermined coefficients or terms (and hence make the guess into a solution).

(2.9)

The details of the first "guessing" step vary according to the type of equation under consideration. It takes some experience to make a "reasonable" guess (sometimes called an "ansatz") and to become proficient with the method. ("reasonable" here means, of course, that the guess ultimately works and a solution formula is found!)

Here is an example of the procedure. Based on the exponential form (2.7) of the general solution of the simple example equation (2.6), it seems reasonable to wonder if exponential functions, of some kind or other, might play a role in the general solution of the general homogeneous equation (2.8). Thinking along these lines, we could guess for a solution formula of the form

$$x(t) = ce^{P(t)}, \quad c = \text{ an arbitrary constant} \tag{2.10}$$

where $P(t)$ is an undetermined term and c is an undetermined constant. That is to say, the challenge – the "name of the game" – is to find a function $P(t)$ and c so that this guess really is a solution of (2.6). (If it turns out to be impossible, then the guess was not so "reasonable" after all.)

Following Step 2 we substitute our guess into the left side

$$x'(t) = cP'(t)\, e^{P(t)}$$

and then the right side

$$p(t)\, x(t) = cp(t)\, e^{P(t)}$$

of the differential equation (2.8) and equate the answers. In the result

$$cP'(t)\, e^{P(t)} = cp(t)\, e^{P(t)}$$

we cancel $e^{P(t)}$ from both sides to obtain

$$cP'(t) = cp(t). \tag{2.11}$$

Our guess will be a solution if this equation is satisfied.

The final Step 3 is to solve this equation for the undetermined term $P(t)$ and undetermined coefficient c. One way to satisfy this equation is to choose $c = 0$. Our guess (2.10) then becomes a solution

$$x(t) = 0.$$

This constant function (called an *equilibrium* or steady state) is clearly a solution of the differential equation (2.8). To obtain other solutions from our guess, we look to satisfy (2.11) with $c \neq 0$, in which case we can cancel c in (2.11) to obtain the equation

$$P'(t) = p(t)$$

for the undetermined term $P(t)$. It is a calculus problem to solve for

$$P(t) = \int^t p(s)\, ds.$$

This procedure has successfully led us from our guess (2.10) to solutions $ce^{P(t)}$ (on the interval $a < t < b$ where the coefficient $p(t)$ is continuous) where $P(t)$ is an anti-derivative of $p(t)$ and where $c =$ an arbitrary constant.

We have found infinitely many solutions (since c is an arbitrary constant) to the homogeneous linear equation (2.8), but are there any more solutions?

We can handle this question in the same way we did for the motivating example above. Suppose $x(t)$ is *any* solution of (2.8) on some interval $\alpha < t < \beta$ (necessarily contained or equal to $a < t < b$). Pick any t_0 in the interval $\alpha < t_0 < \beta$ and define x_0 to equal $x(t_0)$. Then $x(t)$ satisfies the initial value problem

$$\begin{aligned} x' &= p(t)\, x \\ x(t_0) &= x_0. \end{aligned}$$

However, so does $ce^{P(t)}$ with $c = x_0 e^{-P(t_0)}$ and, as a result of the uniqueness part of the Fundamental Theorem, $x(t)$ and $ce^{P(t)}$ are identical. We conclude that any other possible solution is a constant multiple of $e^{P(t)}$.

Theorem 2.1 ***Suppose*** $p(t)$ ***is continuous on*** $a < t < b$. ***The general solution of the homogeneous linear equation***

$$x' = p(t)\, x$$

is

$$x(t) = ce^{P(t)}, \quad P(t) = \int^t p(s)\, ds \tag{2.12}$$

where $c =$ ***an arbitrary constant. The general solution is defined on the whole interval*** $a < t < b$.

Remark 1. We know from calculus that there are infinitely many anti-derivatives of $p(t)$, and we could write

$$P(t) = k + \int^t p(s)\, ds$$

where k is an arbitrary constant. However, the resulting *set* of solutions

$$x(t) = ce^{k+P(t)} = ce^k e^{P(t)}$$

obtained for all values of c and k is the same as the *set* of solutions (2.12). No new solutions are obtained by adding the arbitrary constant k to the anti-derivative.

Remark 2. From a linear algebraic point-of-view the general solution (2.12) is the span of the solution $e^{P(t)}$. Therefore, we see that the *general solution of a homogeneous linear equation is a one dimensional vector space.*

Example 2.1 *The autonomous homogeneous equation*

$$x' = \left(-1.245 \times 10^{-4}\right) x$$

describes the radioactive decay of a sample $x = x(t)$ of radioactive carbon-14 (C^{14}). An anti-derivative of the coefficient

$$p(t) = -1.245 \times 10^{-4}$$

is

$$P(t) = \left(-1.245 \times 10^{-4}\right) t$$

and the general solution is therefore given by the formula

$$x = ce^{\left(-1.245 \times 10^{-4}\right) t}.$$

where c is an arbitrary constant.

We can calculate a formula for the solution of an initial value problem

$$\begin{aligned} x' &= p(t)\, x \\ x(t_0) &= x_0 \end{aligned}$$

using the general solution (2.12) of the differential equation. Placing $t = t_0$ into (2.12) we find that

$$x_0 = ce^{P(t_0)} \Longrightarrow c = x_0 e^{-P(t_0)}$$

and hence

$$\begin{aligned} x(t) &= x_0 e^{-P(t_0)} e^{P(t)} \\ &= x_0 e^{P(t) - P(t_0)}. \end{aligned}$$

Noting that

$$P(t) - P(t_0) = \int^t p(s)\, ds - \int^{t_0} p(s)\, ds = \int_{t_0}^t p(s) ds$$

we arrive at the following theorem.

Theorem 2.2 ***Suppose*** $p(t)$ ***is continuous on*** $a < t < b$. ***The unique solution of the initial value problem***

$$\begin{aligned} x' &= p(t)\, x \\ x(t_0) &= x_0 \end{aligned}$$

(for $a < t_0 < b$***) is***

$$x(t) = x_0 e^{P(t)}, \quad P(t) = \int_{t_0}^t p(s) ds. \tag{2.13}$$

This solution is defined on the whole interval $a < t < b$.

Example 2.2 *The initial value problem*

$$x' = ae^{-bt}x, \quad x(0) = x_0 > 0$$

arises from a model of tumor growth. The coefficients a and b are positive constants and $x(t)$ is the size of the tumor, which initially is x_0. From

$$p(t) = ae^{-bt}$$

we calculate

$$P(t) = \int_0^t ae^{-bs}ds = \frac{a}{b}\left(1 - e^{-bt}\right).$$

Using formula (2.13) we obtain the solution formula

$$x = x_0 \exp\left(\frac{a}{b}\left(1 - e^{-bt}\right)\right).$$

As a biological punch line, we note that $\lim_{t\to+\infty} x = x_0 \exp(a/b)$. That is to say, this model implies the tumor monotonically increases to a limiting size

$$\lim_{t\to+\infty} x(t) = x_0 \exp\left(\frac{a}{b}\right).$$

2.2.2 Nonhomogeneous Linear Equations

We now consider the problem of calculating a formula for the general solution of a nonhomogeneous linear equation

$$x' = p(t)x + q(t). \tag{2.14}$$

Remember that the general solution is a set of solutions. We begin by asking ourselves what the solutions in this set have in common? Or, put another way, in what way do two solutions differ?

Suppose $x_p(t)$ is a known solution, obtained in some way or another. The subscript p stands for "particular", so think of $x_p(t)$ as one particular solution that we happen to have in hand. (We'll return in a minute to the question of how we might find a solution.) Suppose that $x(t)$ is *any other* solution of (2.14). How do all other solutions $x(t)$ differ from the one solution $x_p(t)$ we know? Consider the difference

$$x(t) - x_p(t).$$

A calculation shows

$$\begin{aligned}[x(t) - x_p(t)]' &= x'(t) - x_p'(t) \\ &= [p(t)x(t) + q(t)] - [p(t)x_p(t) + q(t)]\end{aligned}$$

or, after some algebraic rearrangement

$$[x(t) - x_p(t)]' = p(t)[x(t) - x_p(t)].$$

In other words, the difference $x(t) - x_p(t)$ satisfies the homogeneous equation, which as we saw in the previous section, has general solution $ce^{P(t)}$. Since $x(t)$ is any solution to the nonhomogeneous equation (2.14), we conclude that the set of all solutions has the form

$$x(t) = ce^{P(t)} + x_p(t).$$

We have just reduced the problem of calculating the general solution of a nonhomogeneous equation to two steps:

Step 1. Calculate the general solution $ce^{P(t)}$ of the associated homogeneous equation

Step 2. Find just one solution $x_p(t)$ of the nonhomogeneous equation

Step 3. Add the two answers: $x(t) = x_h(t) + x_p(t)$

Example 2.3 *The associated homogeneous equation $x' = -x$ to the equation*

$$x' = -x + 1$$

has general solution

$$x_h(t) = ce^{-t}$$

where c is an arbitrary constant. Note that the constant function $x_p(t) = 1$ is a solution (check it!). Thus, the general solution is

$$x(t) = ce^{-t} + 1.$$

(Note: a constant solution is called an "equilibrium".)

In general, we cannot expect to find – or "guess" – a particular solution $x_p(t)$ so easily as we did in the example above. There is, fortunately, a procedure to calculate a particular solution. The procedure is called the *Variation of Constants Formula*. This procedure starts with the "judicious guess"

$$x_p(t) = c(t)\, e^{P(t)}$$

where $c(t)$ is an undetermined coefficient. This is another application of the Method of Undetermined Coefficients, but one in which the yet-to-be determined coefficient $c(t)$ is a function of t and not a constant. Indeed, if one chooses $c(t)$ to be a constant then $x_p(t)$ will certain *not* satisfy the nonhomogeneous equation (2.14). This is because when $c(t)$ is a constant $ce^{P(t)}$ satisfies the homogeneous equation, not the nonhomogeneous equation!.

Following the steps (2.9) of the Method of Undetermined Coefficients, we substitute $x_p(t) = c(t)\, e^{P(t)}$ into bother sides of the nonhomogeneous equation, equate the answers, and let the result tell how we choose $c(t)$ to succeed in making our guess into a solution. The details go like this. We equate

$$\begin{aligned} x_p'(t) &= c(t)\, P'(t)\, e^{P(t)} + c'(t)\, e^{P(t)} = c(t)\, p(t)\, e^{P(t)} + c'(t)\, e^{P(t)} \\ p(t)\, x_p(t) + q(t) &= p(t)\, c(t)\, e^{P(t)} + q(t) \end{aligned}$$

to get

$$c(t)\,p(t)\,e^{P(t)} + c'(t)\,e^{P(t)} = p(t)\,c(t)\,e^{P(t)} + q(t)$$

which simplifies to

$$c'(t) = e^{-P(t)}q(t)$$

or

$$c(t) = \int^t e^{-P(u)}q(u)\,du.$$

This is the function $c(t)$ that turns our guess into a solution of the nonhomogeneous equation:

$$x_p(t) = e^{P(t)}\int^t e^{-P(u)}q(u)\,du.$$

Theorem 2.3 ***(The Variation of Constants Formula for general solutions). Suppose*** $p(t)$ ***and*** $q(t)$ ***are continuous on an interval*** $a < t < b$. ***A formula for the general solution of the nonhomogeneous differential equation***

$$x' = p(t)x + q(t)$$

is

$$x(t) = ce^{P(t)} + e^{P(t)}\int^t e^{-P(u)}q(u)\,du. \tag{2.15}$$

The general solution is a solution on the whole interval $a < t < b$.

Example 2.4 *The coefficients of the linear differential equation*

$$x' = -3x + 2e^{-t}$$

are

$$p(t) = -3, \quad q(t) = 2e^{-t}.$$

Since

$$e^{P(t)} = e^{\int^t(-3)ds} = e^{-3t}$$

from the Variation of Constants Formula (2.15) we find the general solution

$$\begin{aligned} x(t) &= ce^{-3t} + e^{-3t}\int^t e^{3u}\left(2e^{-u}\right)du \\ &= ce^{-3t} + e^{-3t}\int^t 2e^{2u}du \\ &= ce^{-3t} + e^{-3t}e^{2t} \\ &= ce^{-3t} + e^{-t}. \end{aligned}$$

Example 2.5 *The velocity* $v = v(t)$ *of an object falling near the surface of the earth, subject to gravity and friction due to air resistance, satisfies the equation*

$$v' = g - kv$$

where k is the (per unit mass) coefficient of friction and g is the (constant) acceleration due to gravity. This is a linear (nonhomogeneous and autonomous) equation with constant coefficients

$$p(t) = -k \quad \textit{and} \quad q(t) = g.$$

An integral of $p(t)$ is

$$P(t) = -kt.$$

Applying the Variation of Constants Formula (2.15) we obtain a formula for the general solution as follows:

$$\begin{aligned} v(t) &= ce^{-kt} + e^{-kt}\int^t e^{ku} g\, du \\ &= ce^{-kt} + e^{-kt}\frac{g}{k}e^{kt} \\ &= ce^{-kt} + \frac{g}{k}. \end{aligned}$$

It is interesting to note that this formula implies $\lim_{t\to+\infty} v = g/k$, that is, the object approaches a limiting velocity g/k (until it hits the ground, of course).

One way to solve an initial value problem

$$x' = p(t)x + q(t), \quad x(t_0) = x_0 \tag{2.16}$$

is to first find the general solution and then use the initial condition $x(t_0) = x_0$ to determine the correct value for the arbitrary constant c.

Example 2.6 *To find a formula for the initial value problem*

$$x' = -3x + 2e^{-t}, \quad x(0) = 5$$

we can use the general solution

$$x(t) = ce^{-3t} + e^{-t}$$

calculated in Example 2.4. From this formula we find that

$$x(0) = c + 1$$

and therefore the initial condition requires $c + 1 = 5$ or $c = 4$. The solution of the initial value problem is

$$x = 4e^{-3t} + e^{-t}.$$

Another way to solve the initial value problem (2.16) is to incorporate the initial condition into the Variation of Constants Formula. In this way, the resulting formula calculates solutions of initial value problems directly, without first having to calculate the general solution.

This is done by using definite integrals instead of anti-derivatives. Specifically, we choose the anti-derivatives

$$P(t) = \int_{t_0}^t p(s)ds \quad \text{and} \quad \int_{t_0}^t e^{P(u)} q(u)\, du$$

to use in the Variation of Constants Formula (2.15). Note that with these choices, both integrals vanish when $t = t_0$. Then from the Variation of Constants Formula

$$x(t) = ce^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(u)} q(u)\, du.$$

evaluated at $t = t_0$, we get $x(t_0) = c$ and it follows that to solve the initial value problem we need to take $c = x_0$.

Theorem 2.4 ***(The Variation of Constants Formula for Initial Value Problems) Suppose*** $p(t)$ ***and*** $q(t)$ ***are continuous on an interval*** $a < t < b$***.*** [1] ***Then the unique solution of the initial value problem***

$$x' = p(t)x + q(t), \quad x(t_0) = x_0$$

where $a < t_0 < b$***, is given by the Variation of Constants Formula***

$$x(t) = x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(u)} q(u) du \tag{2.17}$$

where

$$P(t) = \int_{t_0}^{t} p(s) ds.$$

Example 2.7 *From the Variation of Constants formula (2.17) for the initial value problem*

$$x' = -3x + 2e^{-t}, \quad x(1) = -1$$

with

$$P(t) = \int_{1}^{t} (-3)\, ds = -3(t-1)$$

we get the solution formula

$$\begin{aligned} x(t) &= (-1) e^{-3(t-1)} + e^{-3(t-1)} \int_{1}^{t} e^{3(u-1)} \left(2e^{-u}\right) du \\ &= -e^{-3(t-1)} + e^{-3(t-1)} \int_{1}^{t} 2e^{2u-3} du \\ &= -e^{-3(t-1)} + e^{-3(t-1)} \left[e^{2u-3}\right]_{u=1}^{u=t} \\ &= -e^{-3(t-1)} + e^{-3(t-1)} \left[e^{2t-3} - e^{-1}\right] \end{aligned}$$

or

$$x(t) = \left(-1 - e^{-1}\right) e^{-3(t-1)} + e^{-t}.$$

[1] The continuity of p and q guarantees the differentiability of $P(t)$ and $x(t)$ for on the interval $a < t < b$.

Example 2.8 *According to Newton's Law of Cooling the temperature $x = x(t)$ of an object, residing in an environment of temperature b, satisfies the nonhomogeneous linear equation*

$$x' = a\,(b - x)$$

where coefficient a is positive. Suppose x_0 is the initial temperature of the object. To solve the initial value problem $x(0) = x_0$ we can use either the Variation of Constants Formula (2.17) with $p(t) = -a$ and

$$P(t) = \int_0^t (-a)dt = -at.$$

Using formula (2.17) with $q(t) = ab$ we obtain

$$\begin{aligned} x &= x_0 e^{-at} + e^{-at} \int_0^t e^{as} ab ds \\ &= (x_0 - b)\, e^{-at} + b. \end{aligned}$$

Note $\lim_{t\to+\infty} x(t) = b$ (since $a > 0$). Thus, Newton's Law of Cooling predicts, in the long run, that the temperature of the object will (exponentially) approach that of its environment.

As a final observation, we note that the Variation of Constants Formula (2.17) shows the solution of a linear differential equation exists on the whole interval on which the coefficients $p(t)$ and $q(t)$ of the equation are continuous. This is because the continuity of $p(t)$ and $q(t)$ on an interval $a < t < b$ guarantees the integrals appearing in the formula define differentiable functions on the interval $a < t < b$.

Corollary 2.1 ***If the coefficient*** $p(t)$ ***and nonhomogeneous term*** $q(t)$ ***are continuous on an interval*** $a < t < b$ ***containing*** t_0**, then the solution of the initial value problem** $x' = p(t)x + q(t),\ x(t_0) = x_0,$ ***exists on the whole interval*** $a < t < b$**.**

2.3 Properties of Solutions

Suppose $x_p(t)$ is any particular solution of the nonhomogeneous equation

$$x' = p(t)x + q(t)$$

and suppose $x(t)$ is any other solution of this same equation. Let y denote the difference between these two solutions, i.e.,

$$y = x - x_p.$$

Then

$$\begin{aligned} y' &= x' - x_p' \\ &= p(t)x + q(t) - [p(t)x_p + q(t)] \\ &= p(t)\,(x - x_p) \\ &= p(t)y. \end{aligned}$$

In other words, the difference y is a solution of the homogeneous equation

$$x' = p(t)x.$$

It follows that $y = x - x_p$ must be found in the general solution $ce^{P(t)}$, $P'(t) = p(t)$, of this homogeneous equation. We conclude that *any* solution of the nonhomogeneous equation can be written in the form $x(t) = ce^{P(t)} + x_p(t)$ for some constant c where $x_p(t)$ is any particular solution.

Theorem 2.5 ***The general solution of the nonhomogeneous linear equation***

$$x' = p(t)x + q(t)$$

has the additive decomposition

$$x(t) = x_h(t) + x_p(t)$$

where

$$x_h(t) = ce^{P(t)}, \quad P'(t) = p(t)$$

is the general solution of the associated homogeneous equation

$$x' = p(t)x$$

and x_p is any particular solution of the nonhomogeneous equation.

The additive decomposition

$$x(t) = x_h(t) + x_p(t)$$

of the general solution often provides a shortcut for its calculation.

STEP 1: Calculate the general solution $x_h(t) = ce^{P(t)}$ of the homogeneous equation

$$x' = p(t)x.$$

STEP 2: Find *any* particular solution $x_p(t)$ of the *nonhomogeneous* equation

$$x' = p(t)x + q(t).$$

STEP 3: Add the answers from Step 1 and Step 2 to get the *general* solution $x(t) = x_h(t) + x_p(t)$.

To accomplish Step 1 we must calculate the antiderivative

$$P(t) = \int p(t)\,dt$$

of the coefficient $p(t)$. One way to accomplish Step 2 is to use the particular solution provided by the variation of constants (2.15), namely

$$x_p(t) = e^{P(t)} \int e^{-P(t)} q(t) dt. \tag{2.18}$$

The reason we know this is a solution of the nonhomogeneous equation is because it is the solution we obtain from (2.15) when we choose the arbitrary constant $c = 0$. However, it frequently occurs that shortcuts for finding a particular solution $x_p(t)$ are available that are simpler than calculating the integral required in (2.18). Sometimes there is even an "obvious" solution that can be found be inspection. For example, the equation

$$x' = x - 1$$

has the constant solution (equilibrium solution) $x_p(t) = 1$, as is easily checked by inspection. Therefore, the general solution (noting that $p(t) = 1$ and $P(t) = e^t$) is

$$x(t) = ce^t + 1.$$

Example 2.9 *Consider the nonhomogeneous equation*

$$v' = g - kv$$

for the velocity v of a falling object subject to gravity and a frictional force. (See Example 2.5). The general solution of the associated homogeneous equation $v' = -kv$ is $v_h(t) = ce^{-kt}$. The constant solution $v_p(t) = g/k$ is found by inspection. Therefore the general solution is the sum

$$v(t) = ce^{-kt} + \frac{g}{k}.$$

Perhaps the constant solution in the preceding example ("found by inspection") would not have immediately occurred to the reader. Sometimes there are shortcuts for finding a particular solution $x_p(t)$ which are more systematic than simply guessing. In the next section we illustrate one such method. The method (which involves only algebraic calculations and thereby avoids having to calculate integrals) starts by making a "reasonable guess" for $x_p(t)$ and using the Method of Undetermined Coefficients (2.9) in Section 2.2.1.

Example 2.10 *Suppose a population $x = x(t)$ grows exponentially according to the equation $x' = x$. If we harvest this population at a rate $g(t)$, then*

$$x' = x - h(t).$$

For example if $h(t) = e^{-t}$, then we harvest the population at an exponentially decreasing rate, as time goes by. In this case we have the nonhomogeneous equation

$$x' = x - e^{-t}.$$

The homogeneous equation $x' = x$ associated with this equation has the general solution

$$x_h(t) = ce^t.$$

To find a particular solution $x_p(t)$ of the nonhomogeneous equation we reason as follows. If $x_p' - x_p$ is equal to $-e^{-t}$, then x_p must somehow involve the exponential function e^{-t}. Suppose, then, we try to find a solution $x_p(t)$ that is a constant multiple of e^{-t}. That is, suppose we try to find a constant k such that $x_p(t) = ke^{-t}$ solves the differential equation.

Following the steps in the Method of Undetermined Coefficients (2.9) we substitute this guess into the left side of the differential equation

$$x_p' = -ke^{-t}$$

and into the right side of the differential equation

$$x_p - e^{-t} = (k-1)e^{-t}$$

and equate the results:

$$-ke^{-t} = (k-1)e^{-t}.$$

After canceling e^{-t} from both sides we have $-k = k-1$ or $k = 1/2$. This choice for k yields the particular solution $x_p(t) = e^{-t}/2$ and consequently the general solution

$$x = ce^t + \frac{1}{2}e^{-t}.$$

2.3.1 The Method of Undetermined Coefficients for Linear Differential Equations

To calculate a formula for the general solution

$$x(t) = ce^{P(t)} + x_p(t)$$

of a nonhomogeneous linear equation

$$x' = p(t)\,x + q(t)$$

we need to do two things: calculate the integral

$$P(t) = \int p(t)\,dt$$

and find a particular solution. The variation of constants formula provides a method to calculate

$$x_p(t) = e^{P(t)} \int^t e^{-P(u)} q(u)\,du.$$

The guessing method illustrated in Example 2.10 obtained a particular solution $x_p(t)$ without having to perform any integrations. This shortcut method is available for restricted kinds of nonhomogeneous linear equations, namely, for those equations with the following two properties:

(1) The coefficient $p(t) = p$ is a constant.
(2) The nonhomogeneous term $q(t)$ produces a finite number of independent functions upon repeated differentiations.

Note in Example 2.10 that $p(t) = 1$ is a constant and $q(t) = -e^{-t}$ produces no new independent functions upon repeated differentiations.

When criteria (1) and (2) hold, then we can calculate a particular function $x_p(t)$ as follows.

(a) Calculate the general solution $x_h(t) = e^{pt}$ of the associated homogeneous equation $x' = px$.
(b) List the independent functions obtained from the nonhomogeneous term $q(t)$ by repeated differentiations.
(c) If e^{pt} appears in the list (2), multiple every function in the list by t.
(d) Construct a guess $x_p(t)$ by constructing a linear combination of the list in (c) with constants k_i that are then determined by the Method of Undetermined Coefficients.

Example 2.11 *The nonhomogeneous equation*

$$x' = 2x + te^{-t}$$

has a constant coefficient $p(t) = 2$ *and* $x_h(t) = ce^{2t}$. *A differentiation of* $q(t) = te^{-t}$ *produces* $q'(t) = -te^{-t} + e^{-t}$ *and hence a new independent function* e^{-t}. *Thus, we begin with the list*

$$te^{-t},\ e^{-t}.$$

To complete Step (b) we differentiate e^{-t} *and find that no new independent functions arise. Therefore, our list is complete. We note that* e^{2t} *does not appear in this list. Therefore, our guess is the linear combination*

$$x_p(t) = k_1 te^{-t} + k_2 e^{-t}$$

to which we apply the Method of Undetermined Coefficients (2.9) in Section 2.2.1. Equating

$$x_p'(t) = k_1\left(-te^{-t} + e^{-t}\right) - k_2 e^{-t}$$

to

$$2x_p(t) + te^{-t} = 2\left(k_1 te^{-t} + k_2 e^{-t}\right) + te^{-t}$$

we get

$$k_1\left(-te^{-t} + e^{-t}\right) - k_2 e^{-t} = 2\left(k_1 te^{-t} + k_2 e^{-t}\right) + te^{-t}.$$

Gathering together like terms, we obtain

$$(3k_1 + 1)\,te^{-t} + (-k_1 + 3k_2)\,e^{-t} = 0$$

which (because te^{-t} *and* e^{-t} *are independent) implies*

$$\begin{aligned} 3k_1 + 1 &= 0 \\ -k_1 + 3k_2 &= 0. \end{aligned}$$

These two algebraic equations have the unique solution pair $k_1 = -1/3$ *and* $k_2 = -1/9$. *Thus, we arrive at the particular solution*

$$x_p(t) = -\frac{1}{3}te^{-t} - \frac{1}{9}e^{-t}$$

and the general solution

$$x(t) = ce^{2t} - \frac{1}{3}te^{-t} - \frac{1}{9}e^{-t}.$$

Example 2.12 *Suppose a population $x = x(t)$ grows exponentially according to the equation $x' = x$. If individuals immigrate or emigrate at a rate $q(t)$, then the population changes according to the equation*

$$x' = x + q(t).$$

If $q(t) > 0$, then at time t individuals are being added to the population at the rate $q(t)$. If $q(t) < 0$, then at time t individuals are removed from the population at the rate $q(t)$.

Consider the case when immigration and emigration alternate periodically at the rate $q(t) = \cos t$. Then we have the linear nonhomogeneous equation

$$x' = x + \cos t$$

for x.

Since the associated homogeneous equation $x' = x$ has general solution $x_h(t) = ce^t$, the general solution of the nonhomogeneous equation has the form $x(t) = ce^t + x_p(t)$ where $x_p(t)$ is any particular solution. Since repeated differentiations of $\cos t$ produce only two independent solutions, namely, $\cos t$ and $\sin t$, we begin the Method of Undetermined Coefficients with the list

$$\cos t, \ \ \sin t.$$

Since e^t does not appear in this list, we do not need to multiple the list by t. Our guess is the linear combination

$$x_p(t) = k_1 \sin t + k_2 \cos t. \tag{2.19}$$

Applying the Method of Undetermined Coefficients, we substitute this guess into the differential equation. The result from the left side of the differential equation is

$$x_p'(t) = k_1 \cos t - k_2 \sin t$$

and from the right side is

$$x_p(t) + \cos t = k_1 \sin t + (k_2 + 1) \cos t.$$

Equating these we obtain

$$k_1 \cos t - k_2 \sin t = k_1 \sin t + (k_2 + 1) \cos t$$

or gathering like terms

$$(-k_1 + k_2 + 1) \cos t + (k_1 + k_2) \sin t = 0.$$

Since $\cos t$ and $\sin t$ are independent, we have

$$\begin{aligned} -k_1 + k_2 + 1 &= 0 \\ k_1 + k_2 &= 0 \end{aligned}$$

which is an algebraic system of equations for k_1 and k_2 whose unique solution is

$$k_1 = \frac{1}{2}, \quad k_2 = -\frac{1}{2}.$$

Using these coefficients in (2.19) we obtain the particular

$$x_p(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t$$

and the general solution

$$x(t) = ce^t + \frac{1}{2}\sin t - \frac{1}{2}\cos t.$$

Example 2.13 *The nonhomogeneous equation*

$$x' = x + e^t$$

has a constant coefficient $p(t) = 1$ and $x_p(t) = ce^t$. A differentiation of e^t produces e^t and hence no new independent functions. Thus, we begin with the list of only one function

$$e^t.$$

We note that the solution of the homogeneous equation e^t does appear in our list. Therefore, we multiple every function in the list by t to obtain

$$te^t$$

from which we construct our guess

$$x_p(t) = kte^t.$$

Applying the Method of Undetermined Coefficients (2.9) in Section 2.2.1, we substitute this guess into the differential equation. The left side of the equation produces

$$x_p'(t) = k\left(te^t + e^t\right)$$

and the right side produces

$$x_p(t) + e^t = kte^t + e^t$$

Equating these, we get

$$k\left(te^t + e^t\right) = kte^t + e^t$$

from which we obtain, after gathering together like terms (note the cancellation of the te^t terms!)

$$(k-1)e^t = 0.$$

Thus $k = 1$ and we arrive at the particular solution

$$x_p(t) = te^t$$

and the general solution

$$x(t) = ce^t + te^t.$$

Example 2.14 *The homogeneous equation* $x' = 2x$ *associated with the nonhomogeneous equation*

$$x' = 2x - 2te^t \sin t$$

has general solution $x_h(t) = ce^{2t}$. *The nonhomogeneous term* $q(t) = -2te^t \sin t$ *is a multiple of* $te^t \sin t$. *Using the Method of Undetermined Coefficients we construct a particular solution* $x_p(t)$ *from* $te^t \sin t$ *and all new independent functions (up to linear combinations) that arise by repeated differentiation of* $te^t \sin t$. *From the first derivative of* $te^t \sin t$, *namely*

$$te^t \cos t + te^t \sin t + e^t \sin t$$

we obtain two new independent functions $te^t \cos t$ *and* $e^t \sin t$. *We therefore begin with the list*

$$te^t \sin t,\ te^t \cos t,\ e^t \sin t.$$

We have left to consider the derivatives of the last two functions. The derivative of $te^t \cos t$

$$-te^t \sin t + te^t \cos t + e^t \cos t$$

results in one new independent function, $e^t \cos t$ *and our list expands to*

$$te^t \sin t,\ te^t \cos t,\ e^t \sin t,\ e^t \cos t.$$

Left to consider are independent functions arising from derivatives the last two functions. However, neither derivative results in new independent functions not already in the list. Therefore, the list is complete. Moreover, the homogeneous equation solution e^{2t} *does not appear in the list (so there is no need to multiply by* t*). Our guess is constructed as a linear combination of this finalized list:*

$$x_p(t) = k_1 te^t \sin t + k_2 te^t \cos t + k_3 e^t \sin t + k_4 e^t \cos t.$$

Applying the Method of Undetermined Coefficients, we substitute this guess into the differential equation, obtaining

$$\begin{aligned} x_p'(t) = {} & (k_1 - k_2)\, te^t \sin t + (k_1 + k_2)\, te^t \cos t \\ & + (k_1 + k_3 - k_4)\, e^t \sin t + (k_2 + k_3 + k_4)\, e^t \cos t \end{aligned}$$

from the left side and

$$\begin{aligned} 2x_p(t) - 2te^t \sin t = {} & (2k_1 - 2)\, te^t \sin t + 2k_2 te^t \cos t \\ & + 2k_3 e^t \sin t + 2k_4 e^t \cos t \end{aligned}$$

from the right side. If you equation these two expressions, gather up all like terms and set their coefficients equal to 0, *you obtain the four linear algebraic equations*

$$\begin{aligned} k_1 - k_2 &= 2k_1 - 2 \\ k_1 + k_2 &= 2k_2 \\ k_1 + k_3 - k_4 &= 2k_3 \\ k_2 + k_3 + k_4 &= 2k_4 \end{aligned}$$

to solve for

$$k_1 = 1, \quad k_2 = 1, \quad k_3 = 0, \quad k_4 = 1.$$

Thus, a particular solution is

$$x_p(t) = te^t \sin t + te^t \cos t + e^t \cos t$$

and the general solution $x(t) = x_h(t) + x_p(t)$ *is*

$$x = ce^{2t} + te^t \sin t + te^t \cos t + e^t \cos t.$$

Remark 3. If the list generated by the method used in this section is long, and hence the linear combination of the list that serves as the guess in the Method of Undetermined Coefficients is long, then the calculations involved in this method can be lengthy and tedious. The method is guaranteed to work, but it might not be such a shortcut in this case when compared to using the Variation of Constants Formula to calculate $x_p(t)$. A main advantage of the Method of Undetermined Coefficients is that it involves no integration (as does the Variation of Constants Formula).

Remark 4. The Method of Undetermined Coefficients described in this section cannot be replied upon to work when the coefficient $p(t)$ is not a constant. Thus, we would not use this method to calculate a particular solution of

$$x' = x \sin t + e^t,$$

for example.

Remark 5. Not all functions satisfy the criterion (2). For example, $\ln t$ produces infinitely many independent functions upon repeated differentiation. Thus, we would not use this method to calculate a particular solution of $x' = x + \ln t$, for example.

2.3.2 The Superposition Principle

Another shortcut for calculating a particular solution $x_p(t)$ of a nonhomogeneous linear differential equation involves breaking the equation into two or more "simpler" nonhomogeneous equations. Then particular solutions of the simpler equations are put back together, in just the right way, to obtain a particular solution of the original equation.

This idea is based on what's called the Superposition Principle.

We begin with an example. The nonhomogeneous term of the equation

$$x' = x + 3e^t - 2\cos t, \tag{2.20}$$

namely,

$$q(t) = 3e^t + (-2)\cos t \tag{2.21}$$

is a linear combination of e^t and $\cos t$. We use these two functions to construct the two nonhomogeneous differential equations

$$\begin{aligned} x' &= x + e^t \\ x' &= x + \cos t. \end{aligned}$$

Examples 2.13 and 2.19 in Section 2.3.1 we calculated particular solutions for each of these equations (using the Method of Undetermined Coefficients). The first equation has a particular solution te^t and the second equation has a particular solution

$$\frac{1}{2}\sin t - \frac{1}{2}\cos t.$$

If we form the same linear combination of these two solutions as is formed to construct $q(t)$ in (2.21) we get

$$\begin{aligned} x_p(t) &= 3te^t + (-2)\left(\frac{1}{2}\sin t - \frac{1}{2}\cos t\right) \\ &= 3te^t - \sin t + \cos t. \end{aligned}$$

That this function is actually a solution of the original nonhomogeneous differential equation (2.20) follows from the Superposition Principle, as state in the following theorem. (You can also directly check that this function is a solution by substituting it into the equation.)

Theorem 2.6 ***(Superposition Principle) Suppose, on an interval*** $a < t < b$, $x = x_1(t)$ ***is a solution of the equation***

$$x' = p(t)x + q_1(t)$$

and $x = x_2(t)$ ***is a solution of the equation***

$$x' = p(t)x + q_2(t).$$

Then the linear combination

$$x(t) = k_1x_1(t) + k_2x_2(t)$$

solves the equation

$$x' = p(t)x + [k_1q_1(t) + k_2q_2(t)]$$

on the interval $a < t < b$ ***for any constants*** k_1***.and*** k_2***.***

The student is challenged to prove this theorem in Exercise 2.76.

Notice all three nonhomogeneous equations in this theorem have the same associated homogeneous equation $x' = p(t)x$ and differ only in their nonhomogeneous terms $q_1(t)$, $q_2(t)$ and $k_1q_1(t) + k_2q_2(t)$.

Example 2.15 *Suppose in Example 2.10 the population is harvested at the rate*

$$h(t) = 1 + 9e^{-t}.$$

In this model, the harvesting rate initially starts at $h(0) = 10$ *and then exponentially decreases over time to* 1. *This yields the nonhomogeneous equation*

$$x' = x - h(t)$$

or

$$x' = x - 1 - 9e^{-t}$$

for the population size $x(t)$.

The general solution of the associated homogeneous equation $x' = x$ *is*

$$x_h = ce^t.$$

To find the general solution of the nonhomogeneous equation we need only find a particular solution $x_p(t)$.

We begin by noting that the nonhomogeneous term

$$q(t) = -1 - 9e^{-t}$$

is as a linear combination $k_1 q_1(t) + k_2 q_2(t)$ *of*

$$q_1(t) = 1 \quad \textit{and} \quad q_2(t) = e^{-t}.$$

Specifically

$$q(t) = (-1)\, q_1(t) + (-9)\, q_2(t).$$

We can therefore use the Superposition Principle (Theorem 2.6) to construct a particular solution x_p *by forming the same linear combination of solutions of the equations*

$$x' = x + 1$$
$$x' = x + e^{-t}.$$

The first equation has the constant solution -1. *From Example 2.10,* $e^{-t}/2$ *is a solution of the second equation. Using the Superposition Principle we construct a particular solution as the linear combination*

$$x_p(t) = (-1)(-1) + (-9)\frac{1}{2}e^{-t},$$

that is,

$$x_p = 1 + \frac{9}{2}e^{-t}.$$

Finally, the general solution is the sum $x(t) = x_h(t) + x_p(t)$, *or*

$$x(t) = ce^t + 1 + \frac{9}{2}e^{-t}.$$

The final example uses both the Method of Undetermined coefficients together with the Superposition Principle to find a particular solution.

Example 2.16 *The general solution of the homogeneous equation* $x' = x$ *associated with the nonhomogeneous equation*

$$x' = x - a + b\cos t$$

is

$$x_h = ce^t.$$

Therefore, $x = ce^t + x_p$ is the general solution of this equation if x_p is a particular solution. To find a particular solution x_p we notice that

$$q(t) = -a + b\cos t$$

is a linear combination of $q_q(t) = -1$ and $q_2(t) = \cos t$, i.e.,

$$q(t) = a(-1) + b\cos t.$$

By the Superposition Principle the same linear combination of solutions x_1 and x_2 of the two nonhomogeneous equations

$$\begin{aligned} x' &= x - 1 \\ x' &= x + \cos t \end{aligned}$$

namely,

$$x_p = ax_1 + bx_2.$$

The first equation has the constant solution $x_1 = 1$. From Example 2.12 we find that the second equation has solution

$$x_2 = \frac{1}{2}\sin t - \frac{1}{2}\cos t.$$

Thus,

$$x_p = a\,(1) + b\left(\frac{1}{2}\sin t - \frac{1}{2}\cos t\right)$$

and the general solution is

$$x = ce^t + a + \frac{b}{2}\sin t - \frac{b}{2}\cos t.$$

Theorem 2.6 concerns linear combinations of two solutions of two equations. An analogous principle holds for any number of solutions of any number of equations. For example, if $x_1(t)$, $x_2(t)$, and $x_3(t)$ are solutions of the three linear nonhomogeneous equations

$$\begin{aligned} x' &= p(t)x + q_1(t) \\ x' &= p(t)x + q_2(t) \\ x' &= p(t)x + q_3(t) \end{aligned} \tag{2.22}$$

respectively, then the linear combination

$$x = k_1x_1(t) + k_2x_2(t) + k_3x_3(t)$$

solves the linear nonhomogeneous equation

$$x' = p(t)x + \left[k_1q_1(t) + k_2q_2(t) + k_3q_3(t)\right]. \tag{2.23}$$

The same linear combination is made of the solutions as is made of the nonhomogeneous terms q_1, q_2, q_3. See Exercises 2.77 and 2.78.

2.4 Autonomous Linear Equations

A linear equation is called *autonomous* if both $p(t)$ and $q(t)$ are constant functions:

$$x' = px + q \tag{2.24}$$

where p and q are constants. Although autonomous linear equations are a specialized kind of linear equations, we will nonetheless discuss them briefly in order to introduce several concepts that will play important roles later in our study of nonlinear autonomous equations and systems. We focus on the *asymptotic dynamics* of autonomous equations, i.e., on the behavior of solutions as $t \to \pm\infty$. We need to distinguish between equations (2.24) with $p = 0$ and those with $p \neq 0$.

Definition 2.3 *The autonomous equation $x' = px + q$ is **hyperbolic** if $p \neq 0$. It is **non-hyperbolic** if $p = 0$.*

We begin with two hyperbolic examples. In Figure 2.1a appear graphs of several solutions of the equation

$$x' = -x + 1.$$

These graphs appear to have a horizontal asymptote at $x = 1$. That is to say, the solutions tend to 1 as $t \to +\infty$. The horizontal asymptote is itself the graph of a solution, namely, the constant solution $x = 1$. As $t \to -\infty$ the solutions in Fig 2.1a appear to be unbounded. The formula $x = (x_0 - 1)\, e^{-t} + 1$ for the solution of the initial value problem $x(0) = x_0$ shows that these observations hold for all solutions of the equation.

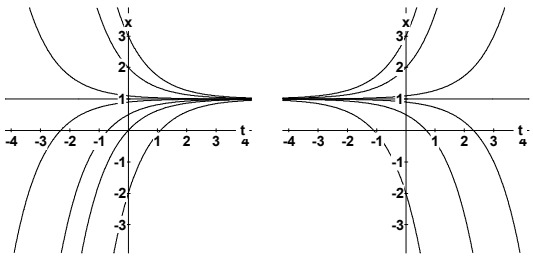
Figure 2.1b shows solution graphs for the equation

$$x' = x - 1.$$

where the asymptotic dynamics are reversed. Solutions appear to have a horizontal asymptote at $x = 1$ as $t \to -\infty$ and are unbounded as $t \to +\infty$. These facts hold for all solutions (except the constant solution $x = 1$), as you can see from the solution formula

$$x\,(t) = (x_0 - 1)\, e^{t} + 1.$$

As we will see, the two examples in Figure 2.1 turn out to be typical for hyperbolic equations.



(a) (b)

Figure 2.1. (a) Selected solutions of $x' = -x + 1$. (b) Selected solutions of $x' = x - 1$

The general solution of the homogeneous equation $x' = px$ associated with equation (2.24) is $x_h = ce^{pt}$. To find the general solution all we need is a particular solution. A hyperbolic equation (2.24) has a very special particular solution, namely the constant solution $x = -q/p$.

Definition 2.4 *A constant solution is called an* ***equilibrium solution*** *or simply an* ***equilibrium***[2]***.***

We will denote equilibrium solutions by x_e. Using the equilibrium

$$x_e = -\frac{q}{p}$$

as a particular solution, we can have the formula

$$x = ce^{pt} - \frac{q}{p}, \quad c = \text{arbitrary constant.}$$

for general solution of (2.24). From this general solution we obtain the formula

$$x = \left(x_0 + \frac{q}{p}\right) e^{p(t-t_0)} - \frac{q}{p} \tag{2.25}$$

for the solution of the initial value problem $x(t_0) = x_0$.

Using (2.25) we discover the following general facts about hyperbolic linear equations. If $p < 0$ then all solutions are strictly monotonic (i.e., strictly increasing or decreasing, depending on the sign of the coefficient $x_0 + q/p$) and tend to the equilibrium $x_e = -q/p$ as $t \to +\infty$. (This is because the exponential term is strictly monotonic and tends to 0 when $p < 0$.) In this case, the equilibrium is called an *attractor* (or a sink) and is called *stable*.

On the other hand, if $p > 0$ then all non-equilibrium solutions ($x_0 \neq -q/p$) are strictly monotonic and are exponentially unbounded as $t \to +\infty$. (This is because the exponential term is strictly monotonic and unbounded when $p > 0$.) In this case, the equilibrium is called a repeller (or a *source*) and is called *unstable*.

Theorem 2.7 ***If the linear autonomous equation***

$$x' = px + q$$

is hyperbolic (i.e., if $p \neq 0$***), then non-equilibrium solutions are strictly monotonic (i.e., are strictly increasing or decreasing). The equilibrium*** $x_e = -q/p$ ***is an attractor if*** $p < 0$ ***and a repeller*** *if* $p > 0$.

Note that simply checking the sign of the coefficient p is sufficient to obtain a great deal of information about all solutions. It is not necessary to solve the differential equation in order to determine whether the equilibrium is an attractor or a repeller. Here is an example involving an attractor.

[2] An equilibrium is sometimes called a *rest point*, a *critical point* or a *singular point*.

Example 2.17 *The equation*

$$v' = g - kv$$
$$g > 0, \quad k > 0$$

for the velocity $v = v(t)$ of an object falling under the influence of gravity and a frictional force kv is linear and autonomous. Since the coefficient of friction $k > 0$ is positive, the equation coefficient $p = -k$ is negative and the equilibrium $v_e = g/k$ is an attractor. Thus, all solutions $v(t)$ tend to v_e as $t \to +\infty$ in a strictly monotonic fashion. If, for example, the object is dropped, so that $v(0) = 0$, its velocity strictly increases to the "limiting velocity" v_e as $t \to +\infty$.

The next example involves a repeller.

Example 2.18 *The linear autonomous equation*

$$x' = rx - h$$
$$r > 0, \qquad h \geq 0$$

describes the dynamics of a harvested population $x = x(t)$. The constant h is the rate at which the population is harvested. The population has a positive (per unit) growth rate r when unharvested. That is, when $h = 0$ the population grows exponentially according to the equation $x' = rx$. Since the coefficient $p = r$ is positive, the equilibrium $x_e = h/r$ is a repeller. This means all non-equilibrium populations are exponentially unbounded, growing without bound for initial population densities $x(0) > x_e$ and decreasing without bound for $x(0) < x_e$. In the latter case, $x(t)$ equals 0 and the population becomes extinct at some finite time $t > 0$.

In the non-hyperbolic case, when $p = 0$, the autonomous equation (2.24) becomes

$$x' = q.$$

The solution of the initial value problem $x(t_0) = x_0$ associated with this equation is

$$x = q(t - t_0) + x_0.$$

By inspection of this formula we obtain the following theorem for the non-hyperbolic case.

Theorem 2.8 ***If the linear autonomous equation***

$$x' = px + q$$

is non-hyperbolic (i.e., if $p = 0$***), then all solutions are equilibrium solutions if*** $q = 0$ ***or are (linearly) unbounded if*** $q \neq 0$.

Theorems 2.7 and 2.8 completely account for the asymptotic dynamics of the linear autonomous equation (2.24). They show that *the asymptotic dynamics an autonomous linear equation can be determined from an inspection of the coefficient p and nonhomogeneous term q alone.* There is no need to solve the equation (i.e., find a formula for solutions).

The coefficient p plays a determining role. If $p < 0$ then all solutions tend to the equilibrium $x_e = -q/p$; one the other hand, if $p > 0$ then all non-equilibrium solutions tend away from the equilibrium and are unbounded.

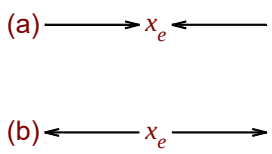


Figure 2.2. Phase line portraits of (a) an attractor and (b) a repeller.

These facts are conveniently summarized in a graphical manner by a *phase line portrait*, which is drawn as follows. Locate the equilibrium x_e on an x-axis (called the "phase space" or, in this case, the "phase line"). The equilibrium point separates the phase line into two half lines. If $p < 0$ place two arrows on these half lines that point toward the equilibrium. These directed half lines are called *orbits*. The resulting picture is the phase line portrait of an attractor. See Figure 2.2a. If $p > 0$ the two arrows are reversed and the resulting phase line portrait is that of a repeller. See Figure 2.2b. Phase line portraits are also important for nonlinear equations and are studied in more detail in Chapter 3.

As an example, the phase line portraits for the equations in the Examples 2.17 and 2.18 are, respectively,

$$\rightarrow \frac{g}{k} \leftarrow \qquad \text{and} \qquad \leftarrow \frac{h}{r} \rightarrow$$

Imagine varying (or "tuning") the coefficient p in the linear autonomous equation (2.24) from $-\infty$ to $+\infty$. For $p < 0$ the equilibrium is an attractor. However, as soon as p is increased through 0 and becomes positive, the attractor changes into a repeller and the arrows reverse their directions in the phase line portrait.

A drastic change in asymptotic dynamics as a coefficient passes through a critical value is called a *bifurcation*. The critical value of the coefficient is called a *bifurcation value*. In this context the coefficient is called a *bifurcation parameter*. With respect to the bifurcation parameter p, the linear autonomous equation $x' = px + q$ undergoes a bifurcation at the bifurcation value $p = 0$ (q being held fixed). Notice the equation is non-hyperbolic at this bifurcation value. This is not a coincidence; it is typically the case, as we will see in Chapter 3 where these bifurcation concepts are extended to nonlinear equations.

2.5 Chapter Summary

A linear first order differential equation for an unknown function $x = x(t)$ has the form $x' = p(t)x + q(t)$. If the coefficients $p(t)$ and $q(t)$ are continuous functions on an interval $a < t < b$, then an extended Fundamental Existence and Uniqueness Theorem implies that any initial value problem $x(t_0) = x_0$ (with $a < t_0 < b$) has a unique solution on the entire interval $a < t < b$. A formula for the general solution is provided by the Variations of Constants Formula. An adaptation of this formula gives the unique solution to an initial value problem. The general solution has the additive decomposition $x(t) = x_h(t) + x_p(t)$ where $x_h(t) = ce^{P(t)}$ is the general solution of the associated homogeneous equation $x' = p(t)x$ and $x_p(t)$ is any particular solution of the nonhomogeneous equation. In some cases shortcut methods exist for finding a particular solution $x_p(t)$ that utilize the Superposition Principle and the Method of Undetermined Coefficients. When p and q are constants the equation is called autonomous. If $p \neq 0$ the equation is hyperbolic. If $p < 0$ all solutions tend to

the equilibrium $x_e = -q/p$, which is then called an attractor. If $p > 0$ all non-equilibrium solutions are unbounded and the equilibrium is called a repeller. These facts are summarized by means of phase line portraits.

2.6 Exercises

Which of the following equations are linear and which are nonlinear? For those equations that are linear, identify the coefficient $p(t)$ and nonhomogeneous term $q(t)$.

Exercise 2.1 $x' - t^2 x = t$

Exercise 2.2 $x' + t^2 x = t^2$

Exercise 2.3 $x' - tx^2 = t^2$

Exercise 2.4 $x' + x = \sin t$

Exercise 2.5 $x' = tx + \sin x$

Exercise 2.6 $x' = -e^t x + \sqrt{x}$

Exercise 2.7 $xx' = 2x + 1$

Exercise 2.8 $x \sin t + 1/t - x' = 1$

Exercise 2.9 $t^2 x - \cos 3t + x \sin t - 5x' = (t^2 + 1)^{-1}$

Exercise 2.10 $t^2 x - \cos 3t + t \sin x - 5x' = (t^2 + 1)^{-1}$

Which of the following equations are linear homogeneous? Which are linear nonhomogeneous? Which are nonlinear? For the linear equations identify the coefficient $p(t)$ and nonhomogeneous term $q(t)$.

Exercise 2.11 $x' = t^2 x - 1$

Exercise 2.12 $x' = (t + x) / (t - 1)$

Exercise 2.13 $x' - t^2 x = 0$

Exercise 2.14 $x' = (t + x)(t - x)^{-1}$

Exercise 2.15 $e^t x' = 3x - t$

Exercise 2.16 $2tx' = -e^t x$

Exercise 2.17 *Identify the coefficient $p(t)$ and nonhomogeneous term $q(t)$ for the equations of (2.5). Which equations are homogeneous and which are nonhomogeneous? Which equations are autonomous?*

Find a formula for the general solution of the following linear homogeneous equations.

Exercise 2.18 $x' = -3x$

Exercise 2.19 $x' = -\frac{1}{2}x$

Exercise 2.20 $x' = t^{-1}x$

Exercise 2.21 $x' = tx$

Exercise 2.22 $x' = e^{-3t}x$

Exercise 2.23 $x' = x \sin 2t$

Exercise 2.24 $x' = t\left(1 + t^2\right)^{-1} x$

Exercise 2.25 $x' = \left(t - t^2\right)^{-1} x$

Exercise 2.26 $\frac{x'}{x} = t \sin t$

Exercise 2.27 $tx - e^t x' = 0$

Find a formula for the general solution of the following linear homogeneous equations. (a and b are constants.)

Exercise 2.28 $x' = x/a$, $a \neq 0$

Exercise 2.29 $x' = bx \cos at$

Exercise 2.30 $x' = e^{at}x$

Exercise 2.31 $x' = a\left(a + t\right)^{-1} x$

Use the Variation of Constants Formula (2.15) to calculate a formula for the general solution of the linear nonhomogeneous equations below. (a and b are constants.)

Exercise 2.32 $x' = -2x + 12$

Exercise 2.33 $x' = 3x - 4$

Exercise 2.34 $x' - tx = t$

Exercise 2.35 $x' = -t^{-1}x + \sin t$

Exercise 2.36 $x' = ax + \cos bt$

Exercise 2.37 $x' = ax + \sin bt$

Exercise 2.38 $tx' = -x + t^{1/3}$

Exercise 2.39 $tx' = -x - t^2$

Exercise 2.40 $x' = tx - 1$

Exercise 2.41 $x' = x\cos t - 2^{-1}\sin 2t$

Exercise 2.42 *Prove all integrals $P(t)$ lead to the same general solution x of $x' = p(t)x + q(t)$.*

Find formulas for the solutions of the following initial value problems. (a and b are constants.)

Exercise 2.43 $x' = \pi x, \quad x(1) = -2$

Exercise 2.44 $x' = -3x/2, \quad x(-2) = 3$

Exercise 2.45 $x' = (1+t^2)^{-1}\, x, \quad x(1) = e^{\pi}$

Exercise 2.46 $x' = (\sec^2 t)\, x, \quad x\,(\pi/3) = e^{\sqrt{3}}$

Exercise 2.47 $x' = (\sin at)\, x, \quad x(0) = 1,\ a \neq 0$

Exercise 2.48 $x' = (a + bt^{-1})\, x, \quad x(1) = 1$

Find formulas for the solutions of the following initial value problems, in which a and b are constants. Do this two ways. (a) Use the Variation of Constants Formula (2.15) to find the general solution and use it so find the unique solution of the initial value problem. (b) Use the Variation of Constants Formula (2.17) for initial value problems.

Exercise 2.49 $x' = 3x - 2,\ x(0) = 5$

Exercise 2.50 $x' = -2x + 6,\ x(0) = -1$

Exercise 2.51 $x' = t^{-1}x + t^3,\ x(2) = 0$

Exercise 2.52 $x' = -t^{-1}x + \sqrt{t},\ x(1) = 1$

Exercise 2.53 $x' = x\cos at + b\cos at,\ x(0) = 0,\ a \neq 0$

Exercise 2.54 $x' = 2atx + bt,\ x(0) = 0\ a \neq 0$

Exercise 2.55 $x' = -t^{-1}x + 2\,(1+t^2)^{-1},\ x(1) = \ln 8$

Exercise 2.56 *Find a formula for the solution of the initial value problem $x' = (b-d)x$, $x(0) = p_0$, for a population of size $x = x(t)$ which has a per capita birth rate $b > 0$ and a per capita death rate $d > 0$.*

Exercise 2.57 *Show the length of time it takes the solution of the initial value problem* $x' = rx$, $x(t_0) = x_0 > 0$ *(where* $r > 0$*) to double its initial size* x_0 *is independent of the initial condition* x_0*. Find a formula for this "doubling" time.*

Exercise 2.58 *Show the length of time it takes the solution of* $x' = -rx$, $x(t_0) = x_0 > 0$ *(where* $r > 0$*) to decrease by* 50% *is independent of the initial condition* x_0*. Find a formula for this "halving" time.*

Exercise 2.59 *Suppose a population* $x = x(t)$ *naturally grows at an exponential rate* r*, i.e.,* $x' = rx$*. However, suppose the population also is subject to harvesting (removal of individuals) at a constant rate* $h > 0$*. Then* $x' = rx - h$*. Find a formula for the solution of the initial value problem*

$$x' = rx - h, \quad x(0) = x_0.$$

Exercise 2.60 *Suppose in Exercise 2.59 that the population is harvested periodically at the rate* $h = h_{av} + \alpha \sin(2\pi t/T)$*. Here* $h_{av} > 0$ *is the average rate over one harvesting period of length* $T > 0$*. Find a formula for the solution of the resulting initial value problem.*

Exercise 2.61 *Suppose the temperature* $x = x(t)$ *of an object satisfies the nonhomogeneous (non-autonomous) linear equation*

$$x' = a\left(b_{av} + \alpha \sin\left(\frac{2\pi}{T}t\right) - x\right), \quad a > 0.$$

This equation arises from Newton's Law of Cooling when the environmental temperature $b_{av} + \alpha \sin(2\pi t/T)$ *oscillates sinusoidally with period* T*, average* b_{av}*, and amplitude* α*. Suppose* x_0 *is the initial temperature of the object.*

(a) Find a formula for the solution $x(t)$ *of the initial value problem* $x(0) = x_0$*.*

(b) For large $t > 0$ *and some constant* θ *show*

$$x(t) = b_{av} + \frac{Ta}{\sqrt{a^2T^2 + 4\pi^2}}\alpha \sin\left(\frac{2\pi}{T}t - \theta\right).$$

(c) Use the result in (b) to discuss the relationship between the oscillating temperature of the environment and that of the object.

Exercise 2.62 *(a) Use a computer to investigate the solutions of the equation* $x' = e^{-0.5t}x$*. What properties do all solutions have in common as* $t \to +\infty$*? What differences do solutions have as* $t \to +\infty$ *?*

(b) Find a formula for the solution of the initial value problem $x(0) = x_0$ *and it to verify your observations in (a).*

Exercise 2.63 *Consider the initial value problem*

$$x' = (500 \sin 600\pi t)\, x, \quad x(0) = 1.$$

(a) Use a computer program to approximate $x(0.2)$*.*

(b) Use a computer program to graph the solution on the interval $0 \le t \le 0.2$. *Describe the important features of the graph.*

(c) Find a formula for the solution of the initial value problem and use it to calculate $x(0.2)$. *Compare your answer with that obtained in (a).*

(d) Describe the important features of the graph of the solution found in (c) and compare your answer with your description in (b).

Exercise 2.64 *Consider the initial value problem*

$$x' = \frac{1}{t}x + te^{-t}, \quad x(-1) = 0.$$

(a) Notice the right hand side of the differential equation is not defined for $t = 0$. *Use a computer to obtain a graph of the solution on the interval* $-1 < t < 0$. *Describe the graph. In particular, what happens as* $t \to 0-$?

(b) Use a computer to obtain approximations to the solution at $t = -0.1$, -0.01, -0.001 *and* -0.0001. *How do these approximations compare to your answer in (a)?*

(c) Find a formula for the solution of the initial value problem and use it to compute $\lim_{t\to 0-} x(t)$. *How does your answer compare with your answers in (a) and (b)?*

Exercise 2.65 *Consider the initial value problem*

$$x' = (\cos 60\pi t)x + 100\cos 60\pi t, \quad x(0) = 0.$$

(a) Use a computer program to approximate the solution x *at* $t = 0.99$.

(b) Use a computer program to graph the solution on the interval $0 \le t \le 1$. *Describe the important features of the graph.*

(c) Find a formula for the solution of the initial value problem and use it to calculate $x(0.99)$. *How does your answer compare with that in (a)?*

(d) Use the formula obtained in (c) to explain the features of the graph found in (b).

Exercise 2.66 *Consider the equation* $x' = -x - 15\sin 5t + 3\cos 5t$.

(a) Use a computer program to find the graph of what appears to be a periodic solution. Do this by investigating solution graphs for many initial conditions $x(0) = x_0$. *What is the approximate period and amplitude of this periodic solution? What relationship do all other solutions have to this periodic solution?*

(b) Find a formula for the solution of the initial value problem $x(0) = x_0$.

(c) Use the formula found in (b) to show there is exactly one periodic solution. (HINT: show the formula gives a periodic function if and only if one special value of x_0 *is chosen.) What is the period and amplitude of this periodic solution? Do your answers compare favorably to your answers in (a)?*

Exercise 2.67 *Given that* $x_p = t^{100}e^t$ *solves* $x' = x + 100t^{99}e^t$ *find the general solution of this equation.*

Exercise 2.68 *Given that* $x_p = t\,(1+t)^{-1}$ *solves* $x' = (\cos t)\,x + (1 + 2t + t^2)^{-1}\,(1 - t\cos t - t^2\cos t)$ *find the general solution of this equation.*

Exercise 2.69 *Given that* $x_1 = 2e^{-3t}$ *solves* $x' = -x - 4e^{-3t}$ *and that* $x_2 = e^t$ *solves* $x' = -x + 2e^t$ *find the general solution of the equation* $x' = -x - 4e^{-3t} + 2e^t$.

Exercise 2.70 *Given that* $x_1 = \sin t$ *solves* $x' = 2x + \cos t - 2\sin t$ *and that* $x_2 = \ln t$ *solves* $x' = 2x - 2\ln t + \frac{1}{t}$ *find the general solution of the equation* $x' = 2x + 2\cos t - 4\sin t + 2\ln t - t^{-1}$.

Exercise 2.71 *Given that* $x_p = 10$ *solves the differential equation, find the solution of the initial value problem* $x' = x - 10$, $x(0) = 5$.

Exercise 2.72 *Given that* $x_p = e^t$ *solves the differential equation, find the solution of the initial value problem* $x' = e^{-t}x + e^t - 1$, $x(0) = 0$.

Exercise 2.73 *Given that* $x_p = h/r$ *solves the differential equation, find the solution of the initial value problem* $x' = rx - h$, $x(0) = 0$.

Exercise 2.74 *Given that* $x_p = 2 + a\left(a^2 + 1\right)(a\sin t - \cos t)$ *solves the differential equation, find the solution of the initial value problem* $x' = -a\left(x - 2 - \sin t\right)$, $x(0) = 0$.

Exercise 2.75 *Suppose* x_1, x_2, ... , x_m *are* m *solutions of the linear homogeneous equation* $x' = p(t)x$. *Show the linear combination* $x = k_1x_1 + k_2x_2 + \cdots + k_mx_m$ *is also a solution for any constants* k_1, k_2,..., k_m.

Exercise 2.76 *Prove Theorem 2.6.*

Exercise 2.77 *If* $x_1(t)$, $x_2(t)$, *and* $x_3(t)$ *are solutions of the three linear nonhomogeneous equations (2.22) respectively, show* $x = k_1x_1(t) + k_2x_2(t) + k_3x_3(t)$ *is a solution of (2.23) for any constants* k_1, k_2, k_3.

Exercise 2.78 *If* $x_i(t)$ *solves* $x' = p(t)x + q_i(t)$ *for* $i = 1, 2, \ldots n$, *show that a linear combination* $x = \sum_{i=1}^{n} k_ix_i(t)$ *solves the equation* $x' = p(t)x + \sum_{i=1}^{n} k_iq_i(t)$.

For each equation below, determine whether or not the Method of Undetermined Coefficients applies.

Exercise 2.79 $x' = 2x + t^2e^{-t}$

Exercise 2.80 $x' = -x + 2t^{-3}\sin 2t$

Exercise 2.81 $x' = tx + \frac{1}{2}t\sin t$

Exercise 2.82 $x' = 5x + t\cos t$

Exercise 2.83 $x' = -\pi x + ae^{2t}$, *a is a constant*

Exercise 2.84 $x' = x - 3e^t$

Exercise 2.85 $x' = -2x - te^{-2t}$

Exercise 2.86 $x' = e^t x - 5e^{2t}$

Exercise 2.87 $x' = x - 3te^{bt}$, *b is a constant*

Exercise 2.88 $x' = px + t^2 e^{bt}$, *p and b are constants*

Use the Method of Undetermined Coefficients to (a) formulate a guess for a particular solution $x_p(t)$ of each equation below. (b) Use your guess to find a particular solution $x_p(t)$. (p and a are constants.)

Exercise 2.89 $x' = 0.5x - 0.3e^{0.2t}$

Exercise 2.90 $x' = 4x - 3e^{\pi t}$

Exercise 2.91 $x' = 3x - 15e^{3t}$

Exercise 2.92 $x' = -2x - 3e^{-2t}$

Exercise 2.93 $x' = -\frac{2}{3}x - \frac{15}{16}e^{-t}\sin t$

Exercise 2.94 $x' = 0.1x + 2te^{-t}\cos t$

Exercise 2.95 $x' = -x + 5t^4 \cos 2t$

Exercise 2.96 $x' = 2x - 3t^3 e^t \sin 5t$

Exercise 2.97 $x' = px + \frac{1}{3}t^3 e^{at}$

Exercise 2.98 $x' = px - 14t^2 e^{at}$

Exercise 2.99 $x' = x + 2\cos 2t$

Exercise 2.100 $x' = 2x + \sin 3t - \cos 3t$

Exercise 2.101 *Find a formula for the general solution of the equation*

$$x' = 2x + ae^t + bte^{2t}$$

where a and b are constants.

Use the Superposition Principle and the Method of Undetermined Coefficients to find a formula for a particular solution $x_p(t)$ of the following equations.

Exercise 2.102 $x' = -x + 2e^t - 3\sin t$

Exercise 2.103 $x' = -x - 0.5t + 3e^{2t}$

Exercise 2.104 $x' = x + 3e^t - 4\cos t$

Exercise 2.105 $x' = 2x - e^{2t} + 0.5e^t$

Under what conditions (i.e., for what values of the parameter a) are the following equations hyperbolic? Under what conditions are they non-hyperbolic? Explain your answers. In all equations a is a constant.

Exercise 2.106 $x' = x \sin a + 5a$

Exercise 2.107 $x' = x \cos a + a$

Exercise 2.108 $x' = a^2 x + 5$

Exercise 2.109 $x' = (1 - a^2)\, x + \sin a$

Without solving the equations, find the equilibrium solutions and determine the asymptotic dynamics of all other solutions as $t \to +\infty$. (a is a constant.)

Exercise 2.110 $x' = -5x - 7$

Exercise 2.111 $x' = -2x + 4$

Exercise 2.112 $x' = 2x - 10$

Exercise 2.113 $x' = 6x + 7$

Exercise 2.114 $x' = (a - 1)x + 2$

Exercise 2.115 $x' = (a^2 - 4)x - 1$

Exercise 2.116 $x' = \sin a$

Exercise 2.117 $x' = \ln(a - 1)\,,\ a > 1$

Describe what solutions do when $t \to -\infty$ when the equation $x' = px + q$ is :

Exercise 2.118 *hyperbolic and the equilibrium is an attractor?*

Exercise 2.119 *hyperbolic and the equilibrium is a repeller?*

Exercise 2.120 *is non-hyperbolic and* $q \neq 0$*?*

Exercise 2.121 *is non-hyperbolic and* $q = 0$*?*

Exercise 2.122 *A population* $x = x(t)$ *with per capita birth and death rates* $b > 0$ *and* $d > 0$ *satisfies the equation* $x' = bx - dx$*. Suppose we harvest the population at constant rate* $h \geq 0$*. Then* $x' = bx - dx - h$*. With respect to the parameter* b*, are there any bifurcation points and, if so, what are they?*

Exercise 2.123 *Fix the coefficient $p < 0$ in the linear autonomous equation (2.24). Does the equation undergo a bifurcation (i.e., a qualitative change in the asymptotic dynamics of its solutions) as the nonhomogeneous term q is varied from $-\infty$ to $+\infty$? Explain your answer. Does your answer change if $p > 0$?*

Exercise 2.124 *Use a computer program to study the asymptotic dynamics of the equation $x' = (a - e^{-a})\, x + 1$ where $a > 0$ is a constant. Formulate conjectures about the dependence of the asymptotic dynamics on a. Are there any bifurcation points? Solve the equation and prove (or disprove) your conjectures.*

Exercise 2.125 *Use a computer program to study the asymptotic dynamics of the equation $x' = (a^2 - 3a + 2)\, x + 1$ where $a > 0$ is a constant. Formulate conjectures about the dependence of the asymptotic dynamics on a. Are there any bifurcation points? Solve the equation and prove (or disprove) your conjectures.*

Find a formula for the general solution of the following homogeneous differential equations. (a, b, and r are constants.)

Exercise 2.126 $x' - atx = 0$

Exercise 2.127 $x' = (1 + t^2)^{-1}\, x$

Exercise 2.128 $x' + te^{-t^2/2}x = 0$

Exercise 2.129 $x' = a(1 + \cos bt)x$

Exercise 2.130 $x' = ae^{rt}x$

Exercise 2.131 $x' = ate^{rt^2}x$

Exercise 2.132 $x' = x \ln t$

Exercise 2.133 $x' = ax \ln bt$

Exercise 2.134 $x' = x \tan t$

Exercise 2.135 $x' = (r + ae^{bt})x$

Find a formula for the general solution of the following nonhomogeneous differential equations by means the Variation of Constants Formula. (a, b and β are constants.)

Exercise 2.136 $x' = x + \cos t$

Exercise 2.137 $x' = -x + \sin t$

Exercise 2.138 $x' = x \cos \beta t + \cos \beta t$

Exercise 2.139 $x' = at^{-1}x + b$

Exercise 2.140 $x' = x \ln t + t^t$

Exercise 2.141 $x' = (1 + t^{-1})x + t$

Exercise 2.142 $x' = x + e^{at}$

Exercise 2.143 $x' = x + te^{at}$

Find a formula for the general solution of the following equations making use of the Method of Undetermined Coefficients.

Exercise 2.144 $x' = x + \sin 10t$

Exercise 2.145 $x' = -2x + te^{2t}$

Exercise 2.146 $x' = -2x + te^{-2t}$

Exercise 2.147 $x' = x + 2t^2 e^{-2t}$

Write down the appropriate guess for a particular solution $x_p(t)$ of the following equations. Do not solve for $x_p(t)$.

Exercise 2.148 $x' = x - 2t^3 e^{-t} \cos 2t$

Exercise 2.149 $x' = -x - 2t^3 e^{-t} \cos 2t$

Use the Superposition Principle together with the Method of Undetermined Coefficients to find a formula for the general solution of the following equations.

Exercise 2.150 $x' = x + 2 \sin 10t + 3 \cos t$

Exercise 2.151 $x' = -2x + 3te^{2t} - 5te^{-2t}$

Exercise 2.152 $x' = 3x + 2 + 3t - t^2 + 6t^3$

Exercise 2.153 $x' = -x + 2 - 3e^{-t} + \cos t$

Exercise 2.154 $x' = -x + 3 \sin t + 2 \sin 2t$

Exercise 2.155 $x' = x - \cos 4t + 2 \cos 2t$

Find a formula for the solution of the following initial value problems.

Exercise 2.156 $x' - tx = 0,\ x(5) = \pi$

Exercise 2.157 $x' = (1 + t^2)^{-1} x,\ x(1) = 1$

Exercise 2.158 $x' = x + \sin 10t,\ x(0) = 0$

Exercise 2.159 $x' = x + \sin 10t$, $x(0) = x_0$

Exercise 2.160 $x' = -2x + te^{2t}$, $x(0) = x_0$

Exercise 2.161 $x' = -2x + te^{2t}$, $x(0) = -1$

Suppose the size of a tumor grows according to the following law: the per unit size rate of growth is a decreasing function of time $p(t)$. Then $x'/x = p(t)$ or $x' = p(t)x$. Solve the initial value problem $x(0) = x_0$ for each of the growth rates below. In all cases $a > 0$, $b > 0$ and p_0 are constants.

Exercise 2.162 $p(t) = ate^{-bt}$

Exercise 2.163 $p(t) = p_0 + ae^{-bt}$

Exercise 2.164 $p(t) = a\,(1+t)^{-1}$

Exercise 2.165 $p(t) = p_0 + a\,(1+t)^{-1}$

Exercise 2.166 $p(t) = a\,(1+t^2)^{-1}$

Exercise 2.167 $p(t) = p_0 + a\,(1+t^2)^{-1}$

Exercise 2.168 *A chemical substance is dissolved in a fluid contained in a container. Fluid flows into the container at a rate $r_{in} > 0$ and out of the container at rate $r_{out} > 0$. The concentration of the substance in the entering fluid is c_{in}. If we denote the amount of the substance in the container at time t by $x(t)$, then x satisfies the equation*

$$x' = c_{in}r_{in} - \frac{x}{rt + V_0}r_{out}$$

where $r = r_{in} - r_{out}$ and $V_0 \geq 0$ is the initial amount of fluid in the container. If the container initially contains no chemical substance, then $x(0) = 0$. Suppose the volume of the container is 10 and the initial volume of fluid is $V_0 = 2$.

(a) Find a formula for the general solution when the volume of fluid in the container remains constant, i.e., $r_{in} = r_{out}$. Then solve the initial value problem. Use your answer to find the long term concentration in the container.

(b) Suppose the inflow rate is twice the outflow rate, i.e., $r_{in} = 2r_{out}$. Find the general solution. Then find a formula for the solution of the initial value problem. What is the concentration at the moment the container is full? Is it more or is it less than the concentration in the incoming fluid?

Suppose a population $x = x(t)$ is grows (decays) exponentially according to the equation $x' = rx$. Suppose this population is subject to immigration (seeding) and emigration (harvesting) at rate $q(t)$. Interpret the immigration/ emigration rate $q(t)$ and then find a formula for the solution of the initial value problem $x(0) = x_0$.

Exercise 2.169 $q(t) = e^{-t}\sin t$

Exercise 2.170 $q(t) = te^{-t}$

Exercise 2.171 $q(t) = 1 + 2\cos t$

Exercise 2.172 $q(t) = -1 + 2\cos t$

Exercise 2.173 *Consider the initial value problem*

$$x' = \left(100e^{\sin 40\pi t}\cos 40\pi t\right)x, \quad x(0) = 1.$$

(a) Use a computer program to approximate the solution x at $t = 2/3$.

(b) Use a computer program to graph the solution. Describe the important features of the graph.

(c) Find a formula for the solution of the initial value problem and use your answer to calculate x at $t = 2/3$. Compare this answer with the results in (a).

(d) Describe the important features of the graph of the solution found in (c) and compare your answer with your description in (b).

Exercise 2.174 *Consider the initial value problem*

$$x' = 100t\cos(100t^2)x, \quad x(0) = 1.$$

(a) Use a computer program to approximate the value $x(1)$ of the solution at $t = 1$.

(b) Use a computer program to graph the solution on the interval $0 \le t \le 1$. Describe the important features of the graph.

(c) Find a formula for the solution of the initial value problem and use is to calculate $x(1)$. Compare the answer with that obtained in (a).

(d) Describe the important features of the graph of the solution found in (c) and compare your answer with your description in (b).

Without finding a formula for the solution of the equation determine the asymptotic dynamics of each of the following autonomous equations. Draw the phase line portrait. In Exercise 2.181 - 2.184, in which a is a constant, find all bifurcation values.

Exercise 2.175 $x' = -0.5x + 1$

Exercise 2.176 $x' = x - 3$

Exercise 2.177 $x' = -x + 2$

Exercise 2.178 $x' = 0.01x - 1$

Exercise 2.179 $x' = -\pi x + 7$

Exercise 2.180 $x' = \frac{3}{2}x - e$

Exercise 2.181 $x' = (2a-1)x + 1$

Exercise 2.182 $x' = -x + e^a$

Exercise 2.183 $x' = (a^2-1)x + 1 + a$

Exercise 2.184 $x' = (\ln a)x - \ln 2$, $a > 0$

Exercise 2.185 *(a) Use a computer program to investigate the asymptotic dynamics as $t \to +\infty$ of the equation $x' = (e^{-a} - a^2)x - 1$, $0 < a < 1$. Describe how the asymptotic dynamics depend on the parameter a. Find (approximately) any bifurcation points for a.*

(b) Formulate a conjecture about the phase line portraits as they depend on a.

(c) Prove or disprove your conjectures by utilizing the theorems in the Sec. 2.4.

For each equation find a formula for a periodic solution. Discuss the asymptotic dynamics as $t \to +\infty$ of all other solutions.

Exercise 2.186 $x' = x + \sin 10t$

Exercise 2.187 $x' = x + \sin 10t + \cos 10t$

Exercise 2.188 $x' = -x + \sin 10t + \cos t$

Exercise 2.189 $x' = -3x + 2\sin t \cos t$

Find a formula for the general solution of the following equations in which p is a constant.

Exercise 2.190 $x' = px + 2\sin 2\pi t$

Exercise 2.191 $x' = px - 3\cos t$

Exercise 2.192 *Consider the initial value problem equation $x' = -ax + 0.5\sin 2\pi t$, $x(0) = 0$, where $a > 0$ is a constant.*

(a) Use a computer program to study the solution for selected values of a, ranging from $a = 0.1$ to 50. The solutions will approach periodic oscillations as $t \to +\infty$. Formulate conjectures about how the period, amplitude and phase of this oscillation depend on a. Relate these properties of the oscillation to the nonhomogeneous nonhomogeneous term $q(t) = 0.5\sin 2\pi t$.

(b) Find a formula for the solution of the initial value problem.

(c) Use the formula in (b) to prove (or disprove) your conjectures in (a).

Exercise 2.193 *In this exercise you are asked to prove the linear equation $x' = px + q(t)$ has exactly one periodic solution of period T when $p \neq 0$ and $q(t)$ is a periodic solution of period T.*

(a) Prove $x(t)$ is a periodic solution if $x(0) = x(T)$. (Hint: show the function $y(t) = x(t+T)$ is a solution.)

(b) Use the Variation of Constants Formula to prove there exists exactly one initial condition x_0 for which the condition in (a) holds.

(c) Prove $p < 0$ implies all other solutions tend to the unique periodic solution as $t \to +\infty$. Prove $p > 0$ implies no other solution tends to the unique periodic solution as $t \to +\infty$ (and are in fact exponentially unbounded).

(d) Apply these results to the equation $x' = (-0.3)x + (2 + \cos 2\pi t)^{-1}$. Use a computer program to study and then describe the periodic solution. Can you find a formula for the periodic solution?

Chapter 3

Nonlinear First Order Equations

In Chapter 2 we learned to solve and analyze linear first order differential equations. We now turn our attention to nonlinear equations. In general, nonlinear equations are more difficult to solve and analyze than are linear equations. The Fundamental Existence and Uniqueness Theorem 1.1 in Chapter 1 tells us, under very general conditions, that nonlinear equations do have solutions. For specialized types of nonlinear equations mathematicians have developed methods for obtaining formulas for the solutions. Although such methods can be useful, applications frequently do not involve equations of these specialized types. In these cases, we must turn to other methods of analysis.

In an application that involves a differential equation one generally wants to answer specific questions about solutions. For example, one may want to know whether the solution has zeros or not; whether it is increasing or decreasing; whether it has maxima or minima; whether it is periodic; whether its graph has an asymptote; and so on. If we can "solve" the equation, in the sense of obtaining a formula for solutions, then we could use the solution formula to answer such questions. This approach is possible only for those special types of equations for which solution methods are available and tractable. Otherwise we will have to use other methods to obtain answers to our questions. It turns out, in fact, that methods are available for certain types of analysis that are often much easier to use even when solution formulas are readily obtainable. This chapter begins with a study of a very important class of nonlinear equations, called "autonomous" equations, for which this is the case. In this chapter we will see that a great deal can be learned about solutions of autonomous equations directly from the equation itself, without the need of a solution formula. These "qualitative" methods of analysis serve as a basic introduction to the modern theory of dynamical systems.

Of course solution formulas, when available, can be useful. Autonomous equations (studied in Sec. 3.1) are a special case of so-called "separable" equations for which a method is available to calculate solution formulas. The solution method for separable equations is covered in Sec. 3.2. Finally, in Sec. 3.3 we study one classic method for calculating formulas that approximate solutions.

3.1 Autonomous Equations

An important class of first order differential equations are those in which the independent variable t does not explicitly appear on the right hand side :

$$x' = f(x).$$

Equations of this kind are called *autonomous*. (See Exercise 3.1.) In Chapter 2 we studied *linear* autonomous equations (when $f(x) = px + q$ for constants p and q). Examples of *nonlinear* autonomous equation are

$$\begin{aligned} x' &= r\left(1 - \frac{x}{K}\right)x \\ x' &= -a\,(b - x)\,|b - x|^p \\ v' &= 9.8 - k_0 v^2, \end{aligned}$$

provided all the parameters (r, K, a, b, p and k_0) are constants. The first equation arises in applications from many disciplines, including population dynamics, and is called the "logistic" equation. The second equation arises in the study of the heating and cooling of objects. The third equations describes the motion of an object falling near the surface of the earth under the influence of gravity and frictional forces.

An equation that is not autonomous is called *non-autonomous* . The independent variable t appears explicitly in a non-autonomous equation. For example, the linear equation $x' = x + \sin t$ is non-autonomous. Another example arises from the logistic equation when one of its constants r or K is replaced by a function of t. For example, the non-autonomous equation

$$x' = r\left(1 - \frac{x}{K_0 + a\sin\left(\frac{2\pi}{T}t\right)}\right)x$$

arises in population dynamics.

3.1.1 Basic Properties of Solutions

Consider an autonomous differential equation

$$x' = f(x). \tag{3.1}$$

We assume $f(x)$ is defined and continuously differentiable on some interval I of x values. Then, for each initial conditions x_0 from the interval I, the fundamental Existence and Uniqueness Theorem 1.1 implies the initial value problem $x(t_0) = x_0$ has a unique solution (for any t_0). For most equations we study, $f(x)$ is continuously differentiable for all x or, in other words, the interval I is the entire real number line.

It is possible for an autonomous equation (3.1) to have a constant solution, that is to say, to have a solution $x(t) = x_e$ where x_e is a real number. (The graph of such a solution is a horizontal straight line.) Since the derivative of a constant equals zero, a constant solution $x(t) = x_e$ must satisfy $f(x_e) = 0$. In other words, constant solutions correspond to the roots of $f(x)$.

Definition 3.1 *A constant solution of (3.1) is called an* ***equilibrium*** *(or a rest point or a critical point). Equilibria are the roots of* $f(x)$.

To find the equilibria of an autonomous equation (3.1) we must solve the *equilibrium equation*

$$f(x) = 0.$$

Example 3.1 *The equilibrium equation of the autonomous differential equation*

$$x' = \frac{1}{3}\left(1 - x^3\right)$$

is

$$\frac{1}{3}\left(1 - x^3\right) = 0.$$

This polynomial has one (real) root $x = 1$ *and therefore the differential equation has one equilibrium* $x_e = 1$.

Another example is the equation

$$x' = x^2 - 1.$$

whose equilibrium equation

$$x^2 - 1 = 0$$

has two roots ± 1. *This differential equation has two equilibria*

$$x_e = 1 \quad \textit{and} \quad x_e = -1.$$

The equilibrium equations of some differential equations are not easily solved by hand. Graphic methods or numerical approximation methods (using a computer or calculator) often help determine the equilibria of a differential equation.

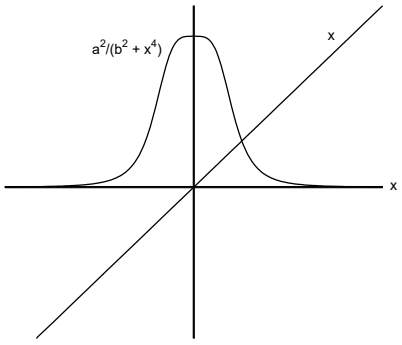


Figure 3.1

Example 3.2 *Consider the autonomous equation*

$$x' = \frac{a^2}{b^2 + x^4} - x$$

in which a *and* b *are nonzero constants. It is not possible to solve the equilibrium equation*

$$\frac{a^2}{b^2 + x^4} - x = 0$$

for x *algebraically. However, rewriting the equation as*

$$\frac{a^2}{b^2 + x^4} = x$$

and plotting the right hand side and the left hand side on the same graph, we see that these graphs intersect in exactly one point (no matter what the nonzero values of a *and* b *are). See Figure 3.1.*

For particular cases in which a *and* b *have specified numerical values, we can use a computer or hand calculator to obtain accurate approximations to the root of the equilibrium equation. For example, with* $a = b = 1$ *the equilibrium equation becomes*

$$\frac{1}{1+x^4} - x = 0.$$

Using a computer or calculator, we find the root of this equation to be approximately $x_e \approx 0.7549$.

We now turn our attention to non-equilibrium solutions of autonomous equations. Unlike equilibrium solutions, non-equilibrium solutions might not be defined for all t. We saw examples in Chapter 1 (Examples 1.3 and 1.7). We denote the *maximal interval* on which a solution $x = x(t)$ is defined by $\alpha < t < \beta$. This means there is no larger interval containing $\alpha < t < \beta$ on which the solution $x = x(t)$ is defined. For equilibria $\alpha = -\infty$ and $\beta = +\infty$. For non-equilibrium solutions one or both of the end points α and β may be finite.

We begin our look at non-equilibrium solutions with two motivating examples.

Example 3.3 *Consider the autonomous equation*

$$x' = \frac{1}{3}\left(1 - x^3\right). \tag{3.2}$$

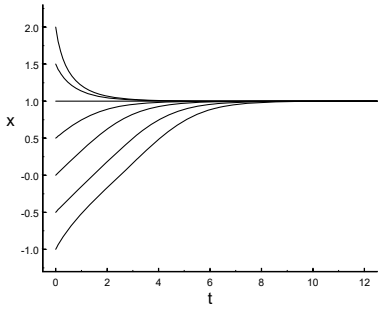


Figure 3.2

Figure 3.2 shows computer generated graphs of solutions for a selection of initial values $x(0) = x_0$. *Further computer experimentation, using a wider selection of initial values, will show these graphs are typical. The non-equilibrium graphs in Figure 3.2 all appear to approach the limit* 1 *as* $t \to +\infty$ *and to do so in a strictly monotonic fashion (i.e., the solutions are either strictly increasing or strictly decreasing). Note that* $x_e = 1$ *is the equilibrium solution of equation (3.2). These numerical examples and observations encourage us to conjecture that all non-equilibrium solutions monotonically approach the equilibrium* $x_e = 1$ *as* $t \to +\infty$. *We must remain cautious in making this conjecture, however, since computer examples cannot prove general statements like this. This is because it is possible to investigate only a finite number of examples and also because one cannot be sure what the graphs of solution are like outside the display window.*

In Example 3.3 we conjectured, on the basis of some computer explorations, that all non-equilibrium solutions of the equation (3.2) monotonically approach an equilibrium as $t \to +\infty$. This is one possibility for non-equilibrium solutions of autonomous equations; there are others, however. For example, all non-equilibrium solutions $x(t) = x_0 e^t$ of the linear autonomous equation $x' = x$ are monotonic, but do not approach the equilibrium $x_e = 0$. Instead they are unbounded, either increasing without bound ("blowing up") if $x_0 > 0$ or decreasing without bound ("blowing down") if $x_0 < 0$ as $t \to +\infty$. For nonlinear autonomous equations some non-equilibrium solutions approach an equilibrium while others are unbounded. Here is an example.

Example 3.4 *The differential equation*

$$x' = x^2 - 1$$

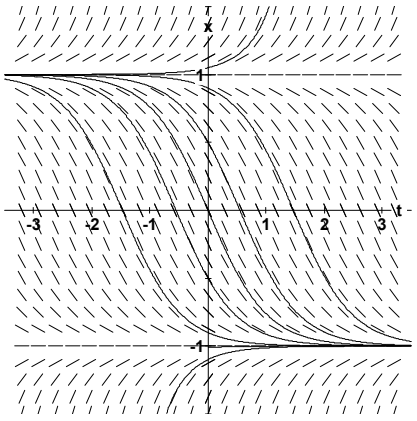


Figure 3.3. The slope field associated with equation $x = x^2 - 1$ and graphs of selected solutions.

has equilibria $x_e = -1$ and $x_e = 1$. We can use a computer to explore the graphs of selected non-equilibrium solutions. Figure 3.3 shows the slope field and several solution graphs. The graphs of the equilibria are horizontal straight lines. The non-equilibrium solutions displayed in Figure 3.3 are monotonic. Those with initial values $x(0) = x_0$ lying between the equilibria -1 and 1 are decreasing and those with initial values outside this interval are increasing. From these computer explorations we conjecture that all non-equilibrium solutions with initial values between -1 and 1 are decreasing (approaching the equilibrium $x_e = -1$ as $t \to +\infty$). We also conjecture that all other non-equilibrium solutions are increasing; those with initial conditions $x_0 < -1$ appear to approach the equilibrium $x_e = -1$ and those with $x_0 > 1$ appear to be unbounded as $t \to +\infty$.

While the computer explorations in Examples 3.4 and 3.3 do not prove the conjectured monotonicity of non-equilibrium solutions for those nonlinear autonomous equations, the conjectures turn out to be true. In fact the conjecture is true for all autonomous differential equations, as the following theorem asserts.

Theorem 3.1 ***(Monotonicity of Solutions) Assume $f(x)$ is continuously differentiable. All non-equilibrium solutions of the autonomous equation $x' = f(x)$ are strictly monotonic, i.e., non-equilibrium solutions are either strictly increasing or strictly decreasing. The solution of the initial value problem***

$$x' = f(x), \quad x(t_0) = x_0$$

is an equilibrium if $f(x_0) = 0$, is strictly increasing if $f(x_0) > 0$ or is strictly decreasing if $f(x_0) < 0$.

To see why this theorem is true consider a non-equilibrium solution $x = x(t)$ of the autonomous equation (3.1) on the (maximal) interval $\alpha < t < \beta$. First, note that this solution can never, for any value of t, equal an equilibrium x_e. (In other words, the graph of the solution $x = x(t)$ in the x, t-plane can never intersect the horizontal straight line graph of an equilibrium x_e.) The reason for this is as follows. Suppose $x(t)$ did equal x_e at some value of t, say at $t = t^*$. Then there would exist two different solutions of the initial value problem

$$x' = f(x), \quad x(t^*) = x_e$$

namely, the non-equilibrium solution $x(t)$ and the equilibrium solution x_e itself. This would contradict the Existence and Uniqueness Theorem 1.1 (specifically, the uniqueness assertion

of the Theorem). From this contradiction we conclude that $x(t)$ cannot equal an equilibrium, that is to say a root of $f(x)$, for any value of t. In other words, we have shown that $f(x(t)) \neq 0$ for all t. This means either $x'(t) = f(x(t)) > 0$ or $x'(t) = f(x(t)) < 0$ on the whole interval $\alpha < t < \beta$. In the first case the solution $x = x(t)$ is strictly increasing and in the second case it is strictly decreasing. This is the conclusion of Theorem 3.1.

Example 3.5 *For the equation $x' = (1 - x^3)/3$ in Example 3.3 an initial value $x_0 < 1$ implies $f(x_0) = (1 - x_0^3)/3 > 0$ and Theorem 3.1 implies the solution is strictly increasing. On the other hand, an initial value $x_0 > 1$ implies $f(x_0) < 0$ and the solution is therefore strictly decreasing. This application of Theorem 3.1 proves the monotonicity conjecture concerning non-equilibrium made in Example 3.3.*

For the equation $x' = x^2 - 1$ in Example 3.4 we see $f(x_0) = x_0^2 - 1 < 0$ for $-1 < x_0 < 1$. By Theorem 3.1 solutions associated with these initial conditions are strictly decreasing. For initial conditions satisfying $x_0 < -1$ or $x_0 > 1$ we see $f(x_0) > 0$ and the solutions are strictly increasing.

In Examples 3.3 and 3.4 we conjectured that non-equilibrium solutions either approach an equilibrium or increase (decrease) without bound as t increases. We now investigate these alternatives for solutions of general autonomous equation.

A function $x = x(t)$, defined on an interval $\alpha < t < \beta$, is said to be *bounded above* if there is a number M such that $x(t) \leq M$ for all t from the interval. It is *bounded below* if there is a number m such that $m \leq x(t)$ holds on the interval.

If $x = x(t)$ is increasing (or decreasing) on an interval $\alpha < t < \beta$ and is bounded above (or below), then the limit

$$\lim_{t \to \beta} x(t) = x_L$$

exists. If $x = x(t)$ is increasing (or decreasing) on an interval $\alpha < t < \beta$ and is not bounded above (or below), then we write

$$\lim_{t \to \beta} x(t) = +\infty \text{ (or } -\infty).$$

Here we allow the possibility that $\beta = +\infty$.

The exponential $x = e^t$ is an example of a solution of an autonomous equation (namely, $x' = x$) that is not bounded above on its interval of definition $-\infty < t < +\infty$. It is, however, bounded below (with $m = 0$ for example). On the other hand, solution $x = -e^t$ is bounded above (with $M = 0$), but not below.

It is also possible that a solution is not be bounded above (i.e., it may "blow up") even when its interval of definition is finite. For example, $x = \tan t$ is a solution of the equation $x' = x^2 + 1$ on the interval $-\pi/2 < t < \pi/2$. This solution is neither bounded above (since $\lim_{t \to \pi/2} \tan t = +\infty$) nor bounded below (since $\lim_{t \to -\pi/2} \tan t = -\infty$). Graphically the solution has vertical asymptotes at $t = \pm\pi/2$.

Now we turn our attention to *bounded* solutions of an autonomous equation $x' = f(x)$. Since non-equilibrium solutions are either strictly increasing or decreasing there are two cases to consider as $t \to \beta$, namely, increasing solutions that are bounded above and decreasing solutions that are bounded below. We consider increasing solutions bounded above. Decreasing solutions bounded below can be treated in a similar way.

Let $x = x(t)$ be an increasing solution of $x' = f(x)$ that is bounded above on its interval of definition $\alpha < t < \beta$. Exercise **??** shows that such a solution must exist for all $t > \alpha$, i.e., $\beta = +\infty$. Let x_L denote the limit

$$\lim_{t\to+\infty} x(t) = x_L.$$

Because $x(t)$ is increasing and approaches a limit (in other words, its graph approaches a horizontal asymptote) the derivative $x'(t)$ must approach 0. Thus,

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} f\left(x(t)\right) = 0.$$

On the other hand, because $f(x)$ is a continuous function of x, we have that

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} f\left(x(t)\right) = f\left(\lim_{t\to+\infty} x(t)\right) = f(x_L).$$

We conclude that $f\left(x_L\right) = 0$. In other words, the limit x_L is a root of $f(x)$ and hence is an equilibrium of the differential equation.

We have shown that an increasing solution of an autonomous equation, bounded above, must approach an equilibrium as $t \to +\infty$. Conversely, if there is an equilibrium x_e greater than the initial value x_0 of an increasing solution, then the solution is certainly bounded above (take $M = x_e$). In summary, an increasing solution is bounded above if and only if there is an equilibrium greater than its initial value x_0.

Similar facts about decreasing functions that are bounded below can be derived in an analogous way. We summarize these findings in the following theorem.

Theorem 3.2 ***(Asymptotic Dynamics) Assume*** $f(x)$ ***is continuously differentiable for all*** x ***and consider the initial value problem***

$$\begin{aligned} x' &= f(x) \\ x(t_0) &= x_0. \end{aligned}$$

Let $x = x(t)$ ***be a non-equilibrium solution (i.e., assume*** $f(x_0) \neq 0$***) and let*** $\alpha < t < \beta$ ***be its maximal interval of existence.***

If $f(x_0) > 0$ ***then*** $x = x(t)$ ***is strictly increasing and one of the following alternatives holds:***

(a) if there is no equilibrium greater than x_0***, then*** $\lim_{t\to\beta} x(t) = +\infty$;

(b) if there is equilibrium greater than x_0***, then*** $\beta = +\infty$ ***and*** $\lim_{t\to+\infty} x(t) = x_e$ ***where*** x_e ***is the smallest such equilibrium.***

If $f(x_0) < 0$ ***then*** $x(t)$ ***is strictly decreasing and one of the following alternatives holds:***

(c) if there is no equilibrium smaller than x_0***, then*** $\lim_{t\to\beta} x(t) = -\infty$***;***

(d) if there is equilibrium x_e ***smaller than*** x_0***, then*** $\beta = +\infty$ ***and*** $\lim_{t\to+\infty} x(t) = x_e$ ***where*** x_e ***is the largest such equilibrium.***

In cases (a) and (c), β may be finite or $+\infty$. In these cases, the solution "blows up" (or "blows down") in either a finite amount of time or as $t \to +\infty$ respectively.

Using Theorem 3.2, we can prove the conjectures made in Examples 3.3 and 3.4.

Example 3.6 *For the equation*

$$x' = \frac{1}{3}\left(1 - x^3\right)$$

in Example 3.3, $f(x) = (1 - x^3)/3$ *has only one root, namely* $x_e = 1$*. Since* $f(x_0) < 0$ *for an initial value* $x_0 > 1$ *the solution, by part (d) of Theorem 3.2, decreases to* $x_e = 1$ *as* $t \to +\infty$*. Since* $f(x_0) > 0$ *for an initial value* $x_0 < 1$*, the solution, by part (b) of Theorem 3.2, increases to* $x_e = 1$ *as* $t \to +\infty$*.*

For the equation

$$x' = x^2 - 1$$

in Example 3.4, $f(x) = x^2 - 1$ *has two roots, namely* -1 *and* 1*. Moreover,* $f(x)$ *is negative between these roots and positive elsewhere. Therefore, a solution with initial value* $x_0 < -1$*, by part (b) of Theorem 3.2, increases to* $x_e = -1$ *as* $t \to +\infty$*. The solution for an initial value* x_0 *between* -1 *and* 1*, by part (d) of Theorem 3.2, decreases to* $x_e = -1$ *as* $t \to +\infty$*. Finally, the solution for an initial value* $x_0 > 1$*, by part (a) of Theorem 3.2, increases without bound as* $t \to \beta$*. (Exercise 3.117 shows* β *is finite in this case.)*

A similar investigation can also be made of non-equilibrium solutions for decreasing t (i.e., as $t \to \alpha$). The result is that non-equilibrium solutions either approach an equilibrium as $t \to -\infty$ or "blow up" (or "down") as $t \to \alpha$.

The monotonicity and limiting properties of solutions described in Theorem 3.2 are called the *asymptotic dynamics* of the equation $x' = f(x)$. Using this theorem, *all we need to do in order to determine the asymptotic dynamics of an autonomous equation is find the roots of* $f(x)$ *and determine the sign of* $f(x)$ *between the roots.*

3.1.2 Phase Line Portraits

In this section we seek a convenient graphical way to summarize the asymptotic dynamics of an autonomous equation. We do this by means of the *phase line portrait*. We begin with an example.

Consider the equation $x' = x^2 - 1$ in Example 3.4, for which $f(x) = x^2 - 1$. From the graph of $f(x)$ in Figure 3.4 we see that the two roots -1 and 1 divide the x-axis into three disjoint intervals :

$$t < -1, \quad -1 < t < 1 \quad \text{and} \quad 1 < t.$$

On the interval $-1 < t < 1$, where $f(x)$ is negative (i.e., where its graph is below the x-axis), place an arrow pointing to the left. This arrow indicates that solutions with initial values in this interval are decreasing. On the two half-line intervals $t < -1$ and $t > 1$ where $f(x)$ is positive (i.e., where its graph of $f(x)$ is above the x-axis) place an arrow pointing to the right. This arrow indicates that solutions with initial values in these intervals are increasing. The result appears in Figure 3.4. Extracting the x-axis from this graph, we obtain a line divided into

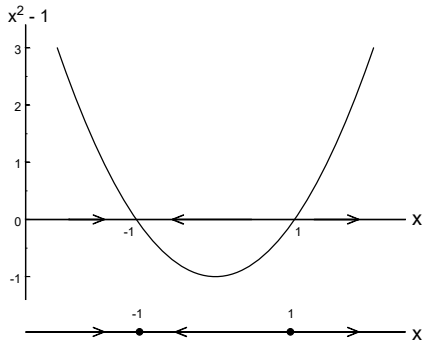


Figure 3.4. The phase line portrait for the equation $x' = x^2 - 1$.

subintervals with orientations indicated by arrows. This line is the "phase line portrait" associated with the equation $x' = x^2 - 1$. It summarizes the asymptotic dynamics of this equation.

We can apply the procedure used to obtain the phase line portrait of the equation $x' = x^2 - 1$ to any autonomous equation $x' = f(x)$. All that we need know are the roots of $f(x)$ and the signs of $f(x)$ between the roots. This even includes the case when $f(x)$ has infinitely many roots, provided they are isolated. A root is *isolated* if it can be placed at the center of an interval in which there are no other roots. The function $f(x) = \sin x$, with roots at integer multiples of π, is an example.

We assume from now on that the roots of $f(x)$ are isolated.

Definition 3.2 *The **phase line portrait** associated with an autonomous equation $x' = f(x)$ consists of those subintervals of the x-axis created by the equilibria (i.e., the roots of $f(x)$) together with arrows pointing to the right if $f(x)$ is positive on a subinterval or to the left if $f(x)$ is negative on a subinterval.*

The phase line portrait summarizes the asymptotic dynamics of an autonomous equation. If the initial value x_0 lies in a subinterval with an arrow pointing to the right, then the solution with this initial value strictly increases and approaches the right hand end point of the subinterval (which may be $+\infty$) as t increases. If the initial value x_0 lies in a subinterval with an arrow pointing to the left, then the solution with this initial condition strictly decreases and approaches the left hand end point of the interval as t increases (which may be $-\infty$). (Note: the asymptotic dynamics for decreasing t are summarized in the phase portrait obtained by reversing the orientation arrows.)

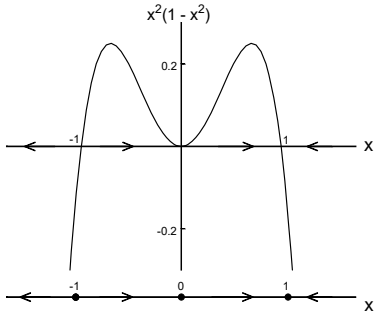


Figure 3.5. The phase line portrait for the equation $x' = x^2\,(1 - x^2)\,.$

Example 3.7 *The roots of $f(x) = x^2(1 - x^2)$ are -1, 0, and 1. These three roots are the equilibria of the equation*

$$x' = x^2(1 - x^2)$$

and they determine four subintervals of the x-axis. The sign of $f(x)$ on each of these intervals can be deduced from the formula $f(x) = x^2(1 - x^2)$ or from the graph shown in Figure 3.5.[1] *Either way we obtain the phase line portrait shown in Figure 3.5.*

From the phase line portrait we see, for example, that an initial condition $x_0 > 1$ implies the solution $x(t)$ is decreases to 1 as $t \to +\infty$. Or, an initial condition satisfying $0 < x_0 < 1$ implies $x(t)$ is increases to 1 as $t \to +\infty$ and approaches 0 as $t \to -\infty$.

There is a connection between the phase line portrait of the equation $x' = f(x)$ and the graphs of solutions $x = x(t)$ in the t, x-plane. Equilibrium points are the (horizontal) projections of the horizontal line plots of the equilibria onto the (vertical) x-axis. The subintervals

[1] Sometimes test points selected from the subintervals are useful. For example, in this case we calculate $f(1/2) = 3/16 > 0$ and conclude that $f(x)$ is positive on the subinterval between 0 and 1.

of the phase line portrait are the projections of non-equilibrium solution graphs onto the x-axis. Thus, the subintervals in the phase line portrait are the ranges of non-equilibrium solutions. This is illustrated in Figure 3.6 for the equation $x' = x^2 - 1$. In Figure 3.6 arrows have been added to the graphs of the solutions, indicating the direction of increasing t, so that one can see how the monotonicities of solutions determine the orientations in the phase line portrait.

Definition 3.3 *The range of a solution $x = x(t)$ of an autonomous equation $x' = f(x)$, together with an orientation in the direction of increasing t, is called the* ***orbit*** *associated with the solution.*

The roots of $f(x)$, which are points on the phase line portrait, are the orbits associated with equilibrium solutions. Notice, however, that many different non-equilibrium solutions project to the same orbit. In general, infinitely many solutions share the same orbit.

An example is seen in Figure 3.6 where all solutions with initial values between -1 and 1 project onto the same orbit, namely the subinterval $-1 < x < 1$. The equation $x' = x^2 - 1$ has infinitely many solutions, but only five orbits: two equilibrium (point) orbits and three non-equilibrium (subinterval) orbits.

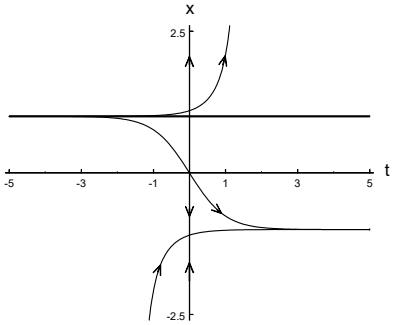


Figure 3.6. Horizontal projections of solution graphs of an equation $x' = f(x)$ produce the phase line portrait on the (vertical) x-axis. The oriented subintervals in the phase line portrait are the orbits of the equation.

Example 3.8 *Orbits of the equation*

$$x' = x^2(1 - x^2)$$

are displayed the phase line portrait in Figure 3.5. There are four non-equilibrium orbits, namely the four intervals

$$x < -1, \quad -1 < x < 0, \quad 0 < x < 1, \quad 1 < x$$

oriented to the left, right, right, and left respectively. There are also three equilibrium orbits at the equilibria $x_e = -1$, 0, and 1 for a total of seven orbits.

In the following example there are infinitely many equilibria and non-equilibrium orbits.

Example 3.9 *The roots of $f(x) = \sin x$ are*

$$x_e = n\pi, \quad n = 0, \ \pm 1, \ \pm 2, \ldots . .$$

These equilibrium orbits of the equation $x' = \sin x$ determine infinitely many non-equilibrium orbits, namely the intervals between integer multiples of π, i.e. $n\pi < x < (n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$. The phase line portrait appears in Figure 3.7.

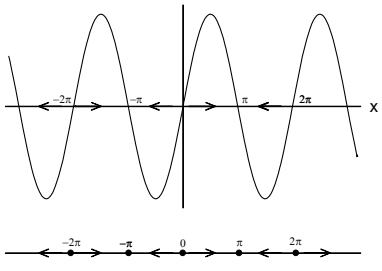


Figure 3.7. Phase line portrait for $x' = \sin x$.

Phase line portraits for linear autonomous equations are particularly simple, as we see in the next example.

Example 3.10 *If $p \neq 0$ the linear equation*

$$x' = px + q$$

has one equilibrium $x_e = -q/p$ and two non-equilibrium orbits, namely the intervals $x < -q/p$ and $-q/p < x$. If $p = 0$ and $q \neq 0$, there is no equilibrium. As a result there is only one orbit, namely the whole x-axis, with a right orientation if $q > 0$ and a left orientation if $q < 0$. (If both $p = 0$ and $q = 0$, then every point is an equilibrium and the equilibria are not isolated.)

$x' = px + q$	*phase line portrait*
$p < 0$	$\longrightarrow \quad -\frac{q}{p} \quad \longleftarrow$
$p > 0$	$\longleftarrow \quad -\frac{q}{p} \quad \longrightarrow$
$p = 0,\ q > 0$	$\longrightarrow$
$p = 0,\ q < 0$	$\longleftarrow$

An (isolated) equilibrium separates two non-equilibrium orbits in a phase line portrait. As a result there are a limited number of possible orbit configurations near this equilibrium (in fact, only three). The possibilities are listed in Figure 3.8.

Equilibrium Type	Phase Line Portrait
attractor	$\longrightarrow \quad x_e \quad \longleftarrow$
repeller	$\longleftarrow \quad x_e \quad \longrightarrow$
shunt	$\longrightarrow \quad x_e \quad \longrightarrow$ $\longleftarrow \quad x_e \quad \longleftarrow$

Figure 3.8. The phase line portraits in the neighborhood of an isolated equilibrium.

Definition 3.4 *Consider an autonomous equation $x' = f(x)$ where $f(x)$ is continuously differentiable for all x. Assume the roots of $f(x)$ are isolated.*

*An equilibrium is called an **attractor** (or a sink) if the orientation arrows of both adjacent orbits point towards it.*

*An equilibrium is called a **repeller** (or a source) if the orientation arrows of both adjacent orbits point away from it.*

*An equilibrium is called a **shunt** if the orientation arrows of both adjacent orbits point in the same direction.*

For linear equations $x' = px + q$ these definitions of an attractor and a repeller are the same are those given in Chapter 2. However, attractors and repellers of linear equations have a property that attractors and repellers of a nonlinear equation might not have. For linear equations *all* orbits move toward an attractor and *all* non-equilibrium orbits move away from a repeller. This is not necessarily true for nonlinear equations.

For example, from the phase line portrait of the equation $x' = x^2(1-x^2)$ in Figure 3.5 we see that the equilibrium $x_e = -1$ is a repeller, the equilibrium $x_e = 0$ is a shunt and the equilibrium $x_e = -1$ is an attractor. Here is another example.

Example 3.11 *The parabolic graph of the quadratic function*

$$f(x) = r\left(1 - \frac{x}{K}\right)x$$

appears in Figure 3.9 for positive constants r and K. From this graph we obtain the phase line portrait in Figure 3.9 for the logistic equation

$$x' = r\left(1 - \frac{x}{K}\right)x.$$

The equilibrium $x_e = 0$ is a repeller and the equilibrium $x_e = K$ is an attractor.

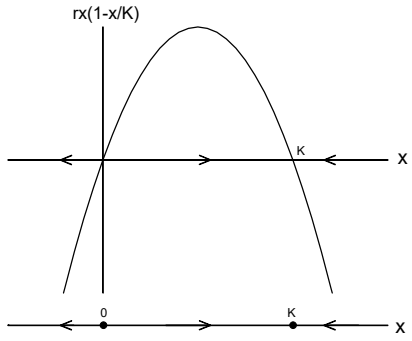


Figure 3.9. The graph of $f(x) = rx(1-x/K)$ together with the phase line portrait of logistic equation.

From the geometric way by which the graph of $f(x)$ is used to construct phase line portraits, we obtain straightforward geometric tests for the three different types of equilibria in Definition 3.4. See Figure 3.10.

Theorem 3.3 ***(Geometric Test) Suppose that $x = x_e$ is an isolated equilibrium of $x' = f(x)$. Then x_e is :***

an attractor (or a sink) if and only if the graph of $f(x)$ decreases through x_e;
a repeller (or a source) if and only if the graph of $f(x)$ increases through x_e;
a shunt if and only if the graph of $f(x)$ has a (local) extremum at x_e.

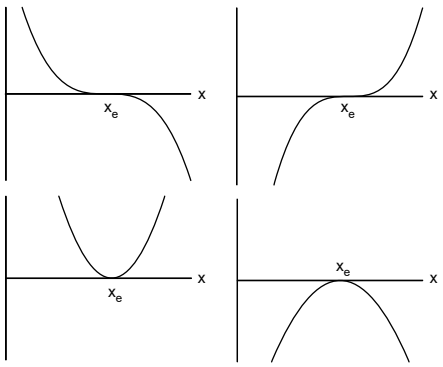


Figure 3.10. If the graph of $f(x)$ decreases or increases through the equilibrium x_e, as in the upper two graphs, then the equilibrium is an attractor or a repeller respectively. If the graph of $f(x)$ has an extremum at the equilibrium x_e, as in the lower two graphs, then the equilibrium is a shunt.

The graph of $f(x) = x^2(1 - x^2)$ shown in Figure 3.5 illustrates this theorem. The graph increases through the root -1, has a (local) minimum at the root 0 and decreases through the root 1. For the differential equation $x' = x^2(1 - x^2)$ this implies the equilibrium $x_e = -1$ is a repeller, $x_e = 0$ is a shunt, and $x_e = 1$ is an attractor.

Another example is provided by the graph of $f(x) = rx\,(1 - x/K)$ in Figure 3.9, which is seen to increase through the root 0 and decrease through the root K. Theorem 3.3 implies $x_e = 0$ is a repeller and $x_e = K$ is an attractor for the logistic equation $x' = rx\,(1 - x/K)$.

The monotonicity of a function $f(x)$ at a point $x = x_e$ is related to its derivative df/dx evaluated at this point, which we denote by

$$\left.\frac{df(x)}{dx}\right|_{x=x_e} \quad \text{or sometimes more concisely by} \quad \left.\frac{df}{dx}\right|_{x_e}.$$

If $df/dx|_{x_e} \neq 0$ then the function $f(x)$ cannot have an extremum at x_e and it either decreases through x_e (if $df/dx|_{x_e} < 0$) or increases through x_e (if $df/dx|_{x_e} > 0$). Thus, we can use the derivative of $f(x)$, evaluated at an equilibrium x_e, to determine whether x_e is an attractor or a repeller, provided this derivative does not vanish.

Definition 3.5 *Suppose x_e is an equilibrium of $x' = f(x)$. If*

$$\left.\frac{df(x)}{dx}\right|_{x=x_e} \neq 0,$$

then x_e is called ***hyperbolic.***

This definition is consistent with that given in Chapter 2 for a linear autonomous equation $x' = px + q$, since in this case $f(x) = px + q$ and $df/dx = p$. From the geometric test in Theorem 3.3 we obtain a derivative test for the equilibrium type.

Theorem 3.4 ***(Derivative Test) If x_e is a hyperbolic equilibrium of $x' = f(x)$, then***

$$\left.\frac{df(x)}{dx}\right|_{x=x_e} < 0 \textbf{ implies } x_e \textbf{ is an attractor}$$

$$\left.\frac{df(x)}{dx}\right|_{x=x_e} > 0 \textbf{ implies } x_e \textbf{ is a repeller.}$$

If the derivative of $f(x)$ evaluated at an equilibrium x_e equals 0 (that is to say, if the equilibrium is non-hyperbolic), then nothing can be deduced from this theorem.

For example, for $f(x) = x^2(1 - x^2)$ we find

$$\frac{df}{dx} = 2x - 4x^3$$

and consequently

$$\left.\frac{df}{dx}\right|_{-1} = 2, \quad \left.\frac{df}{dx}\right|_{0} = 0, \quad \left.\frac{df}{dx}\right|_{1} = -2.$$

From Theorem 3.4 we conclude that $x_e = -1$ is a repeller and $x_e = 1$ is an attractor. The equilibrium $x_e = 0$ is non-hyperbolic, however, and we cannot conclude anything about it from this theorem. (From the geometric test we can see, however, that $x_e = 0$ is a shunt.)

Do not make the mistake of concluding that a non-hyperbolic equilibrium must necessarily be a shunt. Although $df/dx|_{x_e} = 0$ is a necessary condition, it is not sufficient to imply that $f(x)$ has an extremum at x_e. Here is an example.

Example 3.12 *The equilibria of the equation*

$$x' = x^3 - x^4$$

are the roots of

$$f(x) = x^3 - x^4,$$

namely 0 *and 1. Since*

$$\left.\frac{df}{dx}\right|_1 = -1 < 0,$$

the equilibrium $x_e = 1$ *is a hyperbolic attractor. Since*

$$\left.\frac{df}{dx}\right|_0 = 0,$$

the equilibrium $x_e = 0$ *is non-hyperbolic (and Theorem 3.4 does not apply). However,* $f(x)$ *does not have an extremum at* 0 *and hence* $x_e = 0$ *is not a shunt. The graph in Figure 3.11 shows that* $f(x)$ *is increasing through* 0 *and therefore by Theorem 3.3 the non-hyperbolic equilibrium* $x_e = 0$ *is a repeller.*

Both adjacent orbits approach an attractor as $t \to +\infty$. For this reason an attractor is called *asymptotically stable* or, more commonly, simply *stable*. In applications a variable located at a stable equilibrium returns to that same equilibrium when slightly perturbed away. This is in contrast to a repeller for which such a perturbation results in motion away from the equilibrium. For this reason a repeller is called *unstable*. A shunt is also called unstable (because both adjacent orbits do not approach it as $t \to +\infty$.) or semi-stable (because one adjacent orbit approaches it and the other does not). For more on stability see Exercises 3.192 and 3.193.

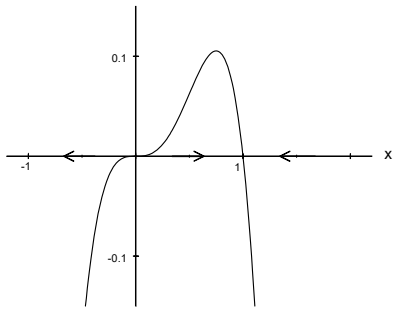


Figure 3.11. The graph of $f(x) = x^3 - x^4$ together with the phase line portrait of $x' = x^3 - x^4$.

3.1.3 The Linearization Principle

The derivative test in Theorem 3.4 is related to the "*linearization principle*", one of the most important principles in applied mathematics. Linearization is a procedure for studying solutions (or orbits) of an equation $x' = f(x)$ near a hyperbolic equilibrium x_e by approximating $f(x)$ with its tangent line at x_e.

Since $f(x_e) = 0$ at an equilibrium x_e, the equation of the tangent line to $f(x)$ at x_e is $y = \lambda(x - x_e)$ where

$$\lambda = \left.\frac{df(x)}{dx}\right|_{x=x_e}.$$

The graph of this tangent line approximates the graph of $f(x)$ near the tangent point and therefore

$$f(x) \approx \lambda(x - x_e) \tag{3.3}$$

for x near x_e. Another way to arrive at this approximation is to recall the formulas for the coefficients in the Taylor series expansion of $f(x)$ centered at x_e:

$$f(x) = f(x_e) + \left.\frac{df(x)}{dx}\right|_{x=x_e} (x - x_e) + O\left((x - x_e)^2\right)$$

where $O\left((x - x_e)^2\right)$ denotes all terms of powers of $(x - x_e)$ of order 2 and higher. We obtain the same approximation (3.3) by ignoring all the higher order terms in $(x - x_e)$ and noting that $f(x_e) = 0$ (because x_e is an equilibrium).

This suggests we may learn about solutions of

$$x' = f(x) \tag{3.4}$$

near an equilibrium x_e from the solutions of the approximating linear equation

$$x' = \lambda (x - x_e). \tag{3.5}$$

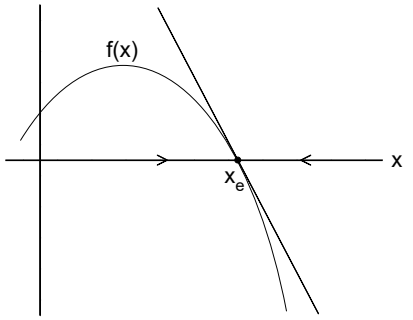


Figure 3.12. The graphs of $f(x)$ and its tangent line produce the same phase line portrait near a hyperbolic equilibrium x_e.

In fact the phase portraits near x_e of these two equations are identical if $\lambda \neq 0$. See Figure 3.12. The equilibrium x_e is an attractor for both equations if $\lambda < 0$; it is a repeller for both equation if $\lambda > 0$.

We call linear differential equation (3.5) the *linearization of the equation (3.4) at the equilibrium* x_e. We can simplify the linearized equation by a change of variables. Let $u = x - x_e$. Then $u' = x'$ and the linearization becomes

$$u' = \lambda u \quad \text{where} \quad \lambda = \left.\frac{df(x)}{dx}\right|_{x=x_e}. \tag{3.6}$$

Referring to Example 3.10 we see that the linear equation $u' = \lambda u$ has an attractor at the equilibrium $u_e = 0$ if $\lambda < 0$ and a repeller at $u_e = 0$ if $\lambda > 0$. This fact and Theorem 3.4 imply that a hyperbolic equilibrium x_e of $x' = f(x)$ has the same type as that of the equilibrium $u_e = 0$ of its linearization. This is the *Linearization Principle.*

Theorem 3.5 ***(Linearization Principle) An autonomous equation*** $x' = f(x)$ ***has an attractor (or repeller) at a hyperbolic equilibrium*** x_e ***if its linearization (3.6) at*** x_e ***has an attractor (or repeller) at*** $u_e = 0$***, that is to say, if*** $\lambda < 0$ ***(or*** $\lambda > 0$***).***

Example 3.13 *The logistic equation*

$$x' = r\left(1 - \frac{x}{K}\right)x, \quad r > 0, \quad K > 0$$

has two equilibria $x_e = 0$ *and* K*. To apply the Linearization Principle at each of these equilibria we evaluate the derivative*

$$\frac{df}{dx} = r\left(1 - 2\frac{x}{K}\right)$$

of $f(x) = rx\,(1 - x/K)$ *at each equilibrium. Since* $df/dx|_0 = r > 0$*, the linearization at* $x_e = 0$ *has a repeller at* 0*. Since* $df/dx|_K = -r < 0$*, the linearization at* $x_e = K$ *has an attractor at* 0*. By the Linearization Principle the logistic equation has a repeller at* $x_e = 0$ *and an attractor at* $x_e = K$*.*

The Linearization Principle does not hold for a non-hyperbolic equilibrium x_e (i.e., when $\lambda = 0$). That is to say, the linearization at a non-hyperbolic equation cannot be used in general to determine the phase portrait near the equilibrium. For example, graphs of x^2, x^3 and $-x^3$ show the equilibrium $x_e = 0$ is a shunt for $x' = x^2$, a repeller for $x' = x^3$ and an attractor for $x' = -x^3$. Yet all three equations have the same linearization $u' = 0$ at 0.

3.1.4 Local Equilibrium Bifurcations

Differential equations that arise in applications often contain unspecified numerical constants called parameters or coefficients. The "radioactive decay" equation

$$x' = -rx$$

has one parameter, r. The logistic equation

$$x' = r\left(1 - \frac{x}{K}\right)x$$

has two parameters, r and K. The "spruce budworm" equation

$$x' = r\left(1 - \frac{x}{K}\right)x - c\frac{x^2}{a + x^2}$$

has four parameters r, K, c and a.

The graph of $f(x)$ and consequently the phase line portrait of an autonomous equation $x' = f(x)$ depend on the values assigned to the parameters appearing in $f(x)$. In applications it is often important to understand how changes in parameter values alter the phase line portrait. Parameters may change, for example, from naturally occurring events or from deliberate (or inadvertent) manipulations by humans. Moreover, in applications parameters have to be estimated numerically (e.g., from data) and therefore we must investigate the phase line portrait throughout a statistical confidence interval for these estimates. *Bifurcation theory* is the study of how changes in parameters alter the phase line portrait and the asymptotic dynamics of an equation.

To introduce some basic ideas, consider the homogeneous linear autonomous equation

$$x' = px \tag{3.7}$$

where p is a constant. The phase line portrait is depends on the sign of p. Specifically, the equilibrium $x_e = 0$ is an attractor if $p < 0$ and it is a repeller if $p > 0$. (See Figure 3.8.) Thus, the phase line portrait changes in a significant way when the parameter p is increased (or decreased) through 0. Such a radical change in the phase line portrait of an equation is called a "bifurcation". Thus, in the linear equation (3.7) a bifurcation occurs when p passes through 0. This critical value 0 of the parameter p is called a "bifurcation value".

The phase line portrait of the linear equation $x' = -x + q$, however, is unaltered if q is changed. The equilibrium $x_e = q$ is an attractor for all q. In this case, we say that the phase portraits remain "qualitatively equivalent" and that no bifurcation occurs.

To make the concept of bifurcation more precise we need a definition. We have been considering autonomous equations $x' = f(x)$ for which the roots of $f(x)$ (the equilibria) are isolated. From now on we assume more, namely that $f(x)$ has at most a finite number of roots (in which case they are necessarily isolated). Between each pair of consecutive roots, the function $f(x)$ is either positive or negative. The sign of $f(x)$ determines the orientation direction of the orbit between two consecutive roots in the phase line portrait. The set of roots of $f(x)$ and the sequence of signs of $f(x)$ between consecutive roots characterize the phase line portrait.

Definition 3.6 *Assume $f(x)$ is continuously differentiable for all x and has at most a finite number of roots. The number of equilibria of $x' = f(x)$ and the orientation directions of the non-equilibrium orbits (more precisely, the sequence of signs of $f(x)$ between consecutive equilibria) determine the "**structure of the phase line portrait**" (or the "**orbit structure**") of the equation.*

Two differential equations might differ considerably, but still have the same orbit structure. For this situation we have the following terminology.

Definition 3.7 *Two phase line portraits are said to be **qualitatively equivalent** if they have the same orbit structure as defined in Definition 3.6.*

Example 3.14 *The function $f(x) = (1 - x)e^{-x}$ has only the root 1. Since*

$$\left.\frac{df}{dx}\right|_1 = -e^{-1} < 0,$$

the equilibrium $x_e = 1$ of the differential equation

$$x' = (1 - x)e^{-x}$$

is an attractor. Figure 3.13 shows the phase line portrait for this equation. The phase line portrait of the equation $x' = -x$ also appears in Figure 3.13. By Definitions 3.6 and 3.7 the phase portraits of these two equations are qualitatively equivalent.

Equation	Phase Line Portrait
$x' = (1 - x)\, e^{-x}$	$\longrightarrow \quad 1 \quad \longleftarrow$
$x' = -x$	$\longrightarrow \quad 0 \quad \longleftarrow$

Figure 3.13

If two differential equations do not have the same number of equilibria, then their phase portraits cannot have the same orbit structure. Thus, for the qualitative equivalence of two phase line portraits it is *necessary* that they have the *same number* of equilibria. Having the same number of equilibria is *not sufficient* for qualitative equivalence, however, because the non-equilibrium orbit orientations might not be identical. The next example illustrates this.

Example 3.15 *Figure 3.14 shows the phase line portraits of the logistic equation*

$$x' = rx(1 - \frac{x}{K})$$

and the equation

$$x' = rx^2(1 - \frac{x}{K})$$

where $r > 0$ and $K > 0$. Both have equilibria $x_e = 0$ and K. However, the orientation of the non-equilibrium orbits differ and therefore the phase line portraits are not qualitatively equivalent.

Equation	Phase Line Portrait
$x' = rx(1 - \frac{x}{K})$	$\longleftarrow \quad 0 \quad \longrightarrow \quad K \quad \longleftarrow$
$x' = rx^2(1 - \frac{x}{K})$	$\longrightarrow \quad 0 \quad \longrightarrow \quad K \quad \longleftarrow$

Figure 3.14

Sometimes phase portraits remain qualitatively equivalent as a parameter in an equation is changed. For example, the phase line portraits of the linear equation $x' = -x + q$ remain qualitatively equivalent for all values of q. Bifurcation theory, on the other hand, is concerned with the loss of qualitative equivalence as a parameter is changed. Here is an example.

Example 3.16 *The equation*

$$x' = x^2 - p \tag{3.8}$$

has no equilibria if $p < 0$. If $p > 0$ this equation has two equilibria $x_e = \pm\sqrt{p}$. The phase line portraits for both cases appear in Figure 3.15. The phase line portraits have the same orbit structure and therefore are qualitatively equivalent for all negative values of p. The same is true for all positive values of p. However, the phase line portraits for negative p are not qualitatively equivalent to those for positive p. They do not have the same number of equilibria. Thus, there is a change in orbit structure as p passes through 0.

$x' = x^2 - p$	Phase Line Portrait
$p < 0$	$\longrightarrow$
$p > 0$	$\longrightarrow \quad -\sqrt{p} \quad \longleftarrow \quad \sqrt{p} \quad \longrightarrow$

Figure 3.15

In the preceding example a "bifurcation" occurs at the "bifurcation value" $p = 0$ because the orbit structure of the equation changes as the parameter p passes through 0. This motivates a general definition of a bifurcation value for an autonomous differential equation with a parameter.

Consider the equation

$$x' = f(x, p) \tag{3.9}$$

where a parameter p is included in the variable list of the function f. There may be other parameters in an equation, but we specify as p only the parameter whose effect on the phase line portrait we want to study. This parameter we designate as the "bifurcation parameter". As the bifurcation parameter p is allowed to vary over a designated interval, we require that our basic assumptions hold: f is continuously differentiable (with respect to x) and has (at most) a finite number of roots.

Definition 3.8 *The phase line portrait of equation (3.9) is* ***stable at*** p_0 *if its orbit structure remains unchanged for all values of p in an interval centered on p_0. If the phase portrait of (3.9) is not stable at p_0, then a* ***bifurcation*** *occurs at $p = p_0$ and p_0 is called a* ***bifurcation value****.*

A bifurcation occurs at $p_0 = 0$ for the linear equation $x' = px$ because the orbit structure is an attractor for $p < 0$ and a repeller for $p > 0$. Any interval centered on $p_0 = 0$ contains both negative and positive values of p and therefore, by Definition 3.8, the phase portrait is unstable at $p_0 = 0$. In Example 3.16 $p_0 = 0$ is a bifurcation value for equation (3.8) for the same reason.

One graphical way to describe bifurcations of an equation (3.9) is to plot the equilibria as a function of the parameter p. This is the same as plotting the graph described by the equation

$$f(x, p) = 0$$

in the p, x-plane. The resulting graph is called an *equilibrium diagram* or, if bifurcations occur, a *bifurcation diagram*.

If, in addition, the equilibria type is indicated on the bifurcation diagram graph, then phase line portraits can be constructed from the graph at a selected value of p. For example, one might simply label the graph with letters or words; or one might indicate attractors (the stable equilibria) by solid lines and repellers and shunts (the unstable equilibria) by dashed lines. For example, Figure 3.16 shows the bifurcation diagram for the equation (3.8) in the Example 3.16. This is the graph of the equation $x^2 - p = 0$ in the p, x-plane. For $p > 0$ the repeller $x_e = \sqrt{p}$ is plotted as a dashed line and the attractor $x_e = -\sqrt{p}$ is plotted as a solid line.

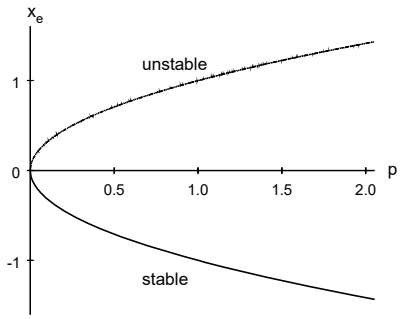


Figure 3.16. The bifurcation diagram for $x' = x^2 - p$ in which the equilibria $x_e = \pm\sqrt{p}$ are plotted against p.

One goal of bifurcation theory is to categorize different kinds of bifurcations. We consider only the three most basic types. The bifurcation in Figure 2.16 (Example 3.16) is called a *blue-sky bifurcation* (or a saddle-node bifurcation or a tangent bifurcation).[2]

> A *blue-sky bifurcation* is characterized by the following properties. For p values on one side of the bifurcation value p_0 there are two equilibria (typically a repeller

[2] We will learn the reason for this odd name in Chapter 8 (Section 8.5). We could also refer to this type of bifurcation as a *blue-sky bifurcation*, a term that colorfully captures the fact that the two equilibria involved suddenly appear, as if out of nowhere, as p passes through the critical value p_0.

and an attractor). These two equilibria merge to a single equilibrium x_e as p approaches p_0 in the limit. For p on the other side of p_0 there are no equilibria (at least near x_e).

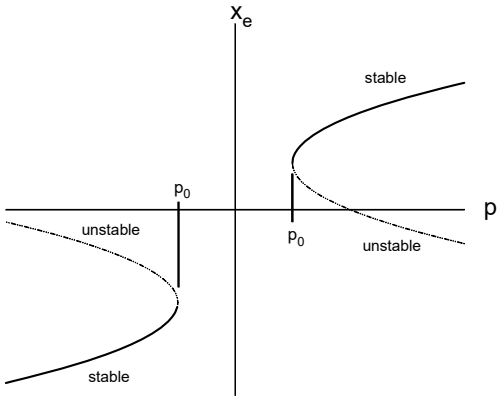


Figure 3.17. Blue-sky bifurcations.

As p passes through p_0 the two equilibria "collide" and "annihilate each other". The bifurcation diagram of a blue-sky bifurcation has a parabolic shape which opens either to the right or the left and has its "nose" at the bifurcation value p_0, as in Figure 3.17. This characterization of a blue-sky bifurcation applies in a neighborhood of the bifurcation point (the "nose" of the bifurcation diagram. See Remark 1 below.

The next example illustrates another basic type of bifurcation.

Example 3.17 *The equation*

$$x' = px - x^3$$

has equilibrium $x_e = 0$ *for all values of* p *and the equilibria* $x_e = \pm\sqrt{p}$ *for* $p > 0$. *Since* $f(x,p) = px - x^3$ *and*

$$\frac{df(0,p)}{dx} = p$$

the equilibrium $x_e = 0$ *is an attractor for* $p < 0$ *and a repeller for* $p > 0$. *For* $p > 0$

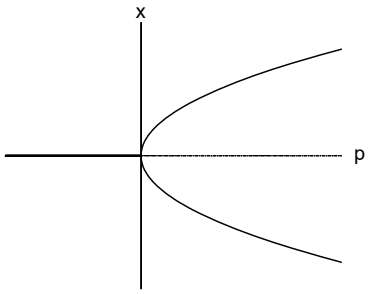
$$\frac{df(\pm\sqrt{p},p)}{dx} = -2p < 0$$

and both equilibria $x_e = \pm\sqrt{p}$ *are attractors. For* $p = 0$ *the equation reduces to* $x' = -x^3$ *whose only equilibrium is the attractor* $x_e = 0$. *A bifurcation occurs at* $p_0 = 0$ *because the orbit structure for* $p < 0$, *consisting of a single attractor, is different from that for* $p > 0$, *which consists of two attractors and a repeller. All these facts are summarized by the bifurcation diagram in Figure 3.18. This graph is found by solving the equation* $px - x^3 = 0$ *and plotting the solutions* $x = 0$ *and* $p = x^2$. *Notice the pitchfork shape of the graph.*

The bifurcation in this example, drawn in Figure 3.18, is a typical *pitchfork bifurcation.*

A *pitch-fork bifurcation* is characterized by the following characteristics. There are three equilibria for p on one side of p_0 and only one on the other side of p_0. The three equilibria merge to a single equilibrium x_e as p approaches p_0. typically the equilibria on the upper and lower branches of the pitchfork have the same stability properties, i.e., are all stable or unstable, and the equilibria on the middle branch have the opposite stability property from those on the outer branches.

A pitchfork bifurcation diagram may open to the right (as in Figure 3.18) or to the left. See Exercises 3.94, 3.95, and 3.96.



$x' = px - x^3$	Phase Line Portrait
$p < 0$	$\rightarrow \quad 0 \quad \leftarrow$
$p = 0$	$\rightarrow \quad 0 \quad \leftarrow$
$p > 0$	$\rightarrow \quad -\sqrt{p} \quad \leftarrow \quad 0 \quad \rightarrow \quad \sqrt{p} \quad \leftarrow$

Figure 3.18 The pitchfork bifurcation of $px - x^3$.

The following example illustrates a third type of basic bifurcation.

Example 3.18 *The equation*

$$x' = px - x^2$$

has equilibria $x_e = 0$ and $x_e = p$ for all values of p. Since $f(x, p) = px - x^2$ and

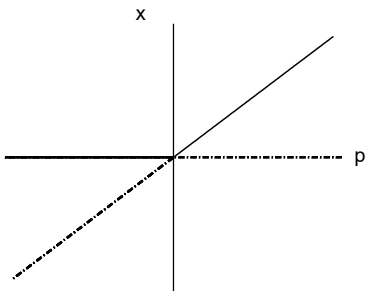
$$\frac{df(0,p)}{dx} = p$$

the equilibrium $x_e = 0$ changes from an attractor when $p < 0$ to a repeller for $p > 0$. Since

$$\frac{df(p,p)}{dx} = -p$$

the equilibrium $x_e = p$ is a repeller for $p < 0$ and an attractor for $p > 0$ (exactly the opposite of the situation for the equilibrium $x_e = 0$). For $p = 0$ the equation reduces to $x' = -x^2$ whose only equilibrium is a shunt at $x_e = 0$.

The orbit structure for $p \neq 0$ is different from the orbit structure for $p = 0$ and therefore a bifurcation occurs at $p_0 = 0$. The bifurcation diagram in Figure 3.19 summarizes all these facts.



	Phase Line Portrait
$p<0$	$\longleftarrow \quad p \quad \longrightarrow \quad 0 \quad \longleftarrow$
$p=0$	$\longleftarrow \quad 0 \quad \longleftarrow$
$p>0$	$\longleftarrow \quad 0 \quad \longrightarrow \quad p \quad \longleftarrow$

Figure 3.19. The transcritical bifurcation of $x' = px - x^2$.

The bifurcation in the preceding example is called *transcritical*.

> A *transcritical bifurcations* is characterized by the crossing of two equilibrium branches in the bifurcation diagram. There are two equilibria on each side of the bifurcation value p_0 which merge to a single equilibrium x_e at p_0. Moreover, the equilibrium type on each branch changes as p passes through p_0 This is called an *exchange of stability* and it is a typical feature of transcritical bifurcations.

Remark 1. The descriptions above for the three basic bifurcations (as based on numerical counts of equilibria) are to be viewed as characterizations *only when one focuses in a neighborhood of the bifurcation point* in the bifurcation diagram. If one zooms out and looks at a more global picture of the bifurcation diagram, then there might well be other equilibria for parameter values p near p_0. For this reason, the three types of bifurcations defined above – blue-sky, pitchfork, and transcritical – are called *local bifurcations*.

Furthermore, from a global point-of-view there can be more than one (local) bifurcations in a bifurcation diagram. The next example illustrates this.

Example 3.19 *Consider the equation*

$$x' = 3x - x^3 - p.$$

The roots of the cubic polynomial $3x - x^3 - p$ and hence the equilibria of this equation are not easily found algebraically. However, as we will see, it is not necessary to calculate the roots in order to draw the bifurcation diagram.

The bifurcation diagram is the graph in the p, x-plane associated with the (equilibrium) equation

$$3x - x^3 - p = 0.$$

In principle, we want to solve this equation for x and graph the result as a function of p. A simpler way to obtain this graph, however, is to do the just opposite: solve the equation for

p in terms of x and graph p as a function of x. We can then obtain the desired bifurcation diagram by reflecting this graph through the 45^0 line $p = x$. Thus, we solve for

$$p = 3x - x^3$$

and graph this cubic polynomial in Figure 3.20a. The bifurcation diagram, obtained by reflecting this graph through the line $p = x$, is shown in Figure 3.20b.

From the bifurcation diagram we observe that there are two saddle node bifurcations. One is located at $p = 2$ and the other at $p = -2$. We also see that there is one equilibrium for $p < -2$ and one equilibrium for $p > 2$. For p between -2 and 2, however, there are three equilibria.

To determine the type of equilibria in the bifurcation diagram we can apply the derivative test. From $f(x) = 3x - x^3 - p$ we obtain

$$\frac{df}{dx} = 3\left(1 - x^2\right).$$

It follows that

$$\left.\frac{df}{dx}\right|_{x_e} = 3\left(1 - x_e^2\right) \quad \begin{cases} < 0 & \text{for those equilibria } x_e < -1 \text{ and } x_e > 1 \\ > 0 & \text{for those equilibria } -1 < x_e < 1. \end{cases}$$

From this we have the following phase line portraits

$$\begin{array}{rl} \textit{For} \quad p < -2: & \longrightarrow \circ \longleftarrow \\ \textit{For} \quad -2 < p < 2: & \longrightarrow \circ \longleftarrow \circ \longrightarrow \circ \longleftarrow \\ \textit{For} \quad p > 2: & \longrightarrow \circ \longleftarrow \end{array}$$

The bifurcation diagram in Figure 3.20 graphically summaries this information.

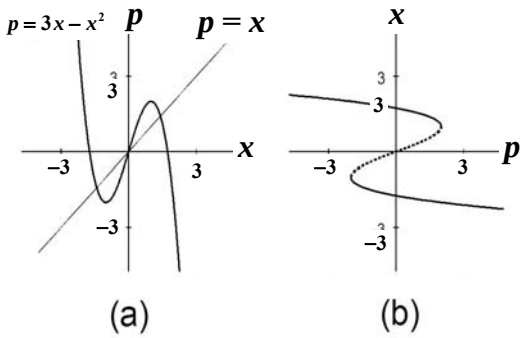


Figure 3.20. The bifurcation diagram for $x' = 3x - x^3 - p$ in (b) is obtained by reflecting the graph of the cubic $p = x^3 - 3x$ in (a) through the line $p = x$.The solid line consists of stable equilibria (attractors). The dashed line consists of unstable equilibria (repellers).

The method we used to obtain the bifurcation diagram in Example 3.19 is often a convenient one. The problem of drawing the bifurcation diagram associated with a first order equation $x' = f(x,p)$ is the problem of drawing the graph defined by the equilibrium equation $f(x,p) = 0$ in the p,x-plane. Ideally we can do this by algebraically solving the equation for x and graphing the answer as a function of p. However, since we are concerned with nonlinear equations, it is usually not easy to solve this equation for x. However, it is frequently the case that the parameter p appears in the equation in a simpler algebraic way than x does. In such a case we can usually more easily solve the equation for p, instead of x. If we do this and plot the answer p as a function of x in the x,p-plane, we obtain the sought after bifurcation diagram by reflecting the resulting graph through the line $p = x^3$.

In applications bifurcations often play an important and crucial role. Here is an example. The equation

$$x' = r\left(1 - \frac{x}{K}\right)x - c\frac{x^2}{a + x^2} \tag{3.10}$$

has been used to describe the dynamics of spruce budworm populations. The variable x denotes the number or density of the budworm population. Outbreaks of this defoliating insect have caused major deforestations in Canada and the United States. One explanation that has been given for the occurrence of outbreaks is based on the multiple bifurcations that occur in the equation (3.10).

As an example, consider equation (3.10) with $a = 0.01$, $c = 1$, and $K = 1$ and $p = r$ as a parameter :

$$x' = rx\,(1 - x) - \frac{x^2}{0.01 + x^2}. \tag{3.11}$$

To obtain the bifurcation diagram for this equation we use the procedure described above. That is, we solve the equilibrium equation

$$rx\,(1 - x) - \frac{x^2}{0.01 + x^2} = 0$$

for the parameter

$$r = \frac{1}{x\,(1 - x)}\frac{x^2}{0.01 + x^2}$$

and plot the result in the x,r-plane. This plot is shown in Figure 3.21a. We obtain the bifurcation diagram in Fig 3.21b by reflecting the graph through the $r = x$ line. In the table below appear phase line portraits at three selected r values in the bifurcation diagram.

$r = 3$	$r = 5$	$r = 7$
$\longleftarrow 0 \longrightarrow x_e \longleftarrow$	$\longleftarrow 0 \longrightarrow x_e \longleftarrow x_e \longrightarrow x_e \longleftarrow$	$\longleftarrow 0 \longrightarrow x_e \longleftarrow$

In Figure 3.21b we observe three bifurcations associated with equation (3.11). A transcritical bifurcation occurs at $r = 0$. For $r < 0$ the equilibrium $x_e = 0$ is an attractor (and the budworm population goes extinct), whereas for $r > 0$ an exchange of stability occurs and there is a positive attractor (and the budworm persists). The remaining two bifurcations

[3] Another way to accomplish the same thing is to relect the graph through the vertical p-axis and then rotate the result 90^0 clockwise.

in Figure 3.21b are blue-sky bifurcations similar to those in Figure 3.20 in Example 3.19. Because of these blue-sky bifurcations, the attractor undergoes discontinuous changes as r passes through the two bifurcation values located approximately at $r \approx 3.84$ and 5.55. For example, if r is increased from small values (where the attractor is also small) to a value larger than 5.55, then the attractor discontinuously jumps to a higher level. This indicates a spruce budworm "outbreak".

The bifurcation diagram in Figure 3.21b contains another important feature. If some kind of control measures are put into effect to decrease r, in an attempt to reverse the spruce budworm outbreak and infestation, the budworm population (now at the higher equilibrium level with $r > 5.55$) will not return to the lower equilibrium level until r is decreased below the smaller critical value 3.84. At that point there is a collapse of the population to the lower attractor. Interestingly, $r = 3.84$ (at which the outbreak is eradicated) is less than $r = 5.55$ (at which the outbreak occurs). This phenomenon is called *hysteresis*. It occurs in many other applications as well.

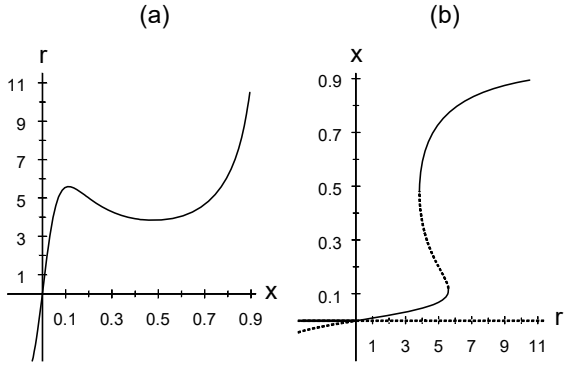


Figure 3.21. (a) shows the plot of r as a function of x. (b) shows the plot of x as a function of r, i.e. the bifurcation diagram for the equation (3.11). Dashed lines indicate unstable equilibria (repellers).

3.2 Separable Equations

In Sec. 3.1 we learned how to study the solutions of an autonomous differential equation $x' = f(x)$ by constructing phase line portraits. It is also sometimes useful to have formulas for solutions. In this section we learn a method that can (at least in principle) produce formulas for the general solution of an autonomous equation (and, in fact, for a more general class of equations called *separable*).

Formally, the four steps are

$$\begin{aligned} x'(t) &= f(x(t)) \\ \frac{1}{f(x(t))}x'(t) &= 1 \\ \frac{d}{dt}\int^{x(t)} \frac{1}{f(x(s))}ds &= 1 \\ \int^{x(t)} \frac{1}{f(x(s))}ds &= t + c. \end{aligned}$$

A more concise, shorthand way of writing this method is

$$\begin{aligned} \frac{dx}{dt} &= f(x) \\ \frac{1}{f(x)}dx &= dt \\ \int \frac{1}{f(x)}dx &= \int dt \\ \int \frac{1}{f(x)}dx &= t + c. \end{aligned}$$

The last equation, if the anti-differentiation can be carried out, yields an equation in x and t. It implicitly defines solutions x. An explicit formula for solutions can be found if this equation can be algebraically solved for x. These formulas, explicit or implicit, yield the general solution when all equilibrium are included. (Sometime an equilibrium will in fact be contained in the calculated formula, but sometimes not.)

We can also use this same method on some non-autonomous equations. Here is an example. The equation

$$x' = -2tx \tag{3.12}$$

is a particular case of the equation

$$x' = -at^p x, \quad a > 0, \quad p > 0 \tag{3.13}$$

which arises in a model for the spread of AIDS in a population infected with the human immunodeficiency virus HIV. We can find solution formulas for equation (3.12) as follows. First we note that $x = 0$ is an equilibrium (it is a root of the right hand side of the equation).

For non-equilibrium solutions we calculate

$$\begin{aligned}
\frac{dx}{dt} &= -2tx \\
\frac{1}{x}dx &= -2tdt \\
\int \frac{1}{x}dx &= -\int 2tdt + k \\
\ln|x| &= -t^2 + k \\
|x| &= e^{-t^2+k} \\
x &= \pm e^k e^{-t^2} \\
x &= ce^{-t^2}
\end{aligned}$$

where $c = \pm e^k$ is an arbitrary, nonzero constant. The solution $x = 0$ is not included in this formula (because of the division by x in the first step of the solution method). We can include the equilibrium solution in the formula, however, by allowing c to equal 0, with the result that the general solution is given by the formula

$$x = ce^{-t^2}, \quad \text{where } c \text{ is an arbitrary constant.} \tag{3.14}$$

The procedure used to solve equation (3.12) is called the *Separation of Variables Method*. This name derives from the first step in which the dependent variable x and the independent variable t are separated to opposite sides of the equation (including the "differentials" dx and dt). This key step in the method is possible for equation (3.12) because the right hand side of the equation $f(t,x) = -2tx$ is "multiplicatively separable", i.e., it is a product of two factors, one depending only on t and the other only on x. The method of separating variables can be applied to any equation $x' = f(t,x)$ for which $f(t,x)$ is multiplicatively separable in this way. This suggests the following definition.

Definition 3.9 *The first order equation $x' = f(t,x)$ is called* ***separable*** *if $f(t,x)$ is multiplicatively separable in the variables t and x, that is to say, if it can be written in the factored form $f(t,x) = g(t)h(x)$.*

We can find a formula for the general solution of a separable equation

$$\frac{dx}{dt} = g(t)h(x)$$

as follows. We first take note of the equilibrium solutions, which are the roots of $h(x) = 0$. For non-equilibrium solutions, we write

$$\begin{aligned}
\frac{dx}{dt} &= g(t)h(x) \\
\frac{1}{h(x)}dx &= g(t)dt \\
\int \frac{1}{h(x)}dx + c_1 &= \int g(t)dt + c_2 \\
\int \frac{1}{h(x)}dx &= \int g(t)dt + c
\end{aligned}$$

where the two constants of integration c_1 and c_2 have been put together as the single arbitrary constant $c = c_2 - c_1$. If the two integrals can be calculated (by hand and/or with the help of integral tables or a computer), the resulting equation in x and t defines a set of solutions.

Theorem 3.6 *The general solution of a separable equation $\frac{dx}{dt} = g(t)h(x)$ consists of the equilibria (roots of $h(x) = 0$) together with the solutions x defined by the equation*

$$\int \frac{1}{h(x)} dx = \int g(t)dt + c$$

where c is an arbitrary constant. This is the "implicitly" defined general solutions. If this equation can be algebraically solved for x, then the resulting formula is the "explicit" general solution.

With a formula for the general solution in hand (implicit or explicit), one can solve any initial value problem

$$\begin{aligned} x' &= g(t)h(x) \\ x(t_0) &= x_0. \end{aligned}$$

Unless the initial condition x_0 is an equilibrium, we obtain a solution formula by choosing the arbitrary constant c so that the general solution formula satisfies the initial condition.

Alternatively, we can instead solve directly the initial value problem (without first calculating the general solution) by using definite integrals in the Separation of Variables Method. We proceed as follows. If x_0 is an equilibrium (i.e., a root of $h(x)$), the solution is this equilibrium: $x(t) = x_0$. If x_0 is not an equilibrium, then we calculate

$$\begin{aligned} \frac{dx}{dt} &= g(t)h(x) \\ \frac{1}{h(x)} dx &= g(t)dt \\ \int_{x_0}^{x} \frac{1}{h(s)} ds &= \int_{t_0}^{t} g(s)ds. \end{aligned}$$

By carrying out the indicated definite integrals (using the Fundamental Theorem of Calculus) we arrive at a formula for the solution of the initial value problem.

Example 3.20 *For example, we can find a formula for the solution of the initial value problem*

$$\begin{aligned} x' &= -x^2 \\ x(1) &= 10 \end{aligned}$$

by the steps

$$\begin{aligned}
\frac{dx}{dt} &= -x^2 \\
-\frac{1}{x^2}dx &= dt \\
-\int_{10}^{x} \frac{1}{s^2}ds &= \int_{1}^{t} ds \\
\frac{1}{x} - \frac{1}{10} &= t - 1 \\
x &= \frac{10}{10t - 9}.
\end{aligned}$$

The next example uses separation of variables to find a formula for the solution of the initial value problem for the famous logistic equation.

Example 3.21 *Consider the initial value problem*

$$\begin{aligned}
x' &= r\left(1 - \frac{x}{K}\right)x \\
x(t_0) &= x_0.
\end{aligned}$$

The roots of $f(x) = rx\,(1 - x/K)$ *are* 0 *and* K*. If* $x_0 = 0$*, the solution is the equilibrium solution* $x_e = 0$*. Similarly, if* $x_0 = K$ *then the solution is the equilibrium solution* $x_e = K$*. If* x_0 *does not equal* 0 *or* K*, then the solution is a non-equilibrium solution. To find a formula for this solution we use the separation of variables method as follows.*

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) \\
\frac{1}{x\left(1 - \frac{x}{K}\right)}dx &= rdt \\
\int_{x_0}^{x} \frac{1}{s\left(1 - \frac{s}{K}\right)}ds &= \int_{t_0}^{t} rds \\
\int_{x_0}^{x} \left(\frac{1}{s} + \frac{\frac{1}{K}}{1 - \frac{s}{K}}\right)ds &= r(t - t_0) \\
\ln\left|\frac{x}{1 - \frac{x}{K}}\frac{1 - \frac{x_0}{K}}{x_0}\right| &= r(t - t_0) \\
\left|\frac{x}{1 - \frac{x}{K}}\right| &= \left|\frac{x_0}{1 - \frac{x_0}{K}}\right| e^{r(t-t_0)}
\end{aligned}$$

This equation defines the solution x *implicitly. To find an explicit formula we must solve this equation for* x*. Eliminating the absolute value signs we obtain*

$$\frac{x}{1 - \frac{x}{K}} = \pm\frac{x_0}{1 - \frac{x_0}{K}}e^{r(t-t_0)}.$$

Which sign "+" or "−" should we use? Setting $t = t_0$ and $x = x_0$ shows that the "+" sign is required. Solving for x we get, after some algebraic manipulations, the explicit solution

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)\, e^{-r(t-t_0)}}. \tag{3.15}$$

The solution formula (3.15) shows that solutions with positive initial conditions tend to K as $t \to +\infty$, while solutions with negative initial conditions tend to $-\infty$. We can obtain these conclusions in a simpler way from the phase line portrait of the equation (see Example 3.15 and Figure 3.14). On the other hand, more details about the solutions are available from the solution formula (for example, numerical values for x at specific numerical values of t).

The initial value problem in Example 3.21 involves an autonomous equation. The next example involves a non-autonomous, separable equation.

Example 3.22 *For all constant growth rates $r > 0$ all solutions of the logistic equation*

$$x' = rx\left(1 - \frac{x}{K}\right)$$

with positive initial conditions $x_0 > 0$ tend to the carrying capacity $K > 0$ as $t \to +\infty$. The equation

$$x' = r\,(1 + \cos t)\, x\left(1 - \frac{x}{K}\right) \tag{3.16}$$

is a modification of the logistic equation in which the growth rate oscillates between $2r$ and 0 with period 2π and an average of r. This modification accounts for oscillations in birth and death rates that might be due, for example, to seasonal fluctuations in life cycles, food and water supplies, temperature, etc.

Figure 3.22 shows the graphs of some numerically computed solutions for the case $r = 0.25$ and $K = 1$. These graphs indicate that solutions with positive initial conditions tend to $K = 1$ as $t \to +\infty$. This suggests a general conjecture: solutions of the modified logistic equation (3.16) with positive initial conditions $x_0 > 0$ tend to K as $t \to +\infty$. We can prove this conjecture using the solution formula obtained by separating variables.

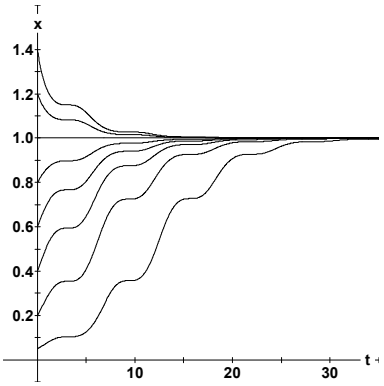


Figure 3.22. Solutions of (3.16) for $r = 0.25$ and $K = 1$ and a selection of initial conditions.

First note $x = 0$ and K are equilibrium solutions. For $x_0 \neq 0$ and $x_0 \neq K$ we separate variables as follows.

$$\begin{aligned}
\int_{x_0}^{x} \frac{1}{\left(1 - \frac{s}{K}\right) s} ds &= \int_0^t r\left(1 + \cos\right) s ds \\
\int_{x_0}^{x} \left(\frac{1}{s} + \frac{1}{K - s}\right) ds &= r\left(s + \sin s\right)\Big|_0^t \\
\ln\left|\frac{s}{K - s}\right|\Big|_{x_0}^{x} &= r\left(t + \sin t\right) \\
\ln\left(\left|\frac{x}{K - x}\right| \left|\frac{x_0}{K - x_0}\right|^{-1}\right) &= r\left(t + \sin t\right).
\end{aligned}$$

The last equation implicitly defines the solution x. To find an explicit formula we solve this equation for x:

$$\begin{aligned}
\left|\frac{x}{K - x}\right| \left|\frac{x_0}{K - x_0}\right|^{-1} &= e^{r(t + \sin t)} \\
\left|\frac{x}{K - x}\right| &= \left|\frac{x_0}{K - x_0}\right| e^{r(t + \sin t)} \\
\frac{x}{K - x} &= \pm \frac{x_0}{K - x_0} e^{r(t + \sin t)}.
\end{aligned}$$

By setting $t = 0$ we see "+" is the appropriate choice of sign. Solving for x we obtain the explicit solution formula[4]

$$x = K \frac{x_0}{x_0 + \left(K - x_0\right) e^{-r(t + \sin t)}}. \tag{3.17}$$

This formula also gives the equilibrium solutions when $x_0 = 0$ and $x_0 = K$.

From (3.17) we find

$$\lim_{t \to +\infty} x = \lim_{t \to +\infty} K \frac{x_0}{x_0 + \left(K - x_0\right) \cdot 0} = K$$

when $x_0 > 0$, as conjectured above.

We give one more example to illustrate a final point. Whether or not one needs a solution formula for a differential equation depends on the questions one wants to answer about the solution. Solution formulas do not always provide an easy way to answer some questions (which can be answered more easily other ways).

Example 3.23 *We can find a formula for the general solution of the autonomous equation*

$$x' = \frac{1}{3}\left(1 - x^3\right)$$

[4] A word of caution: some computer programs that perform symbolic calculus do not successfully obtain this general solution. This is because they use $\int \frac{1}{x} dx = \ln x + c$ instead of $\int \frac{1}{x} dx = \ln |x| + c$.

by separating variables :

$$\frac{3}{1-x^3}dx = dt$$
$$\int \frac{3}{1-x^3}dx = \int dt + c.$$

The integral on the left hand side can be calculated using a computer program, a table of integrals, or by hand (using partial fraction decomposition). From this calculation we obtain

$$\ln\left|\frac{\sqrt{x^2+x+1}}{x-1}\right| + \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = t + c. \tag{3.18}$$

This equation implicitly defines all non-equilibrium solutions. (The only equilibrium is $x_e = 1$.) The equation cannot be algebraically solved explicitly for x. Moreover, because of the complexity of equation (3.18), we cannot easily use it to determine what happens to x as $t \to +\infty$. The answer, however, is rather easily obtained by applying the methods of Sec. 3.1.2 to $f(x) = (1-x^3)/3$ to obtain the phase line portrait

$$\longrightarrow \quad 1 \quad \longleftarrow$$

All solutions approach the equilibrium $x_e = 1$ as $t \to +\infty$ and do so in a strictly monotonic fashion.

3.3 Approximation Formulas By Perturbation Methods

Consider the initial value problem

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0. \end{aligned} \tag{3.19}$$

Under the assumptions of the Fundamental Existence and Uniqueness Theorem 1.1 (i.e., when f and df/dx are continuous at $x = x_0$ and $t = t_0$) there is a unique solution $x(t)$ to this problem. When formulas for the solution $x(t)$ cannot be feasibly obtainable, we can attempt to approximate solutions in some way. For example, in Chapter 1 we learned how to approximate the solution numerically and also how to approximate its graph.

Another approach is to obtain a formula for a function that, while not a solution itself, is a good approximations approximation to the solution $x(t)$. There are many methods available to calculate approximation formulas, each designed for certain types of differential equations and for approximations valid under certain conditions. Many of these methods are based, in one way or another, on series representations of the solution that are truncated to obtain and approximations. Power series and Fourier series as examples.

In this section we will consider only one representative example of an approximation procedure, one that is based on truncated power series, i.e. on Taylor polynomials. This method

is a classic method which has been historically, and continues to be today, of widespread use in the analysis of differential equations. It is designed to approximate solutions of a differential equation by constructing Taylor polynomials in a specific coefficient that appears in the equation.

Before we explain what this means, let's recall some basics about Taylor polynomials from your calculus course.

Consider a function $y = y(x)$ near a point $x = a$. The n^{th} degree Taylor polynomial of $y(x)$ centered at $x = a$ is

$$p_n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

where the coefficients in this polynomial are given by the formulas

$$c_n = \frac{1}{n!} \left. \frac{d^n y}{dx^n} \right|_{x=a}.$$

This notation is probably that used in your calculus course. Of course, the letters y and x can be different. What's important is that x is the independent variable and y is the dependent variable.

In our use of Taylor polynomials, the independent variable will be ε and the center will be $a = 0$. We will also use the letter k_n for the coefficients, instead of c_n. If $g = g(\varepsilon)$ is the dependent variable, then the n^{th} degree Taylor polynomial of $g(\varepsilon)$ centered at $\varepsilon = 0$ is

$$p_n = k_0 + k_1 \varepsilon + k_2 \varepsilon^2 + \cdots + k_n \varepsilon^n \tag{3.20}$$

where the coefficients in this polynomial are given by the formulas

$$k_n = \frac{1}{n!} \left. \frac{d^n g}{d\varepsilon^n} \right|_{\varepsilon=0}. \tag{3.21}$$

The approximation method we consider is applicable for differential equations in which ε appears as a coefficient:

$$x' = f(t, x, \varepsilon). \tag{3.22}$$

In this equation, we consider ε to be a small parameter (hence the choice of the Greek letter ε). Thus, this equation can be consider a "perturbation" of the equation obtained when $\varepsilon = 0$.

For example,

$$x' = -x + \varepsilon x^2$$

is an example of an autonomous, nonlinear equation which is a perturbation of the linear equation

$$x' = -x.$$

Another example, is the non-autonomous perturbation

$$x' = r \left(1 - \frac{x}{K(1 + \varepsilon \sin t)} \right) x$$

of the logistic equation

$$x' = r\left(1 - \frac{x}{K}\right)x.$$

Our goal is to approximate solutions of such perturbed equations, as functions of ε, for small values of ε. Often we want to approximate the unique solution to an initial value problem. Other times, it might be a solution with some other specified property that we wish to approximate (for example, a periodic solution).

Since ε appears in the differential equation, the solutions x of the equation depend on ε. The procedure we use to calculate perturbation approximations will be to construct Taylor polynomials of x in terms of ε (centered at $\varepsilon = 0$), as given by (3.20) and (3.21). This will be, yet again, an application of the Method of Undetermined Coefficients (Chapter 2, Section 2.2) applied by substituting the Taylor polynomial (3.20) into the differential equation, performing the required operations on both sides of the equation, and equating the results in order to determine the coefficients $k_0, k_1, ..., k_n$.

Remark 2. Since the solution x of the differential equation is a function of t (as well as being a function of the coefficient ε), it follows that the coefficients the Taylor polynomial (3.20) are functions of t :

$$p_n(t) = k_0(t) + k_1(t)\varepsilon + k_2(t)\varepsilon^2 + \cdots + k_n(t)\varepsilon^n. \tag{3.23}$$

Remark 3. Recall that two polynomials in ε (or more generally, two power series in ε) are identical if and only if the coefficients of like powers of ε are identical. When applying the Method of Undetermined Coefficients to the Taylor polynomial (3.23), you will need to equate two polynomials in ε (obtained from the left and right sides of the differential equation). You do this by equating coefficients of like powers of ε^n (or, as it turns out, as many as possible, since the polynomials will not necessarily have the same degree). The result will be (differential) equations to solve for the undetermined coefficients $k_0(t), k_1(t), ..., k_n(t)$.

Remark 4. It turns out that, although the procedure is straightforward in principle, the details of the perturbation method described here generally become formidable and intractable for high degree perturbation approximations. For that reason we will focus our attention on low degree perturbation approximations only.

Remark 5. When the coefficients $k_0(t), k_1(t), ..., k_n(t)$ in (3.23) are calculated by the Method of Undetermined Coefficients so as to approximate a solution of the equation (3.22), the Taylor polynomial $p_n(t)$ is called the n^{th} *degree perturbation approximation* of the solution. This method is therefore called a *perturbation approximation method.*

As an example, consider the initial value problem

$$\begin{aligned} x' &= -x + \varepsilon x^2 \\ x(0) &= 1. \end{aligned} \tag{3.24}$$

We seek the first order perturbation approximation

$$p_1(t) = k_0(t) + k_1(t)\varepsilon \tag{3.25}$$

to the (unique) solution. We calculate the coefficients $k_0(t)$ and $k_1(t)$ by substituting this expression into the differential equation (3.24) and equating coefficients of like powers of ε. For notation simplification, we drop the functional notation "(t)".

From the left hand side we get

$$p_1' = k_0' + k_1'\varepsilon. \tag{3.26}$$

From the right hand side we get

$$-p_1 + \varepsilon p_1^2 \;=\; -\left(k_0 + k_1\varepsilon\right) + \varepsilon\left(k_0 + k_1\varepsilon\right)^2.$$

Both of these are polynomials in ε and in order to equate them (i.e., equate the coefficients of like powers of ε) we need to write the right side of the equation as a polynomial in ε. In this example, this is a straightforward algebraic calculation:

$$-p_1 + \varepsilon p_1^2 \;=\; -k_0 + \left(-k_1 + k_0^2\right)\varepsilon + \left(2k_0k_1\right)\varepsilon^2 + \left(k_1^2\right)\varepsilon^3.$$

Since this is a cubic polynomial in ε, there is no way we can make it identical to the left side (3.26) which is a first degree polynomial in ε. So we match all coefficients we can, namely the zeroth and first order ε coefficients, leaving the ε^2 and ε^3 unmatched on the right side. (This is what makes this an approximation and not an exact solution.) We obtain the two (differential) equations

$$\begin{aligned} k_0' &= -k_0 \\ k_1' &= -k_1 + k_0^2 \end{aligned}$$

for the two coefficients $k_0 = k_0(t)$ and $k_1 = k_1(t)$. Notice that these are differential equations.

Remark 6. Since the coefficient matching only involve ε terms up to order one (as will always be the case when calculating $p_1(t)$), we could have ignored the ε^2 and ε^3 terms on the right side of the differential equation and written

$$-p_1 + \varepsilon p_1^2 \;=\; -k_0 + \left(-k_1 + k_0^2\right)\varepsilon + \cdots$$

where the dots denote all the higher order powers in ε. That is to say, in so far as calculating $p_1(t)$ is concerned, there is no need to waste our time in calculating higher order ε terms on the right side of the differential equation. (This observation can say a lot of work in future problems.)

Remark 7. Also notice that the second equation for k_1 involves k_0. Strictly speaking this is a system of two differential equations for two unknowns. We won't take up the study of systems of equations in later chapters. However, we can deal with this system in the following way. Since the first equation for k_0 does not involve k_1 we can solve it for k_0. We can then use the answer in the second equation, which then becomes a single equation of k_1.

We have not yet involved the initial condition in the approximation method. We do so now, in order to obtain initial conditions for the differential equations for k_0 and k_1.

We apply (yet again) the Method of Undetermined Coefficients to the initial condition $x(0) = 1$ in (3.24). That is, we substitute $p_1(t)$ into the initial condition to obtain

$$k_0(0) + k_1(0)\varepsilon = 1 + 0 \cdot \varepsilon.$$

The reason the initial condition 1 has been written as $1 + 0 \cdot \varepsilon$ is to emphasize that we are treating it as a (constant) polynomial in ε, so that we can match coefficients of like powers of ε. The results are $k_0(0) = 1$ and $k_1(0) = 0$, which lead us to the two initial value problems

$$\begin{aligned} k_0' &= -k_0 \\ k_0(0) &= 1 \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} k_1' &= -k_1 + k_0^2 \\ k_1(0) &= 0 \end{aligned} \tag{3.28}$$

to solve for the coefficients k_0 and k_1 respectively. (Notice the "lowest order" initial value problem (3.27) for k_0 is the same as the original initial value problem (3.24) with $\varepsilon = 0$.)

As pointed out in Remark 7 above, we now solve these two initial value problems sequentially, calculating k_0 first and using it in the differential equation (3.28) for k_1. The (linear homogeneous) initial value problem (3.27) has the solution

$$k_0(t) = e^{-t}.$$

This yields the (linear nonhomogeneous) initial value problem

$$\begin{aligned} k_1' &= -k_1 + e^{-2t} \\ k_1(0) &= 0 \end{aligned}$$

whose solution is

$$k_1(t) = e^{-t} - e^{-2t}.$$

Using these coefficients k_0 and k_1 we obtain the first order approximation

$$p_1(t) = e^{-t} + \left(e^{-t} - e^{-2t}\right)\varepsilon \tag{3.29}$$

to the solution of the initial value problem (3.24).

Remark 8. We recall that Taylor polynomials are "nested". That is if

$$p_n = p_{n-1} + k_n \varepsilon^n.$$

This, if

$$p_2 = k_0 + k_1\varepsilon + k_2\varepsilon^2$$

is a second degree perturbation approximation, then

$$\begin{aligned} p_0 &= k_0 \\ p_1 &= k_0 + k_1\varepsilon. \end{aligned}$$

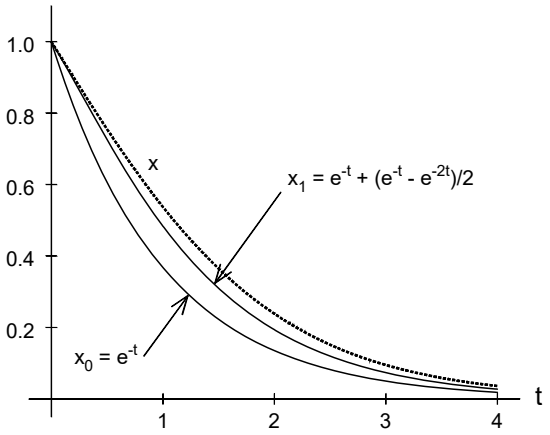


Figure 3.23

The graphs in Figure 3.23 of the first order approximation $p_1(t)$ (with $\varepsilon = 1/2$) and the zeroth order approximation $x_0(t) = k_0 = e^{-t}$ as approximations to the solution of the initial value problem (3.24) with $\varepsilon = 1/2$ show the extent of the accuracy of these approximations to the exact solution of the initial value problem (whose graph is computer estimated in Figure 3.23 as well). Also notice that $p_1(t)$ is a more accurate approximation than is $p_0(t)$, which is to be expected. We expect higher order perturbation to be more accurate than lower order approximations.

One can also use the perturbation method to approximate a solution of differential equation that has a desired property (rather than satisfying an initial condition). For example, we might be interested in approximating a periodic solution of a differential equation. We can use the perturbation method to do this by requiring that the coefficients k_i in (3.23) to be periodic functions. This requires solving the linear differential equations for periodic solutions $k_i = k_i(t)$, rather than for specified initial conditions. See Exercises 3.158 and 3.159 for examples.

3.4 Chapter Summary

A great deal can be learned about the solutions autonomous equation $x' = f(x)$, without having to calculate a formulas for solutions, by means of phase line portraits. A phase line portrait summarizes the monotonicity properties of solutions and their asymptotic properties, as well as the classification of its equilibria (as attractors, repellers, and shunts). We learned several ways to construct phase line portraits, including a graphical method based on a plot of $f(x)$ and a method based on the Derivative Test for equilibria. Phase line portraits form the basis of bifurcation theory, which is the study of how phase line portraits depend on a parameter p that appears in the differential equation. A bifurcation point p_0 is a value of p at which the phase line portrait significantly changes in a way made precise by the notion of the qualitative equivalence of phase line portraits. We studied three basic types of bifurcations (blue-sky, pitchfork, and transcritical) and how they can be graphically represented in a bifurcation diagram.

Autonomous equations are an important special case of separable equations. It is possible to find solution formulas for separable equations (including autonomous equations) by the Separation of Variables Method provided appropriate integrals can be calculated. In general methods are available for calculating solution formulas only for special types of differential equations, such as linear and separable equations. In place of solution formulas, one can instead calculate formulas for approximations to solutions. In this chapter we looked at one example of solution approximation based on Taylor series methods. The perturbation method calculates approximations to solutions of differential equations that contain a small parameter.

3.5 Exercises

Exercise 3.1 *If $x = x(t)$ is a solution of an autonomous equation $x' = f(x)$ for all t, show $y = x(t + k)$ is also a solution for any constant k. This shows that "translations" $x(t + k)$ of solutions of autonomous equations are also solutions. This is a defining characteristic of autonomous equations.*

Find all equilibria of the equations below. Find exact solutions, if possible. Otherwise use a computer or calculator to obtain numerical estimates.

Exercise 3.2 $x' = x^2 + 2x - 3$

Exercise 3.3 $x' = 4x^3 - 4x^2 - x + 1$

Exercise 3.4 $x' = \ln\left(\frac{2x}{1+x}\right)$

Exercise 3.5 $x' = -1 + 3x\left(1 + x^2\right)^{-1}$

Exercise 3.6 $x' = x - 2 - e^{-x}$

Exercise 3.7 $x' = 1 - x - \ln(1 + x)$

How many equilibria do the following equations have? You might find different answers for different values of a. (Hint: use geometric methods to study the roots of the equilibrium equation.)

Exercise 3.8 $x' = e^{ax} - x^2, \quad a > 0$

Exercise 3.9 $x' = ax - e^{-x^2}, \quad a \neq 0$

Exercise 3.10 $x' = a - x - x^2\left(1 + x^2\right)^{-1}, \quad a > 1$

Exercise 3.11 $x' = x^4 - xe^{-ax}, \quad a > 0$

For which initial values x_0 are the solutions of the following equations strictly increasing and for which are they strictly decreasing? Justify your answers using theorems in the text.

Exercise 3.12 $x' = x(1 - x^4)$

Exercise 3.13 $x' = \ln\left(x^2 + 1/4\right)$

Exercise 3.14 $mx' = g - cx^2$ *where* m, g, c *are positive constants.*

Exercise 3.15 $x' = r\left(1 - x/K\right)x$ *where* r, K *are positive constants.*

Exercise 3.16 $x' = 6x^2 - 5x + 1$

Exercise 3.17 $x' = (1 - x)\left(1 - e^{-x}\right)$

For each equation below find all equilibria. Sketch a graph of $f(x)$ and use it to draw the phase line portrait. Identify the type of each equilibrium.

Exercise 3.18 $x' = x^3 - 1$

Exercise 3.19 $x' = x^3 - x$

Exercise 3.20 $x' = x^3 - x^2$

Exercise 3.21 $x' = x - e^{-x}$

For each equation below find all equilibria. Use the Derivative Test to determine which are hyperbolic attractors and which are hyperbolic repellers. Otherwise use the Geometric Test. Draw the phase line portrait. (a is a constant.)

Exercise 3.22 $x' = -x + \cos x$

Exercise 3.23 $x' = x^2(x-1)^3(2-x)$

Exercise 3.24 $x' = x(x+1)(x-0.5)^4$

Exercise 3.25 $x' = x(1-e^x)$

Exercise 3.26 $x' = (1-x^2)(1-e^{1-x})$

Exercise 3.27 $x' = x^2 - a$

Exercise 3.28 $x' = -x^3 + (1+a)x^2 - ax$

Exercise 3.29 $x' = a - x^2(1+x^2)^{-1}$

Exercise 3.30 *Find all equilibria of the equation* $x' = x\sin x$ *and determine which are hyperbolic. Identify the type of all equilibria. Draw the phase line portrait.*

Exercise 3.31 *Find all equilibria of the equation* $x' = \sin^2 x$ *and determine which are hyperbolic. Identify the type of all equilibria. Draw the phase line portrait.*

For each of the phase line portraits drawn below, write down a first order differential equation of the form $x' = f(x)$. There are infinitely many possible answers for each portrait. The simplest approach is to use a polynomial for $f(x)$.

Exercise 3.32 $\longrightarrow -3 \longleftarrow 3 \longrightarrow$

Exercise 3.33 $\longleftarrow -3 \longrightarrow 3 \longleftarrow$

Exercise 3.34 $\longrightarrow 0 \longrightarrow 2 \longleftarrow$

Exercise 3.35 $\longleftarrow 1 \longleftarrow 10 \longrightarrow$

Exercise 3.36 $\longrightarrow 0 \longrightarrow 1 \longrightarrow$

Exercise 3.37 $\longleftarrow -2 \longleftarrow 5 \longleftarrow$

Exercise 3.38 $\longrightarrow a \longleftarrow b \longrightarrow$

Exercise 3.39 $\longleftarrow a \longrightarrow b \longleftarrow$

Exercise 3.40 $\longrightarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow$

Exercise 3.41 $\longleftarrow 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow$

Exercise 3.42 $\longrightarrow a \longleftarrow b \longrightarrow c \longleftarrow d \longleftarrow$

Exercise 3.43 $\longrightarrow a \longrightarrow b \longrightarrow c \longleftarrow d \longleftarrow$

Exercise 3.44 *The velocity $v \geq 0$ of a falling object satisfies the equation*

$$v' = 9.8 - k_0 v^2$$

where $k_0 > 0$ is the (per unit mass) coefficient of friction. Draw the phase line portrait for $v \geq 0$. Classify the positive equilibrium. Is this equilibrium hyperbolic?

Exercise 3.45 *Let $x = x(t)$ denote the temperature of an object. According to a modified Newton's Law of Cooling x satisfies the differential equation*

$$x' = a\,(b - x)^p$$

where b is the constant environment temperature, a is a positive constant, and p is an odd integer $p \geq 1$. Draw the phase line portrait. Classify the equilibrium. Is the equilibrium hyperbolic?

Exercise 3.46 *The concentration $c = c(t)$ of a substrate in a container in which an enzyme is present (in constant concentration $e > 0$) satisfies the differential equation*

$$c' = d\,(c_{in} - c) - \frac{mc}{a + c}e.$$

All coefficients d, c_{in}, m and a are positive constants. In this exercise let $d = c_{in} = a = e = 1$ and $m = 2$. Draw the phase line portrait for $c \geq 0$. Classify the equilibrium. Is the equilibrium hyperbolic?

Find the linearization of the following equations at each of their equilibria.

Exercise 3.47 $x' = x^3 - x$

Exercise 3.48 $x' = \sin x$

Exercise 3.49 $x' = rx\,(1 - x/K)\,$, *where r, K are positive constants*

Exercise 3.50 $mx' = mg - cx^2$, *where m, g, c are positive constants*

Exercise 3.51 $x' = x^3\,(1 + x^2)^{-1}$

Exercise 3.52 $x' = \sqrt{2}x - 4x^2\,(1 + x^2)^{-1}$

Exercise 3.53 $x' = (1 - x)x - h$ *where* $0 \leq h \leq 1/4$

Exercise 3.54 *The velocity $v \geq 0$ of a falling object satisfies the equation*

$$v' = 9.8 - k_0 v^2$$

where $k_0 > 0$ is the (per unit mass) coefficient of friction. Find this equation's positive equilibrium, and then find the linearization at this equilibrium. What kind of equilibrium does the linearization have? If applicable, use the Linearization Principle (Theorem 3.5) to classify the positive equilibrium.

Exercise 3.55 *Let $x = x(t)$ denote the temperature of an object. According to a modified Newton's Law of Cooling x satisfies the differential equation*

$$x' = a\,(b - x)^p$$

where b is the constant environment temperature, a is a positive constant, and $p \geq 1$ is an odd integers. What is the equilibrium and what is the linearization at the equilibrium? If applicable, use the Linearization Principle (Theorem 3.5) to classify the equilibrium.

Exercise 3.56 *The concentration $c = c(t)$ of a substrate in a container in which an enzyme is present (in constant concentration $e > 0$) satisfies the equation*

$$c' = d\,(c_{in} - c) - \frac{mc}{a + c}e.$$

All coefficients d, c_{in}, m and a are positive constants. In this exercise let $d = c_{in} = a = e = 1$ and $m = 2$. Find this equation's positive equilibrium, and then find the linearization at this equilibrium. If applicable, use the Linearization Principle (Theorem 3.5) to classify the positive equilibrium.

Which pairs of equations have qualitatively equivalent phase line portraits? Which do not and why?

Exercise 3.57 $\begin{cases} x' = 2 - 3x \\ x' = 3 - 2x \end{cases}$

Exercise 3.58 $\begin{cases} x' = 2 + 3x \\ x' = 3 + 2x \end{cases}$

Exercise 3.59 $\begin{cases} x' = px + q,\ p < 0 \\ x' = -a^2 x + 1,\ a \neq 0 \end{cases}$

Exercise 3.60 $\begin{cases} x' = -x + q \\ x' = px + q,\ p > 0 \end{cases}$

Exercise 3.61 $\begin{cases} x' = x - e^{-x} \\ x' = -x \end{cases}$

Exercise 3.62 $\begin{cases} x' = x - e^{-x} \\ x' = x \end{cases}$

Exercise 3.63 $\begin{cases} x' = (x-1)(x+2)^2 \\ x' = (x-2)^3(x+1)^{10} \end{cases}$

Exercise 3.64 $\begin{cases} x' = (2x-1)(x-2)^3 \\ x' = (2x-1)^2(x-2)^3 \end{cases}$

Exercise 3.65 $\begin{cases} x' = 1 - 2e^{-x^2} \\ x' = 2 - 2e^{-x^2} \end{cases}$

Exercise 3.66 $\begin{cases} x' = 1 + bx + x^2,\ b > 2 \\ x' = px + q,\ p > 0 \end{cases}$

Exercise 3.67 $\begin{cases} x' = -1 + 5x - x^2 \\ x' = x(1-x) \end{cases}$

Exercise 3.68 $\begin{cases} x' = e^x \frac{x^2-1}{x^2+1} \\ x' = x^4 - 1 \end{cases}$

Exercise 3.69 $\begin{cases} x' = 1 - x + x^2 - x^3 \\ x' = -x \end{cases}$

Exercise 3.70 $\begin{cases} x' = 1 - x^4 \\ x' = 1 - x^3 \end{cases}$

Exercise 3.71 $\begin{cases} x' = x(1+x^2) \\ x' = x\left(1-x^2\right) \end{cases}$

Exercise 3.72 $\begin{cases} x' = -2 + \sin 2x \\ x' = \frac{1}{1+e^x} \end{cases}$

Exercise 3.73 $\begin{cases} x' = 1 \\ x' = e^{-x} \end{cases}$

Exercise 3.74 $\begin{cases} x' = (1+\sin^2 x)(e^x - 1) \\ x' = x \end{cases}$

Draw the bifurcation diagram for each equation below, over the indicated range of the parameter p. Locate and classify all bifurcations. Determine the stability or instability of each equilibrium and indicate the result on your diagram.

Exercise 3.75 $x' = p - x^2, \quad -\infty < p < +\infty$

Exercise 3.76 $x' = p - 1 - (x-1)^2, \quad -\infty < p < +\infty$

Exercise 3.77 $x' = x\left(x^2 - 1 - p\right), \quad -\infty < p < +\infty$

Exercise 3.78 $x' = (x-1)\left(p - (x-1)^2\right), \quad -\infty < p < +\infty$

Exercise 3.79 $x' = (x-1)\left(p-x^2\right), \quad -\infty < p < +\infty$

Exercise 3.80 $x' = \left(x^2-1\right)\left(p-x^2\right), \quad -\infty < p < +\infty$

Exercise 3.81 $x' = px^2 + (x-1)(x-4), \quad -\infty < p < +\infty$

Exercise 3.82 $x' = (x-p)^2, \quad -\infty < p < +\infty$

Exercise 3.83 $x' = p - e^x, \quad 0 < p < +\infty$

Exercise 3.84 $x' = p\left(x-1\right)^2, \quad -\infty < p < +\infty$

Exercise 3.85 $x' = p\left(x-1\right)^3, \quad -\infty < p < +\infty$

Exercise 3.86 $x' = p + x(1-x)(x-2), \quad -\infty < p < +\infty$

Exercise 3.87 $x' = p - x(1-\frac{1}{27}x^2), \quad -\infty < p < +\infty$

Exercise 3.88 $x' = p - x^3, \quad -\infty < p < +\infty$

Exercise 3.89 $x' = p - x^4, \quad -\infty < p < +\infty$

Exercise 3.90 $x' = p - e^{-x^2}, \quad 0 < p < +\infty$

Exercise 3.91 $x' = x^2\left(p - e^{-x^2}\right), \quad 0 < p < +\infty$

Exercise 3.92 *Show all phase line portraits of the logistic equation* $x' = rx\left(1-x/K\right)$, $K > 0$, $r > 0$, *are qualitatively equivalent.*

Exercise 3.93 *A logistically growing population* $x(t)$ *satisfies the differential equation* $x' = rx\left(1-x/K\right)$, $r > 0$, $K > 0$. *If the population is harvested at a constant rate* $h > 0$, *then* $x' = rx\left(1-x/K\right) - h$. *Draw a bifurcation diagram using* $h > 0$ *as a parameter. Identify any bifurcations that occur. Sketch the relevant phase line portraits. What are the biological implications of the bifurcation diagram?*

Exercise 3.94 *Show the bifurcation diagram of the equation* $x' = -px - x^3$ *is a pitchfork that opens to the left. Describe the stability properties of the branches.*

Exercise 3.95 *Show the bifurcation diagram of the equation* $x' = -px + x^3$ *is a pitchfork that opens to the right. Describe the stability properties of the branches.*

Exercise 3.96 *Write down a differential equation whose bifurcation diagram is a pitchfork that opens to the left in which a repeller is exchanged between the branches.*

Exercise 3.97 *Definition 3.7 of qualitatively equivalent phase portraits is based upon the geometry of the portraits. It turns out that although this definition works well for first order equations, it is too simplistic for systems of equation (and higher order equations). For these "higher dimensional" problems a more general definition is needed. One common definition is based upon the topological notion that two phase portraits are equivalent if one can be "continuously distorted" into the other. Mathematicians make this notion precise by using* ***homeomorphisms****. A homeomorphism is a continuous function $h : R \to R$ that has a continuous inverse. Two equations are said to have "qualitatively equivalent" (or "topologically equivalent") phase portraits if by making a change of variables from x to $h(x)$ one phase portrait is mapped to the other such that orbits go to orbits with their orientations preserved. Analytically this can be tested as follows. Consider two autonomous first order equations*

$$x' = f(x)$$
$$x' = g(x)$$

and let $x = x(t, x_0)$ and $x = \psi(t, x_0)$ be the solutions with initial value $x(0) = x_0$. The phase portraits are qualitatively equivalent if there exists a homeomorphism $h(x)$ such that

$$h(x(t, x_0)) = \psi(t, h(x_0))$$

for all t in the domains of the solutions.

(a) Show

$$h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2 & \text{if } x < 0 \end{cases}.$$

defines a homeomorphism.

(b) Use the homeomorphism in (a) to show the phase portraits of the equations

$$x' = -x$$
$$x' = -2x$$

are qualitatively equivalent.

Exercise 3.98 *Using the homeomorphism in Exercise 3.97 as a guide construct a homeomorphism $h(x)$ and use it to show the phase portraits of the two linear equations*

$$x' = ax, \quad a \neq 0$$
$$x' = bx, \quad b \neq 0$$

are qualitatively equivalent if a and b have the same sign.

Find formulas for the general solution of the following equations. Implicit formulas are acceptable when explicit formulas are not possible.

Exercise 3.99 $x' = 1 + x^2$

Exercise 3.100 $x' = 1 - x^2$

Exercise 3.101 $x' = x - x^{-1}$

Exercise 3.102 $x' = 1 - x^4$

Exercise 3.103 $x' = \cot x$

Exercise 3.104 $x' = x^2(1 - x)$

Find formulas for the solutions of the following initial value problems.

Exercise 3.105 $x' = -x^4$, $x(1) = 1$

Exercise 3.106 $x' = 1 + x^2$, $x(\pi) = 1$

Exercise 3.107 $x' = -x^{-1}e^{-x}$, $x(0) = -1$

Exercise 3.108 $x' = (x^4 - 1)\, x^{-3}$, $x(0) = \sqrt{2}$

Exercise 3.109 $x' = x - x^{-1}$, $x(0) = 1/2$

Exercise 3.110 $x' = x - x^{-1}$, $x(0) = 2$

Find explicit formulas for the solutions of the following initial value problems.

Exercise 3.111 $x' = x - x^{-1}$, $x(0) = -1/2$

Exercise 3.112 $x' = x - x^{-1}$, $x(0) = -3$

Exercise 3.113 *The velocity $v = v(t) \geq 0$ of a falling object subject to constant force of gravity and a quadratic law for the frictional force of air resistance satisfies the equation*

$$v' = 9.8 - k_0 v^2.$$

Here 9.8 (meters/sec^2) is acceleration due to gravity and k_0 is the (per unit mass) coefficient of friction.

(a) Find an explicit formula for the general solution of this equation.

(b) If the object is dropped, then $v(0) = 0$. Find a formula for the solution of this initial value problem.

(c) Use your answer in (b) to calculate the limiting velocity $\lim_{t\to+\infty} v(t)$.

Exercise 3.114 *A modified Newton's Law of Cooling yields the equation*

$$x' = a\,(b - x)\,|b - x|^p$$

for the temperature of an object. Here the constant b is the environmental temperature. The positive constants a and p depend on properties of the object (its material, geometry, etc.).

(a) Find a formula for the general solution when $p = 1/3$.

(b) Let $x_0 > b$ be the initial temperature of the object. Find a formula for the solution of the initial value problem.

(c) Use your answer in (b) to calculate the long term temperature of the object, i.e., $\lim_{t\to+\infty} x$.

Exercise 3.115 *Suppose $f(x)$ is a continuously differentiable function of x and suppose $x = y(t)$ is a solution of the autonomous equation $x' = f(x)$. Show that if $f(y(t^*)) = 0$ for some value of $t = t^*$ then $y(t)$ must be an equilibrium. (Hint: note that $y(t^*)$ is a root of $f(x)$ and apply Theorem 1.1 to the initial value problem $x' = f(x)$, $x(t^*) = y(t^*)$.)*

Exercise 3.116 *Find the solution of the initial value problem $x' = x^2 - x$, $x(0) = x_0$ and determine its maximal interval of existence $\alpha < t < \beta$. Show that if $x_0 > 1$ then $\beta < +\infty$.*

Exercise 3.117 *Find the solution of the initial value problem $x' = x^2 - 1$, $x(0) = x_0$ and determine its maximal interval of existence $\alpha < t < \beta$. Show that if $x_0 > 1$ then $\beta < +\infty$.*

Find formulas for the general solutions of the following equations.

Exercise 3.118 $x' = t^2 x$

Exercise 3.119 $x' = t^{-1} x$

Exercise 3.120 $x' = t^{-2} x^2$

Exercise 3.121 $x' = e^{t+x}$

Exercise 3.122 $x' = t - tx^2$

Exercise 3.123 $x' = t^2 \tan x$

Exercise 3.124 $x' = (x^2 - 3x + 2)e^t$

Exercise 3.125 $x' = (t + 1)x^{-4}$

Exercise 3.126 $x' = (a^2 - x^2)\cos t$

Exercise 3.127 $x' = x^a t^b$

Find formulas for the solutions of the following initial value problems.

Exercise 3.128 $x' = t^2 x$, $x(0) = 1$

Exercise 3.129 $tx' + \sqrt{x} = 0$, $x(1) = 2$

Exercise 3.130 $x' = (t + 1)x^{-4}$, $x(0) = -1$

Exercise 3.131 $x' = e^{t+x}$, $x(0) = 0$

Exercise 3.132 $x' = 2tx^{-a}$, $x(1) = b$

Exercise 3.133 $x' = 2at(x^2 + a^2)$, $x(0) = 0$

Exercise 3.134 *Find a formula for the solution of the initial value problem $x(0) = x_0 > 0$ for equation (3.13). Use your formula to calculate $\lim_{t\to+\infty} x(t)$.*

Exercise 3.135 *Suppose $x(t)$ is a solution of $x' = g(t)h(x)$ and suppose $h(x(t^*)) = 0$ for some t^*. Show $x(t)$ must be an equilibrium.*

Find the first order perturbation approximation $p_1(t) = k_0(t) + k_1(t)\varepsilon$ for the solution of the following initial value problems.

Exercise 3.136 $x' = (1 + \varepsilon e^{-t})x,\ x(0) = 2$

Exercise 3.137 $x' = (\sin t + \varepsilon \cos t)x,\ x(0) = -1$

Exercise 3.138 $x' = 2x - \varepsilon \sin t,\ x(0) = 1$

Exercise 3.139 $x' = -x + \varepsilon e^{-2t},\ x(0) = -7$

Exercise 3.140 $x' = x - x^2 + \varepsilon \sin t,\ x(0) = 1$

Exercise 3.141 $x' = x - x^2 + \varepsilon e^{-t},\ x(0) = 1$

Exercise 3.142 $x' = 1 + \varepsilon x\,(1 - x)\,,\ x\,(0) = 1$

Exercise 3.143 $x' = 2 + \varepsilon x\,(1 - x)\,,\ x\,(0) = 0$

Exercise 3.144 $x' = \sin\,(\varepsilon x)\,,\ x\,(1) = -1$

Exercise 3.145 $x' = \sin\,(\varepsilon x)\,,\ x\,(1) = -2$

Exercise 3.146 $x' = 1 - x \exp\,(\varepsilon x)\,,\ x\,(0) = 0$

Exercise 3.147 $x' = 1 - x \exp\,(\varepsilon x)\,,\ x\,(0) = 1$

Exercise 3.148 $x' = -x + \varepsilon e^t x^3,\ x\,(0) = 2$

Exercise 3.149 $x' = -x + \varepsilon e^t x^4,\ x\,(0) = -1$

Exercise 3.150 $x' = x + \varepsilon \frac{1+3e^t}{1+x},\ x\,(0) = 3$

Exercise 3.151 $x' = \frac{x+1}{1+\varepsilon(x+1)},\ x\,(0) = 0$

Exercise 3.152 *Consider the initial value problem*

$$\begin{aligned} x' &= 4 - \frac{1}{4}x^2 + \varepsilon \sin(2t) \\ x(0) &= 4 \end{aligned}$$

(a) Find the coefficients $k_0(t)$, $k_1(t)$ in the first order perturbation approximation $p_1(t) = k_0(t) + k_1(t)\varepsilon$.

(b) Use a computer to graph the solution with $\varepsilon = 1$, together with the perturbation approximation obtained in (a). How do they compare?

Find the initial value problems for the coefficients $k_0(t)$, $k_1(t)$, $k_2(t)$ in the second order perturbation approximation $p_2(t) = k_0(t) + k_1(t)\varepsilon + k_2(t)\varepsilon^2$ to the solution of the following initial value problems. Solve these initial value problems and construct the approximation $p_2(t)$.

Exercise 3.153 $x' = -(1 + \varepsilon \sin t)x$, $x(t_0) = x_0$

Exercise 3.154 $x' = x - \varepsilon x^2$, $x(t_0) = x_0$

Exercise 3.155 *Find the recursive formulas for the coefficients $k_i(t)$ of the n^{th} order perturbation approximation $p_n(t) = \sum_{i=0}^{n} k_i(t)\varepsilon^i$ of the solution of the initial value problem*

$$x' = -x + \varepsilon \sin t, \ x(0) = 1.$$

Show all coefficients $k_i(t)$ except $k_0(t)$ and $k_1(t)$ are equal to 0 and that therefore the solution has the form $x(t) = k_0(t) + k_1(t)\varepsilon$.

Exercise 3.156 *Consider the initial value problem*

$$x' = a\left(b + \varepsilon \sin\left(\frac{2\pi}{T}t\right) - x\right), \ x(0) = x_0.$$

obtained from Newton's Law of Cooling for an object placed in an environment with a sinusoidally fluctuating temperature $b + \varepsilon \sin(2\pi t/T)$.

(a) Using the amplitude ε of the environmental temperature oscillations as a small parameter, find the initial value problems for the coefficients $k_i(t)$ in the second order perturbation approximation $p_2(t) = c_0(t) + c_1(t)\varepsilon + c_2(t)\varepsilon^2$ of the solution.

(b) Solve the initial value problems for $p_1(t)$ and $p_2(t)$.

(c) Show all coefficients $k_i(t) \equiv 0$, $i \geq 2$, and therefore that the first order perturbation approximation in fact is the exact solution.

Exercise 3.157 *Consider an object falling under the influence of gravity. If we assume the acceleration due to gravity is a constant g and air resistance exerts a frictional force proportional to the object's velocity $v = v(t)$, then $v' = g - cv$, where $c > 0$ is the coefficient of friction. Suppose also present is an additional small frictional force that is proportional to v^2. Then $v' = g - cv - \varepsilon v^2$ where $\varepsilon > 0$ is a small constant. Assume the object is dropped, so that we have the initial condition $v(0) = 0$.*

(a) Find the first order perturbation approximation $p_1(t) = k_0(t) + k_1(t)\varepsilon$ to the initial value problem for v.

(b) An approximation to the terminal velocity v_e of the object is the limit as $t \to +\infty$ of the first order perturbation approximation found in (a). Find this approximation.

(c) Draw the phase line portrait associated with the equation $v' = g - cv - \varepsilon v^2$.

(d) Find a formula for the terminal velocity v_e.

(e) Compare your answers in (b) and (d). (Hint: find the first terms in the Taylor series of your answer in (d), with respect to ε, and compare it to your answer in (b).)

Exercise 3.158 *Consider the differential equation*

$$x' = x\left(1 - \frac{2}{9}x\right) - (1 + \varepsilon \sin 2\pi t)\,.$$

(a) Show the coefficients of the first order perturbation approximation $p_1(t) = k_0(t) + k_1(t)\varepsilon$ *satisfy the differential equations*

$$k_0' = k_0\left(1 - \frac{2}{9}k_0\right) - 1$$
$$k_1' = k_1\left(1 - \frac{4}{9}k_0\right) - \sin 2\pi t.$$

(b) In order to approximate periodic solutions $x(t)$ *we require that* $k_0(t)$ *and* $k_1(t)$ *be periodic solutions of the equations in (a). Find all periodic solutions of the equation for* $k_0(t)$*. (Hint: equilibria are periodic solutions).*

(c) For each periodic solution $k_0(t)$ *from (b) find a periodic solution of the equation for* $k_1(t)$ *in (a).*

(d) Use your answers in (b) and (c) to construct first order perturbation approximations to periodic solutions $x(t)$*.*

(e) Use a computer program to obtain graphs of the periodic solutions when $\varepsilon = 0.5$ *and compare them with the graphs of the perturbation approximations obtained in (d).*

Exercise 3.159 *Consider the differential equation*

$$x' = rx\left(1 - \frac{x}{K}\right) + \varepsilon \sin 2\pi t$$

with $r > 0$*,* $K > 0$ *and* $\varepsilon \geq 0$*. This equation models a population that normally grows according to the logistic equation* $x' = rx(1 - x/K)$*, but which is periodically harvested and seeded with sinusoidal rate* $\varepsilon \sin 2\pi t$*.*

(a) When $\varepsilon = 0$ *there are two equilibria,* $x = 0$ *and* K*. Draw the phase line portrait. What do solutions with* $x(0) > 0$ *do as* $t \to +\infty$*?*

(b) For small $\varepsilon > 0$*, find the first order approximation to a periodic solution* $p_1(t) = k_0(t) + k_1(t)\varepsilon$ *near* K*.*

(c) Show the first order approximation

(1) has period 1*;*

(2) has average K*;*

(3) has amplitude proportional to, but less than, ε*.*

(d) Use the first order approximation in (b) show the population survives indefinitely for small ε*, but goes extinct for large enough* ε*.*

For which initial values x_0 are the solutions of the following equations strictly increasing and for which are they strictly decreasing?

Exercise 3.160 $x' = 1 - e^{-ax}$*,* $a > 0$

Exercise 3.161 $x' = (x - a)(b - x)$*,* $a, b =$ *constants satisfying* $a \leq b$

For each of the following equations find all equilibria and determine which are hyperbolic. Use the derivative test in Theorem 3.4 to determine which are attractors and which are repellers. Sketch a graph of the right hand side $f(x)$ and use it to obtain the phase line portrait. Identify the type of all equilibria. (a is a constant.)

Exercise 3.162 $x' = \cos^2 x$

Exercise 3.163 $x' = (ax - 1)(a - x)$, $a > 0$

Exercise 3.164 $x' = ax^{-1} - x$, $a =$ *constant*

Exercise 3.165 $x' = \tan x$

Find the linearization of the following equations at each of their equilibria.

Exercise 3.166 $x' = x^2(1 - x)$

Exercise 3.167 $x' = x\,(1 - x)^2$

Exercise 3.168 $x' = b - e^{-ax}$, $a \neq 0$, $b > 0$

Exercise 3.169 $x' = ae^x - be^{-x}$, $a > 0$, $b > 0$

For each of the equations below:
(a) find all equilibria and draw the phase line portrait;
(b) classify all equilibria;
(c) determine which equilibria are hyperbolic and which are nonhyperbolic;
(d) find the linearization at each equilibrium and classify its equilibrium;
(e) apply the linearization principle (Theorem 3.5), if possible, to classify the equilibria. If the Linearization Principle does not apply, explain why.

Exercise 3.170 $x' = x^3\,(2 - x^2)\,(1 + x^2)^{-1}$

Exercise 3.171 $x' = x(1 - x)\,(1 + x^4)^{-1}\,e^x$

Exercise 3.172 $x' = p + x^4$

Exercise 3.173 $x' = p + e^{-x}$

Write down autonomous first order differential equations, using polynomials for $f\,(x)$, that have the following phase portraits.

Exercise 3.174 $\longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longleftarrow$

Exercise 3.175 $\longleftarrow -2 \longrightarrow -1 \longleftarrow 0 \longrightarrow 1 \longrightarrow$

Exercise 3.176 $\longrightarrow p \longleftarrow$ *for* $p > 0$ *and* $\longleftarrow p \longrightarrow$ *for* $p < 0$

Exercise 3.177 $\longrightarrow p \longrightarrow$ *for* $p > 0$ $\longleftarrow p \longleftarrow$ *for* $p < 0$

Which pairs of equations have qualitatively equivalent phase portraits?

Exercise 3.178 $\begin{cases} x' = x^7 \left(1 - x^2\right) \left(1 + x^2\right)^{-1} \\ x' = (x+1)^5 (1.5 - x)^3 (2 + x) \end{cases}$

Exercise 3.179 $\begin{cases} x' = -x \\ x' = xe^{-x} \end{cases}$

Exercise 3.180 $\begin{cases} x' = 1 + x \\ x' = -xe^{x} \end{cases}$

Exercise 3.181 $\begin{cases} x' = 1 - x \\ x' = -xe^{x} \end{cases}$

Determine the bifurcation points for the following equations. Determine the type of bifurcation that occurs. Draw a bifurcation diagram.

Exercise 3.182 $x' = x^3(x - p)$

Exercise 3.183 $x' = p + (x-1)^2(x+1)^2$

Exercise 3.184 $x' = p + e^{-x^2}$

Exercise 3.185 $x' = x\left(p + e^{-x^2}\right)$

Exercise 3.186 $x' = 1 + (x-1)^2 - p$

Exercise 3.187 $x' = px - x\left(1 + x^2\right)^{-1}$, $p > 0$

Exercise 3.188 *Find all the equilibria of the equation* $x' = -x + px^2$ *where* p. *Show all equilibria are hyperbolic for all* p. *Using phase line portraits, explain why a bifurcation occurs at* $p_0 = 0$.

Exercise 3.189 *Find the recursive formulas for the coefficients* $k_i(t)$ *of the series representation*

$$x = \sum_{i=0}^{\infty} k_i(t)\varepsilon^i$$

of the solution x *of the initial value problem*

$$x' = a(T(t) - x),\ x(0) = x_0$$

with $T(t) = \tau(1 + \varepsilon \sin t)$. *Here* $a > 0$, $\tau > 0$, $a > 0$ *are positive constants and* ε *is a small constant. Show all coefficients* $k_i(t)$ *except* $k_0(t)$ *and* $k_1(t)$ *are equal to* 0.

Exercise 3.190 *Find the recursive formulas for the coefficients $c_i(t)$ of the perturbation approximation*

$$x_n(t) = \sum_{i=0}^{n} k_i(t)\varepsilon^i$$

to the solution of the initial value problem

$$x' = (1 + \varepsilon \cos t)x,\ x(0) = 1$$

where ε is a small constant.

Exercise 3.191 *A tank full of water has the shape of an inverted circular cone of height H with a very small hole at the bottom out which water is draining. The circular top of the tank has radius R. Suppose the rate at which water drains out the bottom hole at any time is proportional to the square root of the depth of the water in the tank at that time.*

(a) Derive an initial value problem for the depth $x = x(t)$ of water in the tank at time t. Let $k > 0$ denote the constant of proportionality and $x_0 > 0$ the initial depth of water. (HINT: the volume of water at time t is $v(t) = \pi r^2(t)x(t)/3$ where $r(t)$ is the radius of the circular surface of the water at time t. Apply the balance law, v' = inflow rate − outflow rate, to the volume of water.)

(b) Classify the differential equation derived in (a). Discuss the application to this equation of the Fundamental Existence and Uniqueness Theorem 1.

(c) Determine the dynamics of the equation derived in (a) as time t increases. Discuss your answer with respect to the emptying of the water out of the tank.

(d) Using a computer program explore the behavior of the solutions of the model derived in (a). Formulate conjectures about the amount of time t_{empty} it will take for the tank to empty of water and how this length of time depends on the initial water depth x_0. (For example: is t_{empty} proportional to x_0? If not, what are some properties of the dependence?)

(e) Solve the initial value problem in (a). (An explicit solution is not necessary.)

(f) Use your answer in (d) to determine a formula for t_{empty}.

(g) Use your answer in (e) to address your conjectures in (c).

(h) A conical tank of height 9 meters is initially completely full with 100 cubic meters of water. After one half hour the depth of water in the tank is 8 meters. How long will it take for the tank to be half full? To be empty?

Exercise 3.192 *An equilibrium x_e of $x' = f(x)$ is called **stable** if any solution $x = x(t)$ that starts close to x_e remains close for all $t \geq 0$. Formally, x_e is stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - x_e| < \delta$ implies that $|x(t) - x_e| < \varepsilon$ for all $t > 0$. Prove $x_e = 0$ is stable as an equilibrium solution of the linear equation $x' = px$ if $p \leq 0$.*

Exercise 3.193 *An equilibrium x_e of $x' = f(x)$ is called **asymptotically stable** if it is stable and, in addition, there is a $\delta_0 > 0$ such that $|x_0 - x_e| < \delta_0$ implies that $|x(t) - x_e| \to 0$ as $t \to +\infty$. Prove $x_e = 0$ is asymptotically stable as an equilibrium solution of the linear equation $x' = px$ if $p < 0$.*

Exercise 3.194 *Prove the equilibrium $x_e = 0$ is stable as a solution of the equation $x' = -ax^3$ provided $a \geq 0$. Hint: find a formula for the general solution.*

Exercise 3.195 *Prove the equilibrium $x_e = 0$ is asymptotically stable as a solution of the equation $x' = -ax^3$ provided $a > 0$. Hint: find a formula for the general solution.*

Chapter 4

Systems and Higher Order Equations

Many applications involve more than one first order differential equation for more than one unknown function. For example, suppose one is interested in two quantities x and y that change over time. If these quantities effect each others rates of change, then we could have two differential equations of the form

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{4.1}$$

that describe how x and y effect their rates of change x' and y'. Examples include the interaction of a predator species with a prey, a reaction between two chemicals, and the motions of two planetary bodies.

A common situation that gives rise to a system of equations involves the amounts of a substance, x and y, present in two different locations or "compartments". If the movement of the substance into and out of the compartments includes exchanges between between the compartments, then a system of the form (4.1) arises from the balance laws

$$\begin{aligned} x' &= \text{ inflow rate } - \text{ outflow rate} \\ y' &= \text{ inflow rate } - \text{ outflow rate.} \end{aligned}$$

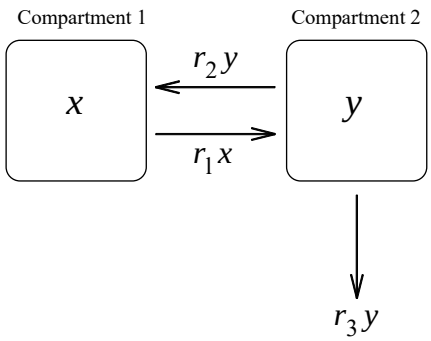


Figure 4.1

For example, if the flow rates for each compartment are proportional to the amounts present in the compartment then a typical compartment model diagram appears in Figure 4.1. This diagram, together with the balance laws, yield the system

$$\begin{aligned} x' &= -r_1 x + r_2 y \\ y' &= r_1 x - (r_2 + r_3)\, y. \end{aligned} \tag{4.2}$$

of differential equations. A specific application As a specific example, the system

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned} \tag{4.3}$$

results when the coefficients are give the values $r_1 = r_2 = 2$ and $r_3 = 3$.

Problems involving three or more quantities changing in time will lead to systems of three or more differential equations. Examples include ecological communities involving three or more species, reactions involving several chemical compounds, the motions of many planetary objects, and compartmental systems with three or more compartments (such as models of epidemics which classify individuals according to different categories with regard to the disease: susceptible, infected, recovered, etc.).

The point-of-view for the following chapters on systems of first order differential equations will be this: does a known theorem or method of analysis for single first order equations remain valid for systems of first order equations? If not, can it be adapted so as to work in some way? We will therefore use the definitions, theorems, and analytic methods learned for single equations as guidelines for a study of systems of equations. We begin, in this chapter by considering the Fundamental Existence and Uniqueness Theorem and see in what why is generalizable to systems of equations. We will also see how to adapt the numerical approximation methods used for single equations to systems of equations.

We also need to keep in mind that higher order differential equations can be written as equivalent first order systems. Therefore, whenever we learn something about systems of first order equations we automatically learn something about higher order equations. As we advance through our study of systems, we will occasionally pause to remind ourselves of this fact and have a look at higher order equations.

In the Chapter we saw how to convert a higher order equation to an equivalent first order system. For example, the second order equation

$$x'' + px' + qx = 0$$

is equivalent to the system

$$\begin{aligned} x' &= y \\ y' &= -qx - py \end{aligned} \tag{4.4}$$

of first order equations. As an application we recall that studies of an object in motion, as described by Newton's Laws of Motion, often give rise to second order differential equations. For example the equation (called the "simple harmonic oscillator" equation)

$$x'' + \frac{k}{m}x = 0 \tag{4.5}$$

arises in the study of the oscillatory motion of a mass m attached to a spring suspended from the ceiling and set in vertical motion in the absence of frictionless forces. The system

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x. \end{aligned} \tag{4.6}$$

is equivalent to this second order equation.

The association between higher and systems of first order equations allows us to apply to higher order equations any solution formulas, approximation techniques, methods of

analysis, and general results derived for systems of equations. Treating higher equations within the context of systems provides a unified approach and an efficiency in presentation. It also provides a natural setting for several important concepts, such as phase space and the"dimension" of a dynamical system. (As a practical matter, most programs for solving differential equations require the user to write a higher equation as an equivalent system.) On the other hand, we will see in 7 that in some circumstances there are shortcut methods for treating higher equations that do not utilize an equivalent first order system.

4.1 The Fundamental Existence and Uniqueness Theorem

A solution of a system of first order equations

$$\begin{aligned} x' &= f(t,x,y) \\ y' &= g(t,x,y) \end{aligned} \tag{4.7}$$

is a *pair* of functions $x = x(t)$, $y = y(t)$. More precisely we have the following definition.

Definition 4.1 *A **solution pair** of the system (4.7) is a pair of functions $x = x(t)$, $y = y(t)$ that are differentiable and reduce both equations to identities on an interval $\alpha < t < \beta$, i.e.,*

$$\begin{aligned} x'(t) &\equiv f(t,x(t),y(t)) \\ y'(t) &\equiv g(t,x(t),y(t)) \end{aligned}$$

for all values of t from the interval.

Note that for a second order equation

$$x'' = g(t,x,x')$$

Definition 4.1, when applied to the equivalent first order system

$$\begin{aligned} x' &= y \\ y' &= g(t,x,y), \end{aligned}$$

implies that a solution $x(t)$ and its derivative $y(t) = x'(t)$ are differentiable, that is to say, that $x(t)$ is twice differentiable on the interval $\alpha < t < \beta$.

For example, the formulas

$$\begin{aligned} x &= 2e^{-t} \\ y &= e^{-t} \end{aligned}$$

define a solution pair for the system (4.3) for all t. To see this we first calculate the derivative

$$x' = -2e^{-t}$$

and find that it is identical to

$$-2x + 2y = -2\left(2e^{-t}\right) + 2\left(e^{-t}\right)$$

for all t, and secondly we calculate the derivative

$$y' = -e^{-t}$$

and find that it is identical to

$$2x - 5y = 2\left(2e^{-t}\right) - 5\left(e^{-t}\right)$$

for all t.

As another example, the simple harmonic oscillator system

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{4.8}$$

is equivalent to the simple harmonic oscillator equation

$$x'' + x = 0. \tag{4.9}$$

The two functions

$$\begin{aligned} x &= \cos t \\ y &= -\sin t \end{aligned}$$

constitute a solution pair for all t. This is true because $x' = -\sin t$ identically equals y, and $y' = -\cos t$ identically equals $-x$, for all t. Or equivalently $x'' + x = -\cos t + \cos t = 0$ for all t.

The system (4.2) describes the rates at which a pesticide is exchanged between a stand of trees and its soil bed. Knowing these rates is not sufficient, however, to determine the amounts of pesticide in the trees and soil at future times. We must also know the initial amounts present, i.e., $x(0) = x_0$ and $y(0) = y_0$. The equations

$$\begin{aligned} x' &= -r_1 x + r_2 y \\ y' &= r_1 x - (r_2 + r_3)\, y \\ x(0) &= x_0, \qquad y(0) = y_0 \end{aligned}$$

constitute an initial value problem for the first order system (4.2).

It is probably not surprising that two initial conditions are necessary in order to specify a unique solution of a system of two first order differential equations. This is because, roughly speaking, two integrations are needed to solve the two differential equations, and as a result two constants of integration arise in the general solution. Or put another way, the differential equations specify the rates of change of x and y, but to determine (predict) future values of x and y initial conditions are required.

An initial value problem for a second order equation $x'' + px' + qx = 0$, or its equivalent first order system (4.4), consists of the two initial conditions $x(0) = x_0$ and $y(0) = x'(0) = x_0$. For example, the initial value problem for the simple harmonic oscillator system is

$$\begin{aligned} x' &= y \\ y' &= -x \\ x(0) = x_0, &\qquad y(0) = y_0. \end{aligned} \tag{4.10}$$

The equations

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \\ x(t_0) = x_0, &\qquad y(t_0) = y_0. \end{aligned} \tag{4.11}$$

describe an *initial value problem* for the general first order system (4.7). The following theorem is an extension to systems of the basic existence and uniqueness Theorem 1.1 (Sec. 1.2) for single equations.

Theorem 4.1 ***(Fundamental Existence and Uniqueness Theorem) Suppose*** $f(t, x, y)$ ***and*** $g(t, x, y)$ ***and all the (partial) derivatives***

$$\frac{df(t, x, y)}{dx}, \quad \frac{df(t, x, y)}{dy}, \quad \frac{dg(t, x, y)}{dx}, \quad \frac{dg(t, x, y)}{dy} \tag{4.12}$$

are continuous in $t, x,$ ***and*** y ***on intervals***

$$a < t < b, \ c_1 < x < d_1, \ c_2 < y < d_2$$

and that the initial conditions lie in these intervals. Then the initial value problem (4.11) has a unique solution pair $x(t)$, $y(t)$ ***on some interval*** $\alpha < t < \beta$ ***that contains*** t_0.

As an example, for system (4.3) the functions $f(t, x, y) = -2x+2y$ and $g(t, x, y) = 2x-5y$ are linear in x and y and therefore continuous for all x and y (and t). Moreover, their partial derivatives

$$\begin{aligned} \frac{df}{dx} &= -2, \quad \frac{df}{dy} = 2, \\ \frac{dg}{dx} &= 2, \quad \frac{dg}{dy} = -5 \end{aligned}$$

are all constant and therefore continuous for all x and y (and t). It follows from Theorem 4.1 that any initial value problem $x(t_0) = x_0$, $y(t_0) = y_0$ for system (4.3) has a unique solution $x = x(t)$, $y = y(t)$ on an interval containing t_0.

Theorem 4.1 applies to the equivalent system of a second order equation and therefore provides the existence and uniqueness of solutions for initial value problems associated with second order equations.

For example, by Theorem 4.1 the initial problem (4.10) for the simple harmonic oscillator has a unique solution. This is because $f(t,x,y)=y$, $g(t,x,y)=-x$ and the derivatives

$$\begin{aligned}\frac{df}{dx}&=0, \quad \frac{df}{dy}=1,\\ \frac{dg}{dx}&=-1, \quad \frac{dg}{dy}=0\end{aligned}$$

are continuous functions for all t, x and y.

As a second example, consider the system

$$\begin{aligned}x'&=y\\ y'&=-x-\alpha\left(x^2-1\right)y\end{aligned}$$

where α is a constant. This system is equivalent to the second order equation

$$x''+\alpha\left(x^2-1\right)x'+x=0. \tag{4.13}$$

(Exercise 4.31.) This famous equation, called the *van der Pol equation*, arises in the theory of electric circuits. Since $f(t,x,y)=y$, $g(t,x,y)=-x-\alpha\left(x^2-1\right)y$ and the derivatives

$$\begin{aligned}\frac{df}{dx}&=0, \quad \frac{df}{dy}=1,\\ \frac{dg}{dx}&=-1-2\alpha x^2y, \quad \frac{dg}{dy}=-\alpha\left(x^2-1\right)\end{aligned}$$

are continuous for all x and y, all initial value problems $x(0)=x_0$, $x'(0)=y_0$ have unique solutions (on an interval containing $t_0=0$).

In order to apply the Fundamental Existence and Uniqueness Theorem 4.1 it is necessary to verify that the functions f and g and all of their partial derivatives (4.12) are continuous at the initial conditions. However, it is often not necessary to calculate these partial derivatives. Instead, one can rely on theorems from calculus for this purpose. For example, we know from calculus that sums, differences, products, and composites of continuous and differentiable functions are themselves continuous and differentiable (and so are quotients, at least where the denominator does not equal 0).

While we will emphasize and focus on systems of two first order equations in this first course on differential equations, we point out that the Fundamental Existence and Uniqueness Theorem 4.1 is straightforwardly extendable to systems of any size/dimension, i.e., to systems of any number of equations.

For example, for systems of three equations

$$\begin{aligned}x'&=f\left(t,x,y,z\right)\\ y'&=g\left(t,x,y,z\right)\\ z'&=h\left(t,x,y,z\right)\end{aligned}$$

the Fundamental Theorem for the existence and uniqueness of a solution to an initial value problem

$$x\left(t_0\right)=x_0, \quad y\left(t_0\right)=y_0, \quad z\left(t_0\right)=z_0$$

requires that f, g and h and their derivatives with respect to (the state variables) x, y and z at the initial be continuous at the initial condition: $t = t_0$, $x_0 = x_0$, $y_0 = y_0$, $z_0 = z_0$.

More generally, an initial value problem of a (so-called n-dimensional) system of n equations in n unknowns $x_1, x_2, \ldots, x_n$

$$\begin{aligned} x_i' &= f_i(t, x_1, x_2, \ldots, x_n) \\ x_i(t_0) &= x_i^0 \end{aligned} \tag{4.14}$$

where t_0 and the n initial conditions x_1^0, x_2^0, ... , x_n^0 are specified real numbers. The Fundamental Existence and Uniqueness Theorem for this system requires that each function f_i and all of its partial derivatives with respect to each x_j are continuous at $t = t_0$ and $x_i = x_i^0$.

It is often notationally convenient to write systems of equations using vector and matrix notation. This will be particularly true in the subsequent chapters on linear systems. To do this we introduce the notation

$$\begin{aligned} \tilde{x} &\triangleq \operatorname{col}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} \\ \tilde{f}(t, \tilde{x}) &\triangleq \operatorname{col}(f(t, x, y), g(t, x, y)) = \begin{pmatrix} f(t, x, y) \\ g(t, x, y) \end{pmatrix}. \end{aligned}$$

Then we write the n-dimensional system (4.7) as

$$\tilde{x}' = \tilde{f}(t, \tilde{x}) \tag{4.15}$$

where by $\tilde{x}'$ we mean the vector of derivatives

$$\tilde{x}' \triangleq \operatorname{col}(x', y') = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

The notation (4.15) can, in fact, be used for systems of any dimension. For example, the initial value problem (4.14) can be written as

$$\begin{aligned} \tilde{x}' &= \tilde{f}(t, \tilde{x}) \\ \tilde{x}(t_0) &= \tilde{x}_0 \end{aligned}$$

where

$$\tilde{x} \triangleq \operatorname{col}(x_i) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \tilde{f}(t, \tilde{x}) \triangleq \operatorname{col}(f_i(t, \tilde{x})) = \begin{pmatrix} f_1(t, \tilde{x}) \\ f_2(t, \tilde{x}) \\ \vdots \\ f_n(t, \tilde{x}) \end{pmatrix}$$

$$\tilde{x}_0 \triangleq \operatorname{col}\left(x_i^0\right) = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}.$$

4.2 Approximation of Solutions

In this section we extend the numerical approximation methods and graphical techniques in Chapter 1 to systems of first order equations.

4.2.1 Numeric Approximations

Consider the problem of numerically approximating the solution $x = x(t)$, $y = y(t)$ of the initial value problem

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \\ x(t_0) = x_0, &\quad y(t_0) = y_0. \end{aligned}$$

at points t_i between the initial time t_0 and a chosen end point T

$$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T.$$

To do this we follow the method used in Sec. 1.3 to approximate the solution of a single first order equation. To get an approximation at the first point t_1 we integrate both of the equations

$$\begin{aligned} x'(t) &= f(t, x(t), y(t)) \\ y'(t) &= g(t, x(t), y(t)) \end{aligned} \tag{4.16}$$

from $t = t_0$ to $t = t_1$. By the Fundamental Theorem of Calculus, together with the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ we obtain

$$\begin{aligned} x(t_1) &= x_0 + \int_{t_0}^{t_1} f(t, x(t), y(t))dt \\ y(t_1) &= y_0 + \int_{t_0}^{t_1} g(t, x(t), y(t))dt. \end{aligned} \tag{4.17}$$

We can use methods for approximating integrals to obtain numerical estimates for $x(t_1)$ and $y(t_1)$. For example, the left hand rectangle rule applied to both integrals yields the approximations

$$\begin{aligned} x(t_1) &\approx x_0 + (t_1 - t_0)f(t_0, x_0, y_0) \\ y(t_1) &\approx y_0 + (t_1 - t_0)g(t_0, x_0, y_0). \end{aligned}$$

If we denote these approximations by x_1 and y_1, i.e.,

$$\begin{aligned} x_1 &= x_0 + (t_1 - t_0)f(t_0, x_0, y_0) \\ y_1 &= y_0 + (t_1 - t_0)g(t_0, x_0, y_0) \end{aligned}$$

then we have the first step of the Euler Algorithm for systems.

As in Sec. 1.3, the left hand rectangle rule yields approximations x_{i+1}, y_{i+1} to the solution values $x(t_{i+1})$, $y(t_{i+1})$ at the time t_{i+1}, assuming we have approximations x_i, y_i at time t_i. Specifically, integrating the equations in (4.16) from $t = t_i$ to t_{i+1} we have

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + \int_{t_i}^{t_{i+1}} f(t, x(t), y(t))dt \\ y(t_{i+1}) &= x(t_i) + \int_{t_i}^{t_{i+1}} g(t, x(t), y(t))dt \end{aligned} \tag{4.18}$$

and from the left hand rectangle rule

$$\begin{aligned} x(t_{i+1}) &\approx x_i + (t_{i+1} - t_i)f(t_i, x_i, y_i) \\ y(t_{i+1}) &\approx y_i + (t_{i+1} - t_i)g(t_i, x_i, y_i). \end{aligned}$$

Thus, approximations to the solutions at $t = t_{i+1}$ are given by the quantities

$$\begin{aligned} x_{i+1} &= x_i + s_i f(t_i, x_i, y_i) \\ y_{i+1} &= y_i + s_i g(t_i, x_i, y_i) \end{aligned}$$

where $s_i = t_{i+1} - t_i$ are the step sizes. In practice, equally spaced step sizes are usually chosen. If $s = t_{i+1} - t_i$ denotes a fixed step size, we obtain the *Euler Algorithm* for systems

$$\begin{aligned} x_{i+1} &= x_i + sf(t_i, x_i, y_i) \\ y_{i+1} &= y_i + sg(t_i, x_i, y_i) \quad \text{for } i = 0, 1, 2, 3, \ldots . \end{aligned} \tag{4.19}$$

Example 4.1 *As an example, the Euler Algorithm formulas for the initial value problem*

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \\ x(0) &= 1, \quad y(0) = 0 \end{aligned} \tag{4.20}$$

are

$$\begin{aligned} x_0 &= 1, \quad y_0 = 0 \\ x_{i+1} &= x_i + s\left(-2x_i + 2y_i\right) \\ y_{i+1} &= y_i + s\left(2x_i - 5y_i\right). \end{aligned} \tag{4.21}$$

To approximate the solution pair at $T = 2$ using step size $s = 0.4$, the Euler Algorithm sequentially generates approximations to the solution at the four points

$$t_1 = 0.4, \;\; t_2 = 0.8, \; t_3 = 1.6, \;\; t_4 = T = 2.$$

We obtain the approximations x_1, y_1 at $t_1 = 0.4$ from (4.21) using $i = 0$ as follows:

$$\begin{aligned} x_1 &= x_0 + s\left(-2x_0 + 2y_0\right) = 1 + 0.4 \times (-2 + 0) = 0.2 \\ y_1 &= y_0 + s\left(2x_0 - 5y_0\right) = 0 + 0.4 \times (2 - 5 \times 0) = 0.8. \end{aligned}$$

Using these approximations for x_1, x_2 we can now calculate the approximations x_2, y_2 at $t_2 = 0.8$ using $i = 1$ in (4.21):

$$x_2 = x_1 + s\,(-2x_1 + 2y_1) = 0.2 + 0.4 \times (-0.4 + 1.6) = 0.68$$
$$y_2 = y_1 + s\,(2x_1 - 5y_1) = 0.8 + 0.4 \times (0.4 - 4) = -0.64.$$

Continuing this process two more times, we obtain approximations at $t_3 = 1.6$ and $t_4 = 2$. The results are (to 6 significant digits)

$$x_3 = x_2 + s\,(-2x_2 + 2y_2) = -0.376000$$
$$y_3 = y_2 + s\,(2_2 - 5y_2) = 1.18400$$

$$x_4 = x_3 + s\,(-2x_3 + 2y_3) = 0.872000$$
$$y_4 = y_3 + s\,(2x_3 - 5y_3) = -1.48480$$

$$x_5 = x_4 + s\,(-2x_4 + 2y_4) = -1.01344$$
$$y_5 = y_4 + s\,(2x_4 - 5y_4) = 2.18240$$

Thus, the Euler approximations to the solution pair at time $t = 2$ with step size $s = 0.4$ are (to 6 significant digits)

$$x(2) \approx -1.01344, \quad y(2) \approx 2.18240.$$

A reduction of the step size s will increase the accuracy of the Euler approximations, but at the same time will increase the number of steps and hence the amount of numerical calculations we must perform. The Table 4.1 below shows the approximations obtained by halving the step size several times.

s	$x(2) \approx$	$y(2) \approx$
0.400	− 1.01344	2.18240
0.200	0.0858994	0.0429496
0.100	0.0972613	0.0486307
0.050	0.102810	0.0514046
0.025	0.105551	0.0527742

Table 4.1. Euler Algorithm approximations to the solution of (4.20) with decreasing step sizes.

The Euler Algorithm for systems has the *same order of convergence* for the Euler Algorithm for a single equation, namely $O(s)$. Recall that this means the errors ($|x(T) - x_n|$ and $|y(T) - y_n|$) are bounded by a constant multiple of the step size s and that they tend to zero at the same rate that s tends to zero. Thus, if the step size s is decreased by a certain factor (one half, one tenth, etc.), then we can expect the errors to decrease by (roughly speaking) this same factor.

For example, in Table 4.1 the errors are approximately halved with each halving of the step size s; see Exercise 4.26.

The Euler Algorithm is straightforwardly extended to systems of any number of first order equations. The formulas for the initial value problem

$$\begin{aligned} \tilde{x}' &= \tilde{f}\,(t, \tilde{x}) \\ \tilde{x}\,(t_0) &= \tilde{x}_0 \end{aligned} \tag{4.22}$$

are

$$\tilde{x}_{i+1} = \tilde{x}_i + s\tilde{f}(t_i, \tilde{x}_i).$$

The order of convergence remains the same, namely order one $O(s)$.

In Sec. 1.3.2 we used the trapezoid rule for approximating integrals to derive the Modified Euler Algorithm for a first order equation. If the trapezoid rule is used to approximate the integrals in (4.18) we obtain *the Modified Euler Algorithm* for systems :

$$x_{i+1} = x_i + \frac{s}{2}\left(f(t_{i+1}, x^*_{i+1}, y^*_{i+1}) + f(t_i, x_i, y_i)\right)$$
$$y_{i+1} = y_i + \frac{s}{2}\left(g(t_{i+1}, x^*_{i+1}, y^*_{i+1}) + g(t_i, x_i, y_i)\right)$$

where

$$x^*_{i+1} = x_i + sf(t_i, x_i, y_i)$$
$$y^*_{i+1} = y_i + sg(t_i, x_i, y_i).$$

This algorithm is second order, i.e., converges at a rate $O(s^2)$.

t_i	x_i	y_i
0.0	1.00000	0.00000
0.4	0.84000	− 0.32000
0.8	0.80800	− 0.69120
1.2	0.89990	− 1.17094
1.6	1.13062	− 1.83362
2.0	1.53648	− 2.78217

Table 4.2

s	$x(2)\approx$	$y(2)\approx$
0.400	1.536480	− 2.78217
0.200	0.110248	0.0544010
0.100	0.108662	0.0543216
0.050	0.108363	0.0541779
0.025	0.108293	0.0541430

Table 4.3

Table 4.2 The Modified Euler Algorithm approximations to the solution of the initial value problem (4.20) using step size $s = 0.4$.
Table 4.3 The rate at which the approximations converge as the step size s decreases. In comparison with the Euler Algorithm approximations in Table 4.1, this rate is considerably faster. (See Exercise 4.27.)

The Modified Euler Algorithm extends to n-dimensional systems (4.22) straightforwardly:

$$\tilde{x}_{i+1} = \tilde{x}_i + \frac{s}{2}\left(\tilde{f}(t_{i+1}, \tilde{x}^*_{i+1}) + \tilde{f}(t_i, \tilde{x}_i,)\right)$$

where

$$\tilde{x}^*_{i+1} = \tilde{x}_i + s\tilde{f}(t_i, \tilde{x}_i).$$

The order of convergence is $O\left(s^2\right)$ regardless of the size of the system.

Runge-Kutta Algorithms, discussed in Sec. 1.3.2, are also available for systems of equations. For example, the fourth order Runge-Kutta Algorithm is a widely used algorithm.

4.2.2 Graphic Approximations

As we did for first order equations, we can use numerical approximations to draw approximate graphs of the solution pairs. For the solution component $x = x(t)$ this is done by drawing straight line segments between the pairs of points (t_0, x_0) and (t_1, x_1), (t_1, x_1) and (t_2, x_2), and so on. Similarly, for the solution component $y = y(t)$ this is done by drawing straight line segments between the points (t_i, y_i). As the step size s is decreased, these broken line graphs will converge to the graphs $x = x(t)$ and $y = y(t)$ respectively.

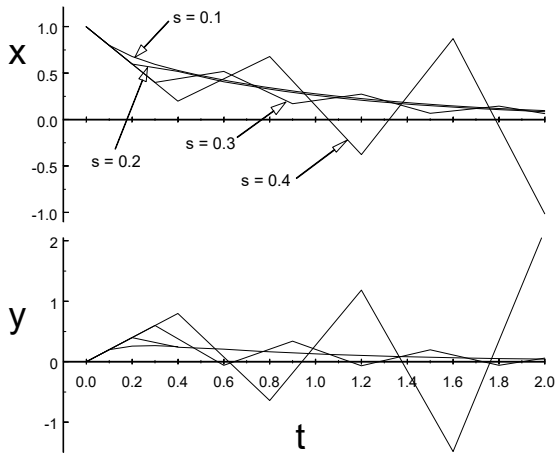


Figure 4.2. Broken line approximations to the graphs of the solution components x and y of the initial value problem (4.20) using the Euler Algorithm with a decreasing sequence of step sizes s

In Figure 4.2 appear graphical approximations to the solutions x and y of the initial value problem (4.20). Notice how these broken line graphs get smoother in appearance and converge to a smooth curve as the step size s decreases.

In practice one should compute approximate solution graphs for several decreasing step sizes until the graphs appear unchanged upon any further decreases. Then one has some confidence that the approximate graphs have sufficiently converged so as to provide an accurate approximation to the graph of the solution.

In Figure 4.3 appear computer drawn graphs of the solutions x and y of the initial value problem

$$\begin{aligned} x' &= y \\ y' &= -x - \frac{1}{2}(x^2 - 1)y \\ x(0) = 2, &\quad x'(0) = 0. \end{aligned} \tag{4.23}$$

(This first order system is equivalent to the van der Pol equation (4.13) with $\alpha = 1/2$.) The graphs were constructed using the Modified Euler Algorithm for two steps sizes. Since the step sizes result in indistinguishable graphs, we conclude that the graphs are accurate approximations.

So far we have graphically represented solutions of initial value problems (4.11) by drawing two graphs, one for each component x and y of the solution pair. Another way is to draw a three dimensional graph using a (t, x, y)-coordinate axis and plotting points $(t, x(t), y(t))$. See Figure 4.4. While sometimes useful, such three dimensional graphs are often difficult to draw and use.

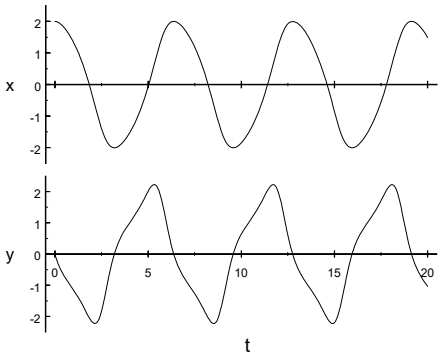


Figure 4.3. Graphs of the solution components x and y of the initial value problem (4.23) obtained using the Modified Euler Method with step sizes $s = 0.05$ and 0.025.

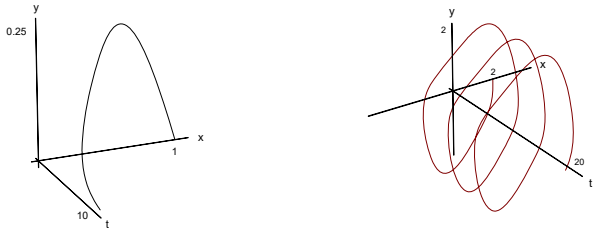


Figure 4.4. Three dimensional graphs of the solution of the initial value problems (4.20) (left) and (4.23) (right).

For certain kinds of differential equations – namely autonomous equations – another useful graphical representation of solutions is available. For systems of autonomous equations a powerful method is to plot the pair $(x(t), y(t))$ in the x, y-plane. This amounts to projecting the three dimensional pictures into the x, y-plane (parallel to the t-axis). We study this method in the next section.

4.3 Phase Plane for Autonomous Systems

Consider the system

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{4.24}$$

(This first order system is equivalent to the second order equation $x'' + x = 0$.) The trigonometric functions

$$x = \sin t, \quad y = \cos t \tag{4.25}$$

form a solution pair of this system. The graphs of these trigonometric functions should be familiar to the reader.

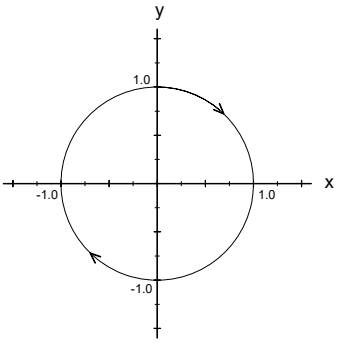


Figure 4.5. The phase plane orbit associated with the solution (4.25) of the system (4.24).

However, instead of graphing each function individually, suppose we plot the points $(x, y) = (\sin t, \cos t)$ in the x, y-plane for each t. The resulting set of points is the circle of radius 1 centered at the origin. This is because

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

for all t. As t increases the point (x, y) given by (4.25) moves continuously around the unit circle in a clockwise manner. Therefore, we place arrows pointing clockwise on the unit circle to indicate the direction of the motion along this circular path as t *increases*. See Figure 4.5. The (clockwise oriented) unit circle is another graphical way to represent the solution (4.25). This circle is an example of an "orbit" associated with the solution of a system of differential equations.

Definition 4.2 *If $x = x(t)$, $y = y(t)$ is a solution pair (on an interval $\alpha < t < \beta$) of an autonomous system*

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{4.26}$$

then the set of points $(x(t), y(t))$ in the x, y-plane is called the ***orbit*** *associated with this solution. An orbit is assigned an* ***orientation*** *in the direction of increasing t. The x, y-plane is called the* ***phase plane*** *and the set of all orbits, together with their orientations, is called the* ***phase plane portrait*** *(or simply its* ***phase portrait****) of the system (4.26).*

Another solution of the system (4.24) is the constant pair $(x, y) = (0, 0)$. The orbit associated with this solution is a single point, namely the origin in the phase plane. A constant solution is called an *equilibrium* and its orbit is called an *equilibrium point* (or a *rest point* or a *critical point*).

Geometrically the orbit of a solution is the projection onto the x, y-plane of the three dimensional graph of the solution obtained by plotting the points $(t, x(t), y(t))$ (as in Figure 4.4). This is the analog of the one dimensional case of a single autonomous equation $x' = f(x)$ where the orbit of a solution is obtained by projecting the two dimensional graph of the solution $(t, x(t))$ onto the x-axis (see Chapter 3.1). Mathematically, the orbit is the range of the solution pair considered as a function: $t \to (x(t), y(t))$.

Autonomous systems (4.26) arise often in applications and a major goal is to determine and sketch their phase portraits.

Example 4.2 *The formulas*

$$\begin{aligned} x &= c_1 \cos t + c_2 \sin t \\ y &= -c_1 \sin t + c_2 \cos t \\ c_1,\ c_2 &= \textit{any constants} \end{aligned}$$

turn out to define the general solution of the system (4.24). Note that

$$\begin{aligned} x^2 + y^2 &= (c_1 \cos t + c_2 \sin t)^2 + (-c_1 \sin t + c_2 \cos t)^2 \\ &= \left(c_1^2 + c_2^2\right)\left(\cos^2 t + \sin^2 t\right) \\ &= c_1^2 + c_2^2. \end{aligned}$$

Thus, every orbit associated with this system is a circle centered at the origin and the phase portrait consists of the collection of all such circles. See Figure 4.6.

As in the previous example, one approach to drawing phase portraits is to find formulas for solutions and plot a selection of orbits using the formulas. This approach is feasible only for specialized systems for which solution methods are available (linear systems are an example, as we will see in Chapter 6).

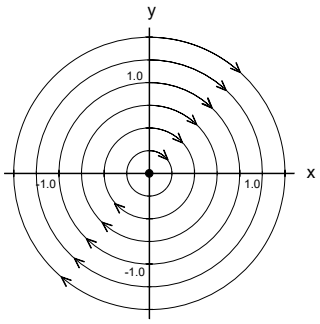


Figure 4.6. The phase portrait of the system (4.24) consists of all circles centered at the origin, oriented clockwise.

Another method for sketching phase portraits, a method that does not require finding formulas for solutions, uses slope fields. At a point (x, y) on an orbit

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{f(x,y)}$$

gives the slope of the tangent line to the orbit curve. See Figure 4.7. (We exclude equilibria, at which the ratio is undefined. If f vanishes at the point, and g doesn't, then the tangent is vertical and the slope is "infinite".) In other words, each point in the (x, y)-plane is associated with a unique slope $g(x, y)/f(x, y)$ and an orbit associated with the system (4.26) must have that slope at each of points through which it passes. The association of a slope with each point in the plane is called a *slope field*. Graphically, we represent the slope field by drawing a small line segment through each point with the assigned slope.

As t increases the orbit passes through a point (x, y) in one of two directions. The direction is that of the vector with components $f(x, y)$ and $g(x, y)$, i.e., the vector

$$\tilde{f} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

We denote this direction by an arrow on the slope field line segment. The resulting association of *directed* line segments with each point (x, y) is called the *vector field* (or *direction field*) associated with the system (4.26).[1]

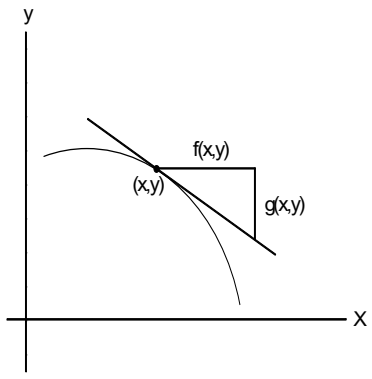


Figure 4.7

The vectors usually are not drawn to scale. That is to say, the arrows indicate the direction, but not the length of the vector $\tilde{x} = \text{col}(f(x, y), g(x, y))$.

The orbits of an autonomous system must "fit" the vector field in the sense that when an orbit passes through a point it must do so in the direction and with the slope assigned to that point by the vector field. Thus, from a sufficiently detailed sketch of the vector field one can usually visualize typical orbits and hence the phase portrait.

[1] The vectors are usually not drawn to scale. That is to say, the arrows indicate the direction, but not the length of the vector $(f(x, y), g(x, y))$.

For example, Figure 4.8 shows the vector field associated with the system (4.24) whose orbits are circles centered at the origin, as we saw in Figure 4.6.

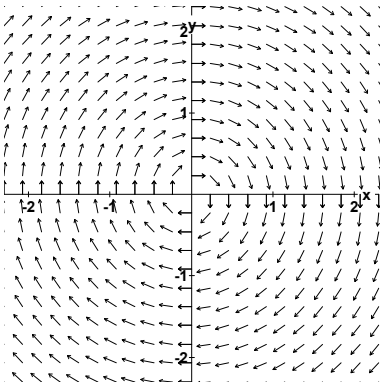


Figure 4.8. The vector field associated with the system (4.24).

Example 4.3 *Figure 4.9 shows a sketch of the vector field of the system*

$$x' = -x + y$$
$$y' = -2x - y$$

together with a selection of orbits. Notice the orbits appear to tend toward the origin as $t \to +\infty$. From these selected graphs it seems reasonable to conjecture that all orbits of this system tend toward the origin as $t \to +\infty$. The reader can test this conjecture by further computer exploration of orbits. (Note, however, that no amount of computer exploration will rigorously prove this conjecture, since there are infinitely many orbits and only a finite number can be numerically calculated.)

Example 4.4 *Figure 4.10 shows the vector field of the system in (4.23) and some typical orbits. From these graphs it appears that orbits beginning near the origin spiral outward and orbits beginning far from the origin spiral inward and that both types of orbits approach a closed loop as $t \to +\infty$.*

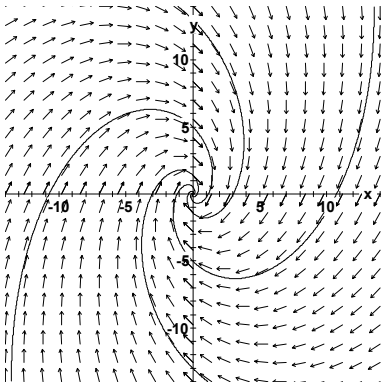


Figure 4.9

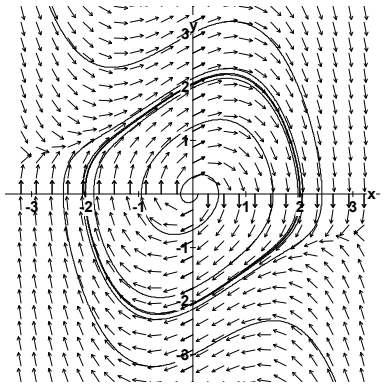


Figure 4.10

It can be useful in sketching direction fields, and thereby analyzing basic properties of the orbits of an autonomous system (4.26), to determine where in the phase plane the direction field points horizontally and where it points vertically. The direction field will point vertically if the first component of the vector (x', y') equals 0. This will occur at, and only at, points in the plane for which

$$f(x, y) = 0.$$

In general this algebraic equation for x and y describes a curve (or curves) in the phase plane. This curve is called an *x-nullcline* of the autonomous system. At each point on the x-nullcline the direction field points vertically, and hence the orbit passing through any such point is moving upward or downward.

Similarly, the equation

$$g(x, y) = 0$$

describes a curve (or curves) called a *y-nullcline.* At each point lying on a y-nullcline the direction field points either left or right.

A point in the phase plane that lies on both nullclines (i.e., at which the x-nullcline and the y-nullcline intersect) is an equilibrium point, since clearly the autonomous system (4.26) is satisfied at such a point.

The x and y-nullclines divide the phase plane into regions or sectors inside of which the sign of x' remains fixed (either positive or negative) and inside of which the sign of y' remains fixed (either positive or negative). Therefore, we can assign one of the four compass directions NE (northeast), SE (southeast), SW (southwest) or NW (northwest) to each of the regions determined by the nullclines. These directions, together with the nullclines and equilibria, usually provide one with a pretty good idea about the geometry of the system's orbits.

Example 4.5 *The nullclines of the autonomous system*

$$\begin{aligned} x' &= x^2 + y^2 - 1 \\ y' &= x - y \end{aligned}$$

are the curves given by the equations

$$\begin{aligned} &x\text{-nullcine:} \quad x^2 + y^2 = 1 \\ &y\text{-nullcline:} \quad y = x. \end{aligned}$$

We recognize the x-nullcline as the circle of radius 1 with center at the origin. The direction field is vertical at points on this circle. The y-nullcline is the straight line $y = x$. The direction field is horizontal at points on this line.

Whether the direction field points vertically up or down at a point on the circle (the x-nullcline) is determined by the sign of y', i.e. by sign of $x - y$, at the point. Thus, the direction field points vertically downward at those points (x, y) that lie on the circle and above the line $y = x$ (the y-nullcline). The direction field points vertically upward at those points (x, y) that lie on the circle and below the line $y = x$ (the y-nullcline).

Whether the direction field points horizontally to the right or to the left a point on the line $y = x$ (the y-nullcline) is determined by the sign of x', i.e. by sign of $x^2 + y^2 - 1$, at the point. Thus, the direction field points horizontally to the right at those points (x, y) that lie on the line $y = x$ and outside of the circle (the x-nullcline). The direction field points horizontally to the left at those points (x, y) that lie on the line $y = x$ and inside the circle (the x-nullcline).

Finally, the four regions created by the two nullclines have a unique compass direction associated with them. They are as in the table below. For example, at points in the region outside the circle $x^2 + y^2 = 1$ and above the line $y = x$ the direction field points because at

such points x' is positive and y' is negative. See Figure 4.11.

$$\left.\begin{array}{c}\text{Outside the circle } x^2+y^2=1\\ \text{Above the line } y=x\end{array}\right\} \quad SE$$

$$\left.\begin{array}{c}\text{Outside the circle } x^2+y^2=1\\ \text{Below the line } y=x\end{array}\right\} \quad NE$$

$$\left.\begin{array}{c}\text{Inside the circle } x^2+y^2=1\\ \text{Above the line } y=x\end{array}\right\} \quad SW$$

$$\left.\begin{array}{c}\text{Inside the circle } x^2+y^2=1\\ \text{Below the line } y=x\end{array}\right\} \quad NW$$

There are exactly two points at which the x- and y-nullcines intersect, i.e. where the straight line $y=x$ and the circle $x^2+y^2=1$ intersect. Thus, there are exactly two equilibrium points for this autonomous system. A little algebra shows that the coordinates of the equilibria are

$$\begin{aligned}(x_1,y_1) &= \left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)\\ (x_2,y_2) &= \left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right).\end{aligned}$$

Taking all of this information together we can make some reasonable conjectures about orbits of this system. For example, consider an initial point that lies in the region marked NE. The orbit starting at this point will, of course, move in the NE direction. One possibility is that it remains in this region, forever moving in the NE direction as $t\to+\infty$, in which case the orbit leaves the window in Figure 4.11. Another possibility is that the orbit intersects and crosses inside the circle, at which it then moves NW. It moves NW until is crosses the line $y=x$ and enters the SW region. It could then leave the circle and move SE until it again crosses the line $y=x$, re-entering the NE region! This route suggests the possibility of a spiral type motion, located around the equilibrium (x_1,y_1).

Both the possibilities described above are illustrated by an orbit in Figure 4.12 (which also shows the direction field using the grid method). The initial point shown as an open circle produces an orbit that remains in region NE and leaves the window of the plot. The initial point shown as a solid circle produces the spiral path described above.

Other orbit paths can be deduced from initial conditions in other regions.

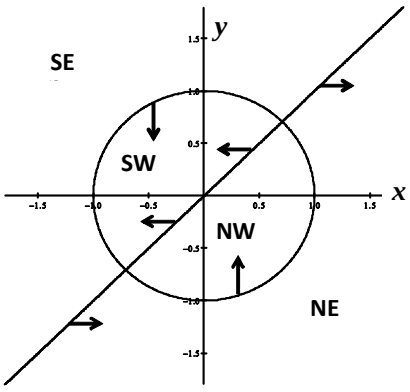


Figure 4.11

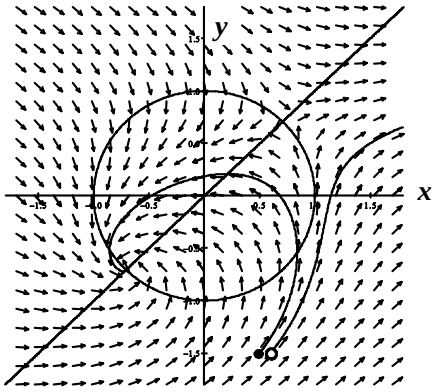


Figure 4.12

4.4 Chapter Summary

A first order system consists of two (or more) first order differential equations for two (or more) solutions. Higher order equations and systems are equivalent to first order systems. The Fundamental Existence and Uniqueness Theorem 4.1 is a generalization to systems of equations of the Fundamental Existence and Uniqueness Theorem 1.1 (Sec. 1.2) for single first order differential equations. Algorithms (Euler, Modified Euler, and Runge-Kutta) for numerically approximating solutions of initial value problems also extend to systems of equations. We can graph solutions of systems in several ways. Each component of the solution can be graphed separately or, in the case of two equations, both components can simultaneously be graphed as functions of t in three dimensional space. For the important special case of autonomous systems a solution pair $(x(t), y(t))$, when graphed in the x, y-plane, produces an orbit. The set of all orbits constitute the phase plane portrait of the system. Vector fields are useful for studying phase portrait.

4.5 Exercises

Which of the following are solution pairs of the system (4.3) and which are not? Justify your answers.

Exercise 4.1 $\begin{cases} x = e^{-6t} \\ y = -2e^{-6t} \end{cases}$

Exercise 4.2 $\begin{cases} x = e^{-6t} \\ y = 2e^{-6t} \end{cases}$

Exercise 4.3 $\begin{cases} x = 4e^{-t} + 4e^{-6t} \\ y = 2e^{-t} + 8e^{-6t} \end{cases}$

Exercise 4.4 $\begin{cases} x = 2e^{-t} + 3e^{-6t} \\ y = e^{-t} - 6e^{-6t} \end{cases}$

Which of the following are solution pairs of the harmonic oscillator system (4.8) and which are not? Justify your answers. (c, c_1 and c_2 are constants.)

Exercise 4.5 $x = 2\cos t, \quad y = -2\sin t$

Exercise 4.6 $x = \sin t, \quad y = \cos t$

Exercise 4.7 $x = \sin 2t, \quad y = \cos 2t$

Exercise 4.8 $x = 2\sin t, \quad y = 2\cos t$

Exercise 4.9 $x = \cos t + \sin t, \quad y = -\sin t + \cos t$

Exercise 4.10 $x = \cos t - \sin t, \quad y = \sin t - \cos t$

Exercise 4.11 $x = c\sin t, \quad y = c\cos t$

Exercise 4.12 $x = c_1\sin t + c_2\cos t, \quad y = c_1\cos t - c_2\sin t$

Exercise 4.13 *(a) Derive an equivalent first order system for the second order equation* $mx'' + kx = 0$ *where* $m > 0$ *and* $k > 0$ *are positive constants. (b) Show that* $x = \cos\omega t$, $y = -\omega\sin\omega t$ *is a solution pair of the system for all* t *where* $\omega = \sqrt{k/m}$.

For each system below, determine those initial conditions t_0, x_0 and y_0 for which Theorem 4.1 applies. Explain your answer. What do you conclude for these initial conditions? What do you conclude for other initial conditions? (a, b, c, d, r_1 and r_2 are constants.)

Exercise 4.14 $\begin{cases} x' = x(1-x) - xy\,(1+x)^{-1} \\ y' = -y + xy\,(1+x)^{-1} \end{cases}$

Exercise 4.15 $\begin{cases} x' = x(1 - xy\,(1+x)^{-1}) \\ y' = (-1+x)y \end{cases}$

Exercise 4.16 $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$

Exercise 4.17 $\begin{cases} x' = r_1\,(1 - ax - by)\,x \\ y' = r_2\,(1 - cx - dy)\,y \end{cases}$

Exercise 4.18 $\begin{cases} x' = x(1 - \frac{x}{2+\sin t}) - xy \\ y' = -y + xy \end{cases}$

Exercise 4.19 $\begin{cases} x' = (2+\cos t)(1-x) - x^2y\,(1+x^2)^{-1} \\ y' = -y + x^2y\,(1+x^2)^{-1} \end{cases}$

For each second order equation below, determine those initial conditions t_0, x_0 and y_0 for which Theorem 4.1 applies. (First find an equivalent first order system.) Explain your answer. What do you conclude for these initial conditions? What do you conclude for other initial conditions?

Exercise 4.20 $x'' + x = \sin t$ *(a forced simple harmonic oscillator)*

Exercise 4.21 $x'' + x' + x = \cos t$ *(a forced oscillator with friction)*

Exercise 4.22 $t^2x'' + tx' + x = 0$ *(a Legendre equation)*

Exercise 4.23 $x'' + px' + qx^3 = 0$ *where* p *and* q *are constants (Duffing equation)*

Exercise 4.24 $x'' + \alpha(x^2-1)x' + x = \beta\sin\theta t$ *where* α, β *and* θ *are constants.*

Exercise 4.25 $mx'' + k\sin x = 0$, *where* $m \neq 0$ *and* k *are constants.*

Exercise 4.26 *Formulas for the solution pair of the initial value problem (4.20) are*

$$x(t) = \frac{1}{5}e^{-6t} + \frac{4}{5}e^{-t} \qquad y(t) = \frac{2}{5}e^{-t} - \frac{2}{5}e^{-6t}.$$

Use these formulas to calculate the (absolute value of) the errors of the Euler approximations to $x(2)$ and $y(2)$ using step sizes $s = 0.4,\ 0.2,\ 0.1,\ 0.05,\ 0.0025$. At what rate do the errors decrease? Is this rate appropriate for the Euler Algorithm?

Exercise 4.27 *Repeat Exercise 4.26 using the Modified Euler Algorithm.*

Exercise 4.28 *Formulas for the solution pair of the initial value problem*

$$\begin{aligned} x' &= -x + y \\ y' &= x - 2y \\ x(0) &= 1, \quad y(0) = 0 \end{aligned}$$

are

$$\begin{aligned} x(t) &= \frac{1}{10}\left(5 + \sqrt{5}\right) e^{-\frac{1}{2}\left(3-\sqrt{5}\right)t} + \frac{1}{10}\left(5 - \sqrt{5}\right) e^{-\frac{1}{2}\left(3+\sqrt{5}\right)t} \\ y(t) &= \frac{1}{5}\sqrt{5}e^{-\frac{1}{2}\left(3-\sqrt{5}\right)t} - \frac{1}{5}\sqrt{5}e^{-\frac{1}{2}\left(3+\sqrt{5}\right)t}. \end{aligned}$$

Use these formulas to calculate the (absolute value of) the errors made by the Euler approximations to $x(1)$ and $y(1)$ using step sizes $s = 0.25,\ 0.125,\ 0.0625,\ 0.03125,\ 0.015625$. At what rate do the errors decrease and is this rate appropriate for the Euler Algorithm?

Exercise 4.29 *Repeat Exercise 4.28 using the Modified Euler Algorithm.*

Exercise 4.30 *(a) Use a computer to graph each component $x(t)$ and $y(t)$ of the solution pair of the initial value problem*

$$\begin{aligned} x' &= -3x + y \\ y' &= \frac{1}{2}x - y \\ x(0) &= \frac{1}{2}, \quad y(0) = 0 \end{aligned}$$

for $0 \le t \le 5$. What are the differences and similarities between these two graphs?

(b) What changes occur in the graphs of $x(t)$ and $y(t)$ if the initial condition $y(0) = 0$ is changed to $y(0) = -1/2$?

Exercise 4.31 *(a) Derive an equivalent system for the general van der Pol equation*

$$x'' + \alpha\left(x^2 - 1\right)x' + x = 0$$

in which α is a constant.

(b) Use a computer to graph the solution $x(t)$ of the initial value problems $x(0) = 2$, $x'(0) = 0$ for $0 \le t \le 50$ when $\alpha = -1$.

(c) In what fundamental way does the graph change if $\alpha = 1$?

(d) Draw the graph of $x(t)$ for some other values of α. At what value of α does the change in (c) appear to occur?

For each system below draw by hand the vector field direction at each of the indicated points.

Exercise 4.32 $\begin{cases} x' = y \\ y' = -x \end{cases}$ *at* $(x, y) = (1, 1)\,,\ (1, 0),\ (0, -2),\ (-1, -1),\ (-1/2, 0)\,.$

Exercise 4.33 $\begin{cases} x' = -x + y \\ y' = -2x - y \end{cases}$ *at* $(x, y) = (1, 1)\,,\ (-1, -1),\ (-2, 1),\ (1, 1/2)\,,\ (-1/2, 1/2)\,.$

For each system below use a computer to obtain the vector field for the indicated rectangle in the (x, y)-plane and the orbits through each of the given points.

Exercise 4.34 $\begin{cases} x' = -2x + 2y \\ y' = 2x - 5y \end{cases}$ *for* $-1 < x < 1$, $-1 < y < 1$. *Obtain orbits passing through the points* $(x, y) = (0, 1)$ *and* $(-1, 0)\,.$

Exercise 4.35 $\begin{cases} x' = 2x - y \\ y' = -3x - 2y \end{cases}$ *for* $-2 < x < 2$, $-2 < y < 2$. *Obtain orbits passing through the points* $(x, y) = (0, 1)$ *and* $(0.5, 1)\,.$

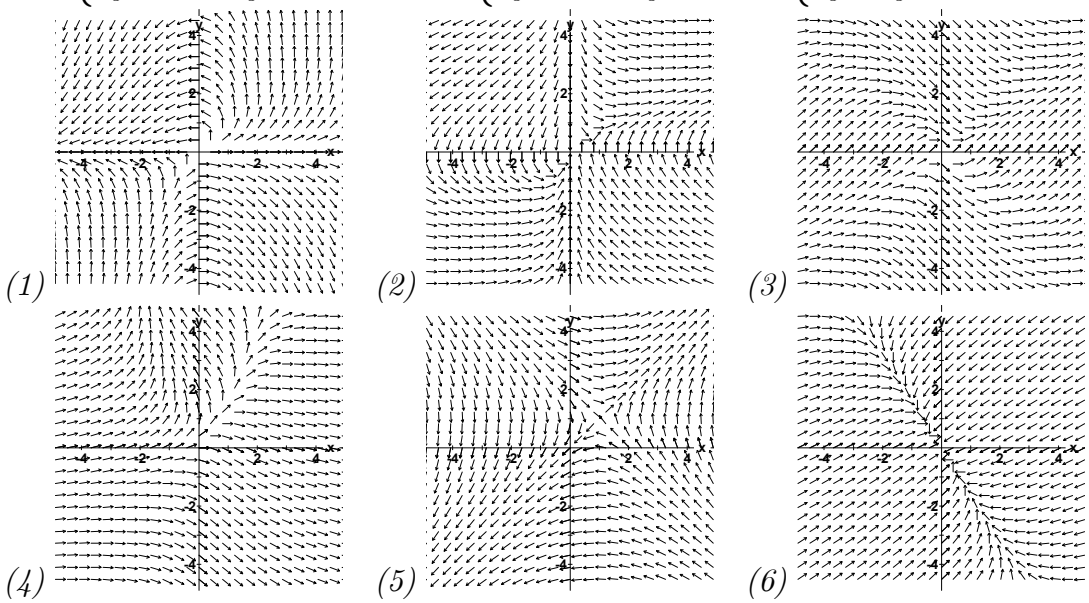
Exercise 4.36 $\begin{cases} x' = -y \\ y' = -x + \left(1 - \frac{1}{3}y^2\right) y \end{cases}$ *for* $-2.5 < x < 2.5$, $-2.5 < y < 2.5$. *Obtain orbits passing through the points* $(x, y) = (2, 1)$ *and* $(2, 2)$ *This system is equivalent to the Rayleigh equation* $x'' - \left(1 - (x')^2/3\right) x' + x = 0.$

Exercise 4.37 $\begin{cases} x' = -y \\ y' = -x + (1 - y^2/3)\, y \end{cases}$ *for* $-5 < x < 5$, $-5 < y < 5$. *Obtain orbits passing through the points* $(x, y) = (-2, -1/2)$ *and* $(-2, 1/2).$

Exercise 4.38 *Without using a computer, match each system with its vector field.*

$$(a)\begin{cases} x' = -2x - y \\ y' = -x - y \end{cases} \qquad (b)\begin{cases} x' = xy \\ y' = x - y \end{cases} \qquad (c)\begin{cases} x' = x - y \\ y' = xy \end{cases}$$

$$(d)\begin{cases} x' = x^2 + y^2 \\ y' = x^2 - y^2 \end{cases} \qquad (d)\begin{cases} x' = x^2 + y^2 \\ y' = x^2 - y^2 \end{cases} \qquad (f)\begin{cases} x' = x^2 - y + 1 \\ y' = y - x \end{cases}$$



(1) (2) (3)

(4) (5) (6)

Identify and sketch the x- and y-nullclines for the following systems. Sketch the vector field by using the nullclines and the regions they define in the x, y-plane (or the portion of the plane indicated). Using your sketch, draw some typical orbits.

Exercise 4.39 $\begin{cases} x' = -y \\ y' = x \end{cases}$

Exercise 4.40 $\begin{cases} x' = y \\ y' = x \end{cases}$

Exercise 4.41 $\begin{cases} x' = -x + y \\ y' = x + y \end{cases}$

Exercise 4.42 $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$

Exercise 4.43 $\begin{cases} x' = x(1 - y) \\ y' = y(1 - x) \end{cases}$ *for* $x \geq 0$, $y \geq 0$

Exercise 4.44 $\begin{cases} x' = x(1 - y) \\ y' = y(-1 + x) \end{cases}$ *for* $x \geq 0$, $y \geq 0$

Exercise 4.45 *The equations*

$$x' = x\,(r_1 - a_{11}x - a_{12}y)$$
$$y' = y\,(r_2 - a_{21}x - a_{22}y)$$

with positive coefficients r_i, a_{ij} are called the "Lotka-Volterra competition" equations. They describe the dynamics of two populations x and y in competition with one another over a limited resource. Only solutions in the first quadrant $x \geq 0$, $y \geq 0$ are of interest.

(a) What kind of curves are the nullclines?

(b) Study the possible configurations of the nullclines in the first quadrant. For each possibility sketch the vector field.

(c) Use your results in (b) to sketch several typical orbits for each possible nullcline configuration you found in (b).

(d) Use a computer to sketch the vector field of an example for each possible nullcline configuration you found in (b). Also use the computer to graph typical orbits for each example.

(e) Discuss the implications of your results in (c) and (d) with regard to the long term fate of both populations.

Exercise 4.46 *The equations*

$$x' = x\,(r_1 - a_{11}x - a_{12}y)$$
$$y' = y\,(-r_2 + a_{21}x)$$

with positive coefficients r_i, a_{ij} are called the "Lotka-Volterra predator-prey" equations. They describe the dynamics of a predator population y and its prey population x. Only solutions in the first quadrant $x \geq 0$, $y \geq 0$ are of interest. Repeat Exercise 4.45 for this system of equations.

Exercise 4.47 *The system*

$$x' = y$$
$$y' = -\alpha(x^2 - 1)y - x\ ,$$

is equivalent to the second order equation $x'' + \alpha\,(x^2 - 1)\,x' + x = 0$ (the van der Pol equation).

(a) Find and sketch the nullclines for $\alpha > 0$. Use the nullclines to sketch the vector field.

(b) Repeat (a) for $\alpha < 0$.

Exercise 4.48 *Find and sketch the nullclines of the system*

$$x' = y$$
$$y' = -x^3 - x$$

This system is equivalent to the second order equation $x'' + x^3 + x = 0$ (the Duffing equation). Use the nullclines to sketch the vector field.

Verify that the pairs below solve the system

$$\begin{aligned} x' &= -2x + y \\ y' &= x - 2y. \end{aligned}$$

Exercise 4.49 $x(t) = e^{-t}, \quad y(t) = e^{-t}$

Exercise 4.50 $x(t) = e^{-3t}, \quad y(t) = -e^{-3t}$

Exercise 4.51 $x(t) = e^{-t} + e^{-3t}, \quad y(t) = e^{-t} - e^{-3t}$

Exercise 4.52 $x(t) = c_1 e^{-t} + c_2 e^{-3t}, \quad y(t) = c_1 e^{-t} - c_2 e^{-3t}$ *for any constants* c_1 *and* c_2

Verify that the pairs below solve the system

$$\begin{aligned} x' &= 4x - 2y \\ y' &= 7x - 5y. \end{aligned}$$

Exercise 4.53 $x(t) = 2e^{-3t}, \quad y(t) = 7e^{-3t}$

Exercise 4.54 $x(t) = e^{2t}, \quad y(t) = e^{2t}$

Exercise 4.55 $x(t) = 2e^{-3t} - e^{2t}, \quad y(t) = 7e^{-3t} - e^{2t}$

Exercise 4.56 $x(t) = 2c_1 e^{-3t} + c_2 e^{2t}, \quad y(t) = 7c_1 e^{-3t} + c_2 e^{2t}$

Exercise 4.57 *Show the pair* $x(t) = 3 + 2\cos t - 2\sin t$, $y(t) = -2\sin t - 2\cos t$ *solves the system*

$$\begin{aligned} x' &= y \\ y' &= -x + 3 \end{aligned}$$

for all t.

Exercise 4.58 *Show the pair* $x(t) = \frac{1}{t}$, $y(t) = \frac{1}{t^2}$ *solves the system*

$$\begin{aligned} x' &= -x\sqrt{y} \\ y' &= -2xy \end{aligned}$$

for $t > 0$.

Theorem 4.1 applies to which initial value problems below? To which does it not apply? Explain your answers. What conclusions can you draw in each case?

Exercise 4.59
$$\begin{cases} x' = \sin(x + y) \\ y' = \sin(x - y) \\ x(0) = 1,\ y(0) = 0 \end{cases}$$

Exercise 4.60 $\begin{cases} x' = (1+x)(1-y)^{-1} \\ y' = (1+y)(1-x)^{-1} \\ x(0) = 0,\ y(0) = 0 \end{cases}$

Exercise 4.61 $\begin{cases} x' = (1+x)(1-y)^{-1} \\ y' = (1+y)(1-x)^{-1} \\ x(0) = 0,\ y(0) = 1 \end{cases}$

Exercise 4.62 $\begin{cases} x' = \sqrt{t-x-y} \\ y' = \sqrt{t+x+y} \\ x(1) = 0,\ y(1) = y_0 \\ \textit{where } -1 < y_0 < 1 \end{cases}$

For which initial conditions x_0, y_0 and initial times t_0 does Theorem 4.1 apply for the systems below. For which does this theorem not apply? Explain your answers. What conclusions can you draw in each case?

Exercise 4.63 $\begin{cases} x' = (1+x)(1-y)^{-1} \\ y' = (1+y)(1-x)^{-1} \end{cases}$

Exercise 4.64 $\begin{cases} x' = ax + by \\ y' = cx + dx \end{cases}$

Exercise 4.65 $\begin{cases} x' = \sqrt{t-x-y} \\ y' = \sqrt{t+x+y} \end{cases}$

Exercise 4.66 $\begin{cases} x' = \ln(1-x-y) \\ y' = \ln(1+x+y) \end{cases}$

For which initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ do the following systems have unique solutions? On what intervals do those solutions exist?

Exercise 4.67 $\begin{cases} x' = x\cos t - 2y \\ y' = 2x - 3y\sin t \end{cases}$

Exercise 4.68 $\begin{cases} x' = \frac{1}{t}x + y - e^t\cos t \\ y' = 5x - \frac{1}{1-t}y \end{cases}$

Exercise 4.69 $\begin{cases} x' = x^2 + y \\ y' = x - y \end{cases}$

Exercise 4.70 $\begin{cases} x' = x^{-2} + y \\ y' = x - y \end{cases}$

For which initial conditions $x(t_0) = x_0$, $x'(t_0) = y_0$ do the following equations have unique solutions? On what intervals do those solutions exist?

Exercise 4.71 *the mass-spring equation* $mx'' + kx = 0$, *where* $m > 0$ *and* $k > 0$ *are constants.*

Exercise 4.72 *the forced mass-spring equation* $mx'' + kx = \sin t$, *where* $m > 0$ *and* $k > 0$ *are constants.*

Exercise 4.73 *the Legendre equation* $t^2x'' + tx' + x = 0$

Exercise 4.74 $x'' - (1 - t^2)^{-1} x' - 2x = e^t$

Exercise 4.75 *Show the initial value problem* $x(t_0) = x_0$, $x'(t_0) = y_0$ *for the second order equation* $x'' + p(t)x' + q(t)x = h(t)$ *has a unique solution on the whole interval* $\alpha < t < \beta$ *if the coefficients* $p(t)$ *and* $q(t)$ *and the nonhomogeneous term* $h(t)$ *are continuous on the interval* $\alpha < t < \beta$.

Consider the following initial value problem

$$\begin{aligned} x' &= x + 2y \\ y' &= -x - y \\ x(0) = 1, &\quad y(0) = 0. \end{aligned}$$

Exercise 4.76 *Use a computer program to approximate the solution at* $T = 1$ *using Euler's Algorithm with step sizes* $s = 0.1$, 0.05, 0.025, 0.0125, *and* 0.00625. *Which digits in these approximations do you think are accurate and why?*

Exercise 4.77 *Repeat Exercise 4.76 using the Modified Euler Algorithm.*

Exercise 4.78 *Repeat Exercise 4.76 using the Runge-Kutta Algorithm.*

Exercise 4.79 *Use a computer program to plot the vector field for this system.*

Exercise 4.80 *Use the Euler Algorithm for* $0 \leq t \leq 20$ *with step size* $s = 0.1$ *to study the orbits in the phase plane. What do you conjecture all orbits do as* $t \to +\infty$*? Repeat using the Modified Euler Algorithm and the Runge-Kutta Algorithm.*

Exercise 4.81 *Repeat Exercises 4.76-4.80 for the initial value problem*

$$\begin{aligned} x' &= x + y - 0.05(x^2 + y^2)x \\ y' &= -x + y - 0.05(x^2 + y^2)y \\ x(0) = 0.1, &\quad y(0) = 0.0. \end{aligned}$$

Consider the following initial value problem

$$\begin{aligned} x' &= x + 0.5y \\ y' &= x - y \\ x(0) = 1, &\quad y(0) = 0. \end{aligned}$$

Exercise 4.82 *Use a computer program to approximate the solution at* $T = 1$ *using Euler's Algorithm with step sizes* $s = 0.1,\ 0.05,\ 0.025,\ 0.0125,$ *and* 0.00625. *Which digits in these approximations do you think are accurate and why?*

Exercise 4.83 *Repeat Exercise 4.82 using the Modified Euler Algorithm.*

Exercise 4.84 *Repeat Exercise 4.82 using the Runge-Kutta Algorithm.*

Exercise 4.85 *Use a computer program to plot the vector field for this system of equations.*

Exercise 4.86 *Use the Euler Algorithm for* $0 \leq t \leq 20$ *with step size* $s = 0.1$ *to study the orbits in the phase plane. What do you conjecture all orbits do as* $t \to +\infty$*? Repeat using the Modified Euler Algorithm and the Runge-Kutta Algorithm.*

Chapter 5

Linear Systems of First Order Equations

Systems of linear, first order differential equations (or "linear systems") constitute an important class of equations. There are at least two reasons for this. First of all, linear systems often arise in applications, as we will see. Secondly, it will turn out that one of the basic methods for the analysis nonlinear equations involves a linearization principle which requires a thorough knowledge of linear systems.

Recall that higher order differential equations can be equivalently written as linear first order systems. It follows that whatever we learn about first order systems will automatically tell us something about higher order equations. In this chapter we will therefore learn about linear higher order equations as well as about linear first order systems.

Our focus in this chapter will be on linear systems of two first order differential equations in two unknowns. Much of what we will learn, however, is applicable to any number of linear equations (in the same number of unknowns). This fact will be enhanced by our use of matrix notation

Linear systems in two unknowns (what we will call "two dimensional systems") are those in which both dependent (or "state") variables x and y appear linearly. That is to say, the right hand sides $f(x, y)$ and $g(x, y)$ in the general system

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned}$$

are linear functions of x and y. That is to say, a (two dimensional) linear system has the form

$$\begin{aligned} x' &= a(t)x + b(t)y + q_1(t) \\ y' &= c(t)x + d(t)y + q_2(t). \end{aligned} \tag{5.1}$$

We call $a(t)$, $b(t)$, $c(t)$, and $d(t)$ the *coefficients* of the system (5.1) and $q_1(t)$ and $q_2(t)$ the *nonhomogeneous terms (or forcing functions).*

Note that the linearity of a system is a property of the dependence of the equations on the state variables x and y. No constraint is placed on the independent variable t (in particular, t is not required to appear linearly). Also note that linearity is *not* about properties of

solutions of the system (in other words, to say a system is linear is *not* to say that its solutions are linear functions). Instead, linearity means that the rates of change x' and y' of the state variables are linear functions of the state variables x and y.

Examples of linear systems include the systems (4.2)

$$\begin{aligned} x' &= -r_1 x + r_2 y \\ y' &= r_1 x - (r_2 + r_3)\, y \end{aligned}$$

and (4.3)

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned}$$

in Chapter 4 (which arise from a pesticide application described there) and the system

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x - \frac{c}{m}y + q\,(t) \end{aligned}$$

which is equivalent to the linear second order equation

$$mx'' + cx' + kx = q\,(t)\,.$$

The latter example includes the system

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x \end{aligned}$$

which is equivalent to the famous simple harmonic oscillator equation

$$mx'' + kx = 0.$$

A convenient way to work with linear systems and their solutions is to use matrix notation. We place the state variables $x = x(t)$ and $y = y(t)$, and their derivatives, into the matrices (column vectors)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\,(t) \\ y\,(t) \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x'\,(t) \\ y'\,(t) \end{pmatrix}.$$

Constructing the column vector

$$\begin{pmatrix} q_1\,(t) \\ q_2\,(t) \end{pmatrix}$$

and recalling the definition of matrix multiplication and addition, we see that

$$\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_1\,(t) \\ q_2\,(t) \end{pmatrix} = \begin{pmatrix} a(t)x + b(t)y + q_1\,(t) \\ c(t)x + d(t)y + q_2\,(t) \end{pmatrix}$$

and that the general linear system (5.1), in matrix format, becomes

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}. \tag{5.2}$$

If both $q_i(t)$ are absent

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.3}$$

then the system is called *homogeneous*. If at least one $q_i(t)$ is present, then the system is called *nonhomogeneous*. Further notational simplification results if we introduce the notation

$$\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}.$$

Then the nonhomogeneous and homogeneous linear systems can be respectively written as

$$\tilde{x}' = A(t)\tilde{x} + \tilde{q}(t) \quad \text{and} \quad \tilde{x}' = A(t)\tilde{x}.$$

The matrix $A(t)$ is called the *coefficient matrix* of these systems.

In fact, a linear system of any number of state variables can by written in this notationally identical form. For n equations in n unknowns, the state vector $\tilde{x}$ is $n \times 1$, coefficient matrix $A(t)$ is $n \times n$ and the nonhomogeneous (or forcing) term $\tilde{q}$ is $n \times 1$. The same is true of any linear differential equation of any order n.

5.1 The Fundamental Existence and Uniqueness Theorem

The initial value problem

$$\begin{aligned} x' &= a(t)x + b(t)y + q_1(t) \\ y' &= c(t)x + d(t)y + q_2(t) \\ x(t_0) &= x_0, \quad y(t_0) = y_0. \end{aligned} \tag{5.4}$$

for a linear system, in matrix notation, takes the form

$$\begin{aligned} \tilde{x}' &= A(t)\tilde{x} + \tilde{q}(t) \\ \tilde{x}(t_0) &= \tilde{x}_0 \end{aligned}$$

where

$$\tilde{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

This is a special case of the initial value problem (4.11) considered in the Sec. 4.1 and, as a result, to which the Fundamental Existence and Uniqueness Theorem 4.1 in Chapter 4, Sec. 4.1 applies. An application of that theorem requires that.

$$\begin{aligned} f(t,x,y) &= a(t)x + b(t)y + q_1(t) \\ g(t,x,y) &= c(t)x + d(t)y + q_2(t) \end{aligned}$$

and the derivatives

$$\frac{df}{dx} = a(t), \qquad \frac{df}{dy} = b(t)$$
$$\frac{dg}{dx} = c(t), \qquad \frac{dg}{dy} = d(t)$$

be continuous functions of t, x and y at t_0, x_0 and y_0. Since f and g are linear in x and y, they certainly are continuous as functions of x and y. To obtain the requirement of continuity in t we need only to assume that the four coefficients $a(t)$, $b(t)$, $c(t)$, and $d(t)$ and the two nonhomogeneous terms $q_1(t)$ and $q_2(t)$ are continuous at $t = t_0$.

In matrix terminology, when we say the coefficient matrix $A(t)$ and the nonhomogeneous term $\tilde{q}(t)$ are continuous at $t = t_0$ we mean all of their components are continuous at t_0. Under this assumption, the Fundamental Theorem 4.1 guarantees that the initial value problem (5.4) has a unique solution pair on some interval containing t_0.

It turns out that for linear systems (just as it did for single linear equations) that a stronger conclusion is possible, namely, that the unique solution exists on an entire interval on which all coefficients are continuous. (This fact is a corollary of the Variation of Constants Formula which we will study in Sec. 5.3 below.) The following theorem is the *Extended Fundamental Existence and Uniqueness Theorem* for linear systems.

Theorem 5.1 ***Suppose the coefficient matrix*** $A(t)$ ***and the nonhomogeneous term*** $\tilde{q}(t)$ ***are continuous on an interval*** $\alpha < t < \beta$. ***For any initial time*** t_0 ***in this interval and for any initial condition*** $\tilde{x}_0$***, the initial value problem***

$$\begin{aligned} \tilde{x}' &= A(t)\,\tilde{x} + \tilde{q}(t) \\ \tilde{x}(t_0) &= \tilde{x}_0 \end{aligned}$$

has a unique solution $\tilde{x}(t)$ ***on the entire interval*** $\alpha < t < \beta$.

In particular, if the coefficient matrix and nonhomogeneous term of a linear system are continuous *for all* t, as is often the case in applications, then solutions of initial value problems *exist for all* t. An important case of this occurs when the components of the coefficient matrix and nonhomogeneous term are constants (in which case the system is called *autonomous*).

Example 5.1 *The initial value problem*

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \\ x(0) &= 1, \quad y(0) = 0 \end{aligned}$$

has a constant coefficient matrix and nonhomogeneous term

$$A(t) = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which are, therefore, continuous on the interval $-\infty < t < +\infty$. *By Theorem 5.1 this initial value problem has a unique solution and it exists on the interval* $-\infty < t < +\infty$.

Example 5.2 *The initial value problem*

$$\begin{aligned} x'' + x &= \sin t \\ x(0) = x_0, &\quad x'(0) = y_0. \end{aligned}$$

is equivalent to the initial value problem

$$\begin{aligned} x' &= y \\ y' &= -x + \sin t \\ x(0) = x_0, &\quad x'(0) = y_0. \end{aligned} \tag{5.5}$$

Since

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$$

are continuous on the interval $-\infty < t < +\infty$. By Theorem 5.1 this initial value problem, and hence the one for the second order differential equation, has a unique solution and it exists on the interval $-\infty < t < +\infty$.

5.2 The Structure of General Solutions

5.2.1 Nonhomogeneous Linear Systems

Recall that the general solution (i.e., the set of all solutions) of a single, nonhomogeneous linear equation

$$x' = p(t)\, x + q(t)$$

has the additive decomposition

$$x = x_h + x_p$$

where x_h is the general solution of the associated homogeneous equation

$$x' = p(t)\, x$$

and x_p is any particular solution of the nonhomogeneous equation. Moreover, the algebraic structured of x_h is that of a linear vector space.

Our goal in this section is to establish these facts for the general solution (i.e. the set of all solutions) of the nonhomogeneous system

$$\tilde{x}' = A(t)\, \tilde{x} + \tilde{q}(t) \tag{5.6}$$

and its associated homogeneous system

$$\tilde{x}' = A(t)\, \tilde{x}. \tag{5.7}$$

As usual, we will focus on systems of two equations (two dimensional systems):

$$\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}.$$

When we say $A(t)$ and $\tilde{q}(t)$ are continuous on an interval $\alpha < t < \beta$ we mean that all their entries are continuous on $\alpha < t < \beta$.

Theorem 5.2 ***Assume*** $A(t)$ ***and*** $\tilde{q}(t)$ ***are continuous on an interval*** $\alpha < t < \beta$. ***Then the general solution of the linear nonhomogeneous system (5.6) on*** $\alpha < t < \beta$ ***has the additive decomposition*** $\tilde{x} = \tilde{x}_h + \tilde{x}_p$ ***where*** $\tilde{x}_h = \tilde{x}_h(t)$ ***is the general solution of the associated homogeneous system (5.7) and*** $\tilde{x}_p = \tilde{x}_p(t)$ ***is a particular solution pair of the nonhomogeneous system (5.6).***

We justify this theorem by showing that if $\tilde{x}(t)$ is a solution of the nonhomogeneous system (5.6) then the difference $\tilde{x}(t) - \tilde{x}_p(t)$ solves (5.7). For a more formal, set theoretic, proof see Exercise 5.19.

Suppose $\tilde{x}(t)$ solves (5.6). Then since differentiation is a linear operator

$$\begin{aligned}(\tilde{x}(t) - \tilde{x}_p(t))' &= \tilde{x}'(t) - \tilde{x}_p'(t) \\ &= A\tilde{x}(t) + \tilde{q}(t) - (A\tilde{x}_p(t) + \tilde{q}(t)).\end{aligned}$$

Using the definition of additive inverse and the additive Commutative Law for vectors, we have

$$(\tilde{x}(t) - \tilde{x}_p(t))' = A\tilde{x}(t) - A\tilde{x}_p(t).$$

Finally the Distributive Law for matrix multiplication implies

$$(\tilde{x}(t) - \tilde{x}_p(t))' = A(\tilde{x}(t) - A\tilde{x}_p(t))$$

which is nothing more than saying that the difference $\tilde{x}(t) - \tilde{x}_p(t)$ satisfies (5.7), that is to say there is a solution $\tilde{x}_h(t)$ of (5.7) such that

$$\tilde{x}_h(t) = \tilde{x}(t) - \tilde{x}_p(t)$$

and hence $\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$.

5.2.2 Homogeneous Linear Systems

Assume the coefficient matrix $A(t)$ (i.e. all of its entries) are continuous on an interval $\alpha < t < \beta$. We claim that the set of all solutions of the homogeneous linear system $\tilde{x}' = A(t)\tilde{x}$ on $\alpha < t < \beta$ (cf. the Extended Fundamental Theorem 5.1) is a linear vector space. That is to say, we claim the general solution $\tilde{x}_h$ is a linear vector space.

To show this, we note:

- The set of continuous functions on an interval $\alpha < t < \beta$ is a linear vector space
- A subset of a linear vector space is itself a linear vector space (i.e., is a linear subspace) if and only if it is *closed under linear combinations.*[1]

Therefore, we can show $\tilde{x}_h$ is a linear vector space by showing that it is closed under linear combinations. This is the same as to say that any linear combination of solutions of a homogeneous system $\tilde{x}' = A(t)\tilde{x}$ is a solution. That is the subject of the following theorem.

[1] A set is *closed under linear cominations* means that any linear combination of vectors from the set is also in the set.

Theorem 5.3 ***(The Superposition Principle) If*** $\tilde{x}_1(t)$ ***and*** $\tilde{x}_2(t)$ ***are solutions of the homogeneous system*** $\tilde{x}' = A(t)\tilde{x}$ ***on*** $\alpha < t < \beta$***, then for any constants*** c_1 ***and*** c_2 ***the linear combination*** $c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t)$ ***is a solution of the homogeneous system on*** $\alpha < t < \beta$.

To prove this theorem we use the fact that differentiation is a linear operator and make use of basic rules of matrix algebra to calculate

$$\begin{aligned}
(c_1\tilde{x}_1 + c_2\tilde{x}_2)' &= c_1\tilde{x}_1' + c_2\tilde{x}_2' \\
&= c_1(A(t)\tilde{x}_1) + c_2(A(t)\tilde{x}_2) \\
&= (c_1A(t))\tilde{x}_1 + (c_2A(t))\tilde{x}_2 \\
&= (A(t)c_1)\tilde{x}_1 + (A(t)c_2)\tilde{x}_2 \\
&= A(t)(c_1\tilde{x}_1) + A(t)(c_2\tilde{x}_2) \\
&= A(t)(c_1\tilde{x}_1 + c_2\tilde{x}_2)
\end{aligned}$$

which shows that the linear combination satisfies $\tilde{x}' = A(t)\tilde{x}$ on $\alpha < t < \beta$.

Corollary 5.1 ***Assume the coefficient matrix*** $A(t)$ ***are continuous on an interval*** $\alpha < t < \beta$. ***The general solution*** $\tilde{x}_h$ ***is a linear vector space.***

Remark 1. A linear homogeneous second order equation

$$c_2(t)x'' + c_1(t)x' + c_0(t)x = 0$$

can be converted to an equivalent system with coefficient matrix

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{c_0(t)}{c_2(t)} & -\frac{c_1(t)}{c_2(t)} \end{pmatrix}.$$

$A(t)$ is continuous on $\alpha < t < \beta$ and hence Theorem 5.3 and Corollary 5.1 apply to the corresponding linear homogeneous system provided

$c_0(t)$, $c_1(t)$, $c_2(t)$ are continuous on an interval $\alpha < t < \beta$
$c_0(t)$ does not equal 0 anywhere on the interval $\alpha < t < \beta$.

Under these conditions, it follows that *the general solution of the second order differential equation is a linear vector space.*

Remark 2. Note that the calculation used above to prove Theorem 5.3 remain valid for systems, vectors and matrices of any size. Therefore, *the general solution of a linear homogeneous system of any dimension is a linear vector space.*

By definition, a linear vector space has an additive identity (i.e. a zero vector). For the general solution of a linear homogeneous system, the zero vector is the *zero or trivial solution*

$$\tilde{x}(t) \equiv \tilde{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ on } \alpha < t < \beta.$$

Since this solution is constant, as a function of t, it is also called an *equilibrium solution* or simply an *equilibrium*.

The linear space of continuous functions on an interval $\alpha < t < \beta$ is not finite dimensional. Our next basic fact about the general solution of linear homogeneous systems is that the linear (sub)space $\tilde{x}_h$ is finite dimensional. Furthermore, for two dimensional systems $\tilde{x}' = A(t)\,\tilde{x}$, i.e., systems with a 2×2 coefficient matrix

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

the general solution $\tilde{x}_h$ is a two dimensional vector space.

Recall the following facts from linear algebra:

1. Two vectors $\tilde{x}_1$ and $\tilde{x}_2$ from a linear vector space are independent if and only if they are not constant multiples of each other, or equivalently if and only if the only linear combination $c_1\tilde{x}_1 + c_2\tilde{x}_2$ that equals the zero vector is the trivial linear combination with $c_1 = c_2 = 0$.
2. The span of a set of vectors is the set of all their linear combinations.
3. A vector space is finite dimensional if and only if it is the span of a finite set of independent vectors. Its dimension is the number of independent vectors that span it.

To accomplish our goal, we need to show that a two dimensional system has two independent solutions $\tilde{x}_1$ and $\tilde{x}_2$ that span the general solution $\tilde{x}_h$. We begin with a test for independence applicable to two solutions of a linear homogeneous system.

Definition 5.1 *If*

$$\tilde{x}_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \quad \textit{and} \quad \tilde{x}_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

are two solutions of the linear homogeneous system $\tilde{x}' = A(t)\,\tilde{x}$ on $\alpha < t < \beta$, then the 2×2 matrix

$$\Phi(t) = \begin{pmatrix} \tilde{x}_1(t) & \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}$$

is called a ***solution matrix****. Equivalently, a matrix $\Phi(t)$ is a solution matrix if and only if it satisfies the "matrix" differential equation*

$$\Phi'(t) = A(t)\Phi(t). \tag{5.8}$$

Note that, by the rules of matrix algebra, a linear combination $c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t)$ of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ can be written as

$$\begin{aligned} c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t) &= c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2\tilde{x}_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1x_1(t) + c_2x_2(t) \\ c_1y_1(t) + c_2y_2(t) \end{pmatrix} \\ &= \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

or

$$c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t) = \Phi(t)\,\tilde{c} \tag{5.9}$$

where we have defined

$$\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Theorem 5.4 ***Assume the coefficient matrix*** $A(t)$ ***is continuous on an interval*** $\alpha < t < \beta$. ***Two solutions*** $\tilde{x}_1(t)$ ***and*** $\tilde{x}_2(t)$ ***of the linear homogeneous system*** $\tilde{x}' = A(t)\,\tilde{x}$ ***on*** $\alpha < t < \beta$ ***are independent if and only if***

$$\det \Phi(t_0) \neq 0 \tag{5.10}$$

for some t_0 ***in the interval:*** $\alpha < t_0 < \beta$. ***If (5.10 holds for some*** t_0, ***then in fact*** $\det \Phi(t) \neq 0$ ***for all*** $\alpha < t < \beta$.

Remark 3. Recall from linear algebra that the independence test (5.10) is the same as saying that $\Phi(t_0)$ is nonsingular or that it is invertible. That is to say, the inverse matrix $\Phi^{-1}(t_0)$ exits:

$$\Phi(t_0)\,\Phi^{-1}(t_0) = \Phi^{-1}(t_0)\,\Phi(t_0) = I$$

where I is the identity matrix.

Since Theorem 5.4 is an "if and only if" statement, there are two implications to prove:

(a) If $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are independent, then $\det \Phi(t_0) \neq 0$ for some t_0 in the interval $\alpha < t < \beta$.
(b) If $\det \Phi(t_0) \neq 0$ for some t_0 in the interval $\alpha < t < \beta$, then $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are independent.

You are asked to prove (a) in Exercise 5.26. We will establish (b) here.

Suppose $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are two solutions on $\alpha < t_0 < \beta$ for which $\det \Phi(t_0) \neq 0$. We want to show they are independent on $\alpha < t_0 < \beta$. That is to say, we want to show the only linear combination $c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t)$ that equals the zero (trivial) solution $\tilde{0}$ is the trivial linear combination with $c_1 = c_2 = 0$.

Suppose

$$c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t) \equiv \tilde{0}$$

on $\alpha < t < \beta$. How can we deduce that $c_1 = c_2 = 0$ must be true? As pointed out above, we can write this identity as

$$\Phi(t)\,\tilde{c} \equiv \tilde{0}$$

on $\alpha < t < \beta$ and hence

$$\Phi(t_0)\,\tilde{c} = \tilde{0}.$$

Since we have assumed $\det \Phi(t_0) \neq 0$ and hence that $\Phi(t_0)$ is invertible, it follows that

$$\tilde{c} = \Phi^{-1}(t_0)\,\tilde{0} = \tilde{0}.$$

This establishes assertion (b)

Remark 4. It follows from Theorem 5.4, of course, that if the determinant $\det \Phi(t_0) = 0$, then the two solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are dependent.

Remark 5. An interesting and important fact about solutions of linear systems is that the determinant $\det \Phi(t)$ in Theorem 5.4 is either identically equal to 0 for all t on the interval $\alpha < t < \beta$ or is *never* equal to 0 on this interval. You are asked to prove this in Exercise 5.27. Consequently, we can just as well say that two solution pairs are independent on an interval $\alpha < t < \beta$ if and only if $\det \Phi(t) \neq 0$ *for all* t in the interval.

Definition 5.2 *If*

$$\tilde{x}_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \quad \textit{and} \quad \tilde{x}_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

are two ***independent*** *solutions of the linear homogeneous system* $\tilde{x}' = A(t)\tilde{x}$ *on the interval* $\alpha < t < \beta$, *then the* 2×2 *matrix*

$$\Phi(t) = \begin{pmatrix} \tilde{x}_1(t) & \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}$$

is called a ***fundamental solution matrix****. Equivalently, a matrix* $\Phi(t)$ *is a fundamental solution matrix if and only if it is nonsingular (invertible) on the interval* $\alpha < t < \beta$ *and satisfies the "matrix" differential equation*

$$\Phi'(t) = A(t)\Phi(t). \tag{5.11}$$

Remember that our goal is to show that the general solution $\tilde{x}_h$ of a linear homogeneous system is a two dimensional vector space. At this point we know it is a vector space. We have yet to establish its dimensionality. What remains to prove are the following two assertions:

(i) There exist two independent solutions.
(ii) Any two independent solutions span $\tilde{x}_h$, i.e., any solution is a linear combination of the two independent solutions.

The first assertion (i) follows straightforwardly from the Extended Fundamental Existence and Uniqueness Theorem 5.1 by applying that theorem to two initial value problems. Let

$$\tilde{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

denote the familiar canonical basis vectors in the plane (the two dimensional Euclidean vector space). Choose any point t_0 from the interval $\alpha < t < \beta$ of continuity. By Theorem 5.1 each of the initial value problems

$$\begin{array}{ll} \tilde{x}' = A(t)\tilde{x} & \tilde{x}' = A(t)\tilde{x} \\ \tilde{x}(t_0) = \tilde{e}_1 & \tilde{x}(t_0) = \tilde{e}_2 \end{array} \tag{5.12}$$

has a (unique) solution on $\alpha < t < \beta$. Denote these two solutions by $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$. They are independent by the independence test (5.10) in Theorem 5.4 since

$$\begin{aligned} \det \Phi(t_0) &= \det \begin{pmatrix} \tilde{x}_1(t_0) & \tilde{x}_2(t_0) \end{pmatrix} \\ &= \det \begin{pmatrix} \tilde{e}_1 & \tilde{e}_2 \end{pmatrix} \\ &= \det I \\ &= 1 \end{aligned}$$

is not equal to 0.

To prove assertion (ii) suppose $\tilde{x}(t)$ is any solution of the homogeneous system on the interval $\alpha < t < \beta$ and let $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ be any two independent solutions (not necessarily the two obtained above). We have to figure out how to determine two constants c_1 and c_2 so that this solution $\tilde{x}(t)$ identically equals the linear combination $c_1\tilde{x}_1(t) + c_2\tilde{x}_2(t)$ on the interval $\alpha < t < \beta$. Here's how we do it.

Pick any point t_0 in the interval $\alpha < t < \beta$ and choose c_1 and c_2 by solving the system of linear algebraic equations

$$\Phi(t_0)\,\tilde{c} = \tilde{x}(t_0)$$

for

$$\tilde{c} = \Phi^{-1}(t_0)\,\tilde{x}(t_0)$$

which we can do since $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are independent and hence $\Phi(t_0)$ is nonsingular (invertible). We then form the linear combination (5.9)

$$\Phi(t)\,\tilde{c} = \Phi(t)\,\Phi^{-1}(t_0)\,\tilde{x}(t_0)$$

of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ by using this $\tilde{c}$. We now claim – and this we prove assertion (ii) – that this linear combination is identically the same as the solution $\tilde{x}(t)$.

That $\tilde{x}(t)$ and $\Phi(t)\,\Phi^{-1}(t_0)\,\tilde{x}(t_0)$ are identical follows from the fact that both are solutions of the linear homogeneous system with the same initial condition at t_0. (Substitute $t = t_0$ into both and you get the same answer!). But the Fundamental Existence and Uniqueness Theorem says there can only by one solution to an initial value problem. Therefore, these have to be the same solution:

$$\tilde{x}(t) \equiv \Phi(t)\,\Phi^{-1}(t_0)\,\tilde{x}(t_0)$$

which says nothing more than that $\tilde{x}(t)$ is a linear combination of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$. This establishes assertion (ii), and finishes the proof of the following theorem.

Theorem 5.5 ***Assume the coefficient matrix*** $A(t)$ ***is continuous on an interval*** $\alpha < t < \beta$. ***If*** $\tilde{x}_1(t)$ ***and*** $\tilde{x}_2(t)$ ***of the linear homogeneous system*** $\tilde{x}' = A(t)\,\tilde{x}$ ***on*** $\alpha < t < \beta$ ***are independent, then the general solution is***

$$\tilde{x}_h(t) = \Phi(t)\,\tilde{c} \tag{5.13}$$

where $\tilde{c}$ ***is a vector of arbitrary constants.***

As a result of this Theorem we obtain the following corollary about the linear algebraic structure of the general solution of a linear homogeneous system.

Corollary 5.2 ***Assume the coefficient matrix*** $A(t)$ ***is continuous on an interval*** $\alpha < t < \beta$. ***The general solution*** $\tilde{x}_h(t)$ ***of the linear homogeneous system*** $\tilde{x}' = A(t)\,\tilde{x}$ ***on*** $\alpha < t < \beta$ ***is a two dimensional linear vector space.***

Remark 6. In establishing the existence of two independent solutions we solved to two initial value problems (5.12) with the canonical Euclidean basis vectors $\tilde{e}_i$. The same proof goes through if any other pair of independent Euclidean vectors $\tilde{v}_1$ and $\tilde{v}_2$ are used instead. All that changes in the proof is that

$$\det \Phi(t_0) = \det \begin{pmatrix} \tilde{v}_1 & \tilde{v}_2 \end{pmatrix}$$

which, although not necessarily equal to 1, is nonetheless nonzero since the vectors $\tilde{v}_1$ and $\tilde{v}_2$ are independent. This change in initial conditions gives rise to a different pair of independent solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ and hence a deferent fundamental solution matrix $\Phi(t)$. This just amounts to a *different basis* for the solution space. One cannot accurately speak of *the* fundamental solution matrix for a linear homogeneous system (any more than one can speak of *the* basis of a vector space).

Remark 7. In following up on Remark 1, we note that Corollary 5.2 implies the general solution of a linear homogeneous second order equation

$$c_2(t)\,x'' + c_1(t)\,x' + c_0(t)\,x = 0$$

is a two dimensional vector space, under the assumptions in Remark 1, namely, that the coefficients satisfy the conditions:

$$\begin{aligned} c_2(t),\ c_1(t),\ c_0(t) &\text{ are continuous on an interval } \alpha < t < \beta \\ c_2(t) &\text{ does not equal 0 anywhere on the interval } \alpha < t < \beta. \end{aligned}$$

If one recalls the definition of matrix multiplication from linear algebra, one can see that a matrix

$$\Phi(t) = \begin{pmatrix} \tilde{x}_1(t) & \tilde{x}_1(t) \end{pmatrix} = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}$$

is a solution matrix of a linear homogeneous system, i.e., that the columns are solutions, can by mathematically expressed by

$$\Phi'(t) = A(t)\,\Phi(t).$$

Thus, an equivalent definition of *a fundamental solution matrix is a nonsingular (invertible) matrix* $\Phi(t)$ *that satisfies this matrix differential equation.*

Before looking at some examples, we point out a useful shortcut for calculating the inverse of an invertible 2×2 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad \det M = m_{11}m_{22} - m_{21}m_{12} \neq 0.$$

The inverse

$$M^{-1} = \frac{1}{m_{11}m_{22} - m_{21}m_{12}} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \tag{5.14}$$

is obtained by three steps:

1. Interchange the diagonal entries m_{11} and m_{22}.
2. Change the sign of the anti-diagonal entries m_{12} and m_{21}.
3. Divide the resulting matrix (i.e. divide each entry) by $\det M$.

In examples and applications, this method is useful for calculating the inverse of the fundamental solution matrix $\Phi(t)$, when needed.

Example 5.3 *The linear homogeneous system*

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned}$$

has coefficient matrix

$$A(t) = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}.$$

That

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}$$

is a fundamental solution matrix on $-\infty < t < +\infty$ follows from the calculations

$$\Phi'(t) = \begin{pmatrix} -2e^{-t} & -6e^{-6t} \\ -e^{-t} & 12e^{-6t} \end{pmatrix}$$

$$\begin{aligned} A(t)\Phi(t) &= \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-t} & -6e^{-6t} \\ -e^{-t} & 12e^{-6t} \end{pmatrix} \end{aligned}$$

and the nonsingularity test

$$\Phi(0) = \det \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = -5 \neq 0.$$

Therefore, the general solution is

$$\begin{aligned} \tilde{x}_h(t) &= \Phi(t)\tilde{c} \\ &= \begin{pmatrix} -2e^{-t} & -6e^{-6t} \\ -e^{-t} & 12e^{-6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} -2c_1e^{-t} - 6c_2e^{-6t} \\ -c_1e^{-t} + 12c_2e^{-6t} \end{pmatrix} \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 5.4 *The pair*

$$\tilde{x}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

is a solution of the linear homogeneous system $\tilde{x}' = A(t)\,\tilde{x}$ *on* $-\infty < t < +\infty$ *with the coefficient matrix*

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as the calculations

$$\tilde{x}_1'(t) = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}$$

$$A(t)\,\tilde{x}_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}$$

demonstrate. Similar calculations show that

$$\tilde{x}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

is another solution of the same system. These two solutions are independent because

$$\det \Phi(0) = \det \begin{pmatrix} \tilde{x}_1(0) & \tilde{x}_2(0) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

is nonzero. Therefore

$$\Phi(t) = \begin{pmatrix} \tilde{x}_1(t) & \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

is a fundamental solution matrix and the general solution is

$$\begin{aligned} \tilde{x}_h(t) &= \Phi(t)\,\tilde{c} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} \end{aligned}$$

where c_1 *and* c_2 *are arbitrary constants.*

The linear homogeneous system in Example 5.4 is equivalent to the second order differential equation

$$x'' + x = 0.$$

Therefore, the general solution of this second order equation is

$$x(t) = c_1 \cos t + c_2 \sin t$$

where c_1 and c_2 are arbitrary constants.

In the preceding examples you will no doubt wonder where the independent solutions and hence the fundamental solution matrices $\Phi(t)$ came from or how they were found. Unfortunately, for general linear homogeneous systems there is no universal method that will produce solution formulas. However, for an important restricted class of systems, namely those with constant coefficient matrices $A(t) = A$ (i.e., so-called *autonomous systems*), there is such a method. We study this method in the next chapter. Both examples above have constant coefficient matrices (i.e. are autonomous). You will learn in the next chapter how to find for yourself the independent solutions given in these Examples.

For some examples of fundamental matrices for systems with nonconstant coefficient matrices $A(t)$ see the Exercises.

From a formula for the general solution (i.e., if a fundamental solution matrix $\Phi(t)$ is available) one can calculate formulas for the solution of any initial value problem. The unique solution of the initial value problem

$$\begin{aligned} \tilde{x}' &= A(t)\,\tilde{x} \\ \tilde{x}(t_0) &= \tilde{x}_0 \end{aligned}$$

lies somewhere in the general solution

$$\tilde{x}_h(t) = \Phi(t)\,\tilde{c}.$$

To find it is simply a matter of determine the correct vector $\tilde{c}$ of arbitrary constants (i.e., the solutions "coordinates" in the solution space). This is done by requiring that

$$\begin{aligned} \Phi(t_0)\,\tilde{c} &= \tilde{x}_0 \\ \tilde{c} &= \Phi^{-1}(t_0)\,\tilde{x}_0. \end{aligned}$$

Consequently, the solution of the initial value problem is

$$\tilde{x}(t) = \Phi(t)\,\Phi^{-1}(t_0)\,\tilde{x}_0. \tag{5.15}$$

Example 5.5 *A fundamental solution matrix for the linear homogeneous system with coefficient matrix*

$$A(t) = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

is

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}.$$

See Example 5.3). A formula for the unique solution of the initial value problem

$$\tilde{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is given by (5.15). To use this formula we need the inverse of the fundamental solution matrix

$$\begin{aligned} \Phi^{-1}(t) &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}^{-1} \\ &= \frac{1}{5}\begin{pmatrix} 2e^{t} & e^{t} \\ e^{6t} & -2e^{6t} \end{pmatrix}. \end{aligned}$$

From (5.15) we obtain solution formula

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}(0)\\ &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t}\end{pmatrix}\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}\\ &= \frac{1}{5}\begin{pmatrix} 4e^{-t}+e^{-6t} & 2e^{-t}-2e^{-6t} \\ 2e^{-t}-2e^{-6t} & e^{-t}+4e^{-6t}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}\\ &= \begin{pmatrix} \frac{4}{5}e^{-t}+\frac{1}{5}e^{-6t} \\ \frac{2}{5}e^{-t}-\frac{2}{5}e^{-6t}\end{pmatrix}.\end{aligned}$$

Example 5.6 *A formula for the solution of an arbitrary initial value problem*

$$\tilde{x}(0) = \begin{pmatrix} x_0 \\ y_0\end{pmatrix}$$

for the linear homogeneous system with coefficient matrix

$$A(t) = \begin{pmatrix} -2 & 2 \\ 2 & -5\end{pmatrix}$$

is

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}(0)\\ &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t}\end{pmatrix}\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2\end{pmatrix}\begin{pmatrix} x_0 \\ y_0\end{pmatrix}\\ &= \frac{1}{5}\begin{pmatrix} 4e^{-t}+e^{-6t} & 2e^{-t}-2e^{-6t} \\ 2e^{-t}-2e^{-6t} & e^{-t}+4e^{-6t}\end{pmatrix}\begin{pmatrix} x_0 \\ y_0\end{pmatrix}\\ &= \frac{1}{5}\begin{pmatrix} x_0(4e^{-t}+e^{-6t})+y_0(2e^{-t}-2e^{-6t}) \\ x_0(2e^{-t}-2e^{-6t})+y_0(e^{-t}+4e^{-6t})\end{pmatrix}\end{aligned}$$

or

$$\tilde{x}(t) = \begin{pmatrix} \frac{2}{5}(2x_0+y_0)e^{-t}+\frac{1}{5}(x_0-2y_0)e^{-6t} \\ \frac{1}{5}(2x_0+y_0)e^{-t}-\frac{2}{5}(x_0-2y_0)e^{-6t}\end{pmatrix}.$$

Example 5.7 *Consider the linear homogeneous system with coefficient matrix (as in Example 5.4)*

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

We can, for example, find a formula for the solution of initial value problem

$$\tilde{x}(0) = \tilde{x}_0 = \begin{pmatrix} 1 \\ -2\end{pmatrix}$$

using the fundamental solution matrix (from Example 5.4)

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}$$

and its inverse

$$\Phi^{-1}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

in formula (5.15):

$$\begin{aligned} \tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}_0 \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

or

$$\tilde{x}(t) = \begin{pmatrix} \cos t - 2\sin t \\ -\sin t - 2\cos t \end{pmatrix}.$$

Example 5.8 *A formula for the solution of an arbitrary initial value problem*

$$\tilde{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for the linear homogeneous system with coefficient matrix

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is

$$\begin{aligned} \tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}(0) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{aligned}$$

or

$$\tilde{x}(t) = \begin{pmatrix} x_0\cos t + y_0\sin t \\ -x_0\sin t + y_0\cos t \end{pmatrix}.$$

Since the coefficient matrix of the equivalent associated system to the second order equation

$$x'' + x + 0$$

is that

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

used in the previous example, the solution of any initial value problem

$$x(0) = x_0, \quad y(0) = y_0$$

is

$$x(t) = x_0 \cos t + y_0 \sin t.$$

Theorem 5.5 reduces the problem of solving a homogenous system of two linear equations (i.e., finding a formula for the general solution) to a search for only two independent solutions. Unfortunately, for general homogeneous systems there is no universal solution method that will always produce two independent solution pairs. However, for an important restricted class of homogeneous systems, namely those with constant coefficients (i.e., "autonomous" systems), there is such a method. We study this method in the next chapter.

A last remark should be familiar from linear algebra.

$$\det \begin{pmatrix} x_1(t) & 0 \\ y_1(t) & 0 \end{pmatrix} = 0$$

for all t and consequently, no solution is independent of the trivial solution $\tilde{0}$. Only nontrivial solutions can be independent. This means we can never use the trivial solution to construct a fundamental solution matrix and the general solution of a linear system.

5.3 The Variation of Constants Formula

In Section 5.2.1 we learned that the general solution of a linear system

$$\tilde{x}' = A(t)\tilde{x} + \tilde{q}(t) \tag{5.16}$$

has the additive decomposition

$$\tilde{x} = \tilde{x}_h + \tilde{x}_p$$

where $\tilde{x}_h$ is the general solution of the associated homogeneous system

$$\tilde{x}' = A(t)\tilde{x} \tag{5.17}$$

and $\tilde{x}_p$ is (any) particular solution of the nonhomogeneous system (5.16).

In Section 5.2.2 we learned that the general solution $\tilde{x}_h$ of the homogeneous system (5.17) has the form

$$\tilde{x}_h(t) = \Phi(t)\,\tilde{c}$$

where $\Phi(t)$ is a fundamental solution matrix (i.e., a matrix whose columns are independent solutions of (5.17)) and $\tilde{c}$ is a column vector of arbitrary constants.

In this section we take up the problem of find a particular solution $\tilde{x}_p$ of the nonhomogeneous system (5.16). We will see that there is a formula that one can use to calculate a particular solution $\tilde{x}_p$, provided that a fundamental solution matrix $\Phi(t)$ of the associated homogeneous system is available. By way of motivating the derivation of this formula, we

recall the variation of constants formula for $x_p(t)$ in the case of a single differential equation $x' = p(t)\,x$, namely

$$x_p(t) = e^{P(t)} \int e^{-P(t)} q(t)\, dt.$$

Notice that $e^{P(t)}$ plays the role of $\Phi(t)$ since $x_h = ce^{P(t)}$ in this single equation case. We recognize that this particular solution $x_p(t)$, while not a constant multiple of $e^{P(t)}$, is a function of t times $e^{P(t)}$ (namely $\int e^{-P(t)} q(t)\, dt$). With this observation in mind, we might, in the case of a system of equations, search for a particular solution $\tilde{x}_p(t)$ that is a function of t times $\Phi(t)$. Specifically, we look for a particular solution of the form

$$\tilde{x}_p(t) = \Phi(t)\,\tilde{c}(t) \tag{5.18}$$

where $\tilde{c}(t)$ is a column vector of yet to be determined functions of t. This is, then, an application of the Method of Undetermined Coefficients as described in Section 2.2.1 of Chapter 2.Following the steps of that general method, we substitute the guess (5.18) into the nonhomogeneous equation (5.16) in order to calculate $\tilde{c}(t)$. Thus, we equate

$$\begin{aligned} \tilde{x}_p' &= \Phi'(t)\,\tilde{c}(t) + \Phi(t)\,\tilde{c}'(t) \\ &= (A(t)\,\Phi(t))\,\tilde{c}(t) + \Phi(t)\,\tilde{c}'(t) \end{aligned}$$

(recall (5.8)) and

$$A(t)\tilde{x}_p + \tilde{q}(t) = A(t)\,(\Phi(t)\,\tilde{c}(t)) + \tilde{q}(t)$$

to get (after using the multiplicative associate law a matrix algebra)

$$A(t)\,(\Phi(t)\,\tilde{c}(t)) + \Phi(t)\,\tilde{c}'(t) = A(t)\,(\Phi(t)\,\tilde{c}(t)) + \tilde{q}(t)$$

or

$$\Phi(t)\,\tilde{c}'(t) = \tilde{q}(t).$$

Since the fundamental solution matrix $\Phi(t)$ is invertible, we have

$$\begin{aligned} \tilde{c}'(t) &= \Phi^{-1}(t)\,\tilde{q}(t) \\ \tilde{c}(t) &= \int^t \Phi^{-1}(s)\,\tilde{q}(s)ds \end{aligned}$$

and arrive at the formula

$$\tilde{x}_p(t) = \Phi(t) \int^t \Phi^{-1}(s)\,\tilde{q}(s)ds.$$

This result, when used in the additive decomposition of the general solution, gives us the important *Variation of Constants Formula* for the general solution of a nonhomogeneous linear systems.[2]

[2]The rather self-contradictory name of this formula comes from the guess (5.18), which derives from the formula for $\tilde{x}_h = \Phi(t)\,\tilde{c}$ by replacing the constant $\tilde{c}$ by the function $\tilde{c}(t)$, i.e., by letting this constant vary with time t.

Theorem 5.6 ***(Variation of Constants Formula) Assume the coefficient matrix*** $A(t)$ ***and the nonhomogeneous term*** $\tilde{q}(t)$ ***are continuous on an interval*** $\alpha < t < \beta$***. If*** $\Phi(t)$ ***is a fundamental solution matrix of the homogeneous system*** $\tilde{x}' = A(t)\tilde{x}$***, then the general solution of the nonhomogeneous system***

$$\tilde{x}' = A(t)\tilde{x} + \tilde{q}(t)$$

is given by the formula

$$\tilde{x}\,(t) = \Phi(t)\tilde{c} + \Phi(t)\int^t \Phi^{-1}(s)\tilde{q}(s)ds \tag{5.19}$$

where $\tilde{c}$ ***is a vector of arbitrary constants.***

From the Variation of Constants Formula (5.19) we see that to calculate the general solution of a nonhomogeneous linear first order system, all we need are two independent solutions of the associated homogeneous system (5.17), that is to say, to find a fundamental solution matrix $\Phi(t)$ of the associated homogeneous system. However, in order to obtain an explicit formula for the general solution, we must carry out the integration in (5.19), which in practice this might be difficult (if not impossible in closed form).

Remark 8. The integral (anti-derivative) in the Variation of Constants formula (5.19) is not, of course, unique. However, any specific integral can be used to obtain the general solution. This is because all integrals differ by a constant, i.e., in (5.19) we could replace

$$\int^t \Phi^{-1}(s)\tilde{q}(s)dt \quad \text{by} \quad \int^t \Phi^{-1}(s)\tilde{q}(s)dt + \tilde{k}$$

where $\tilde{k}$ is an arbitrary constant of integration. However, we can then write

$$\begin{aligned}\tilde{x}\,(t) &= \Phi(t)\tilde{c} + \Phi(t)\left(\int^t \Phi^{-1}(s)\tilde{q}(s)dt + \tilde{k}\right)\\ &= \Phi(t)\left(\tilde{c} + \tilde{k}\right) + \Phi(t)\int^t \Phi^{-1}(s)\tilde{q}(s)ds\end{aligned}$$

which still has the form (5.19) since we can just relabel the arbitrary constant $\tilde{c} + \tilde{k}$ by $\tilde{c}$.

Example 5.9 *In matrix notation the system*

$$\begin{aligned}x' &= -2x + 2y + e^{-2t}\\ y' &= 2x - 5y\end{aligned}$$

has the form (5.16) with

$$A\,(t) = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}.$$

From Example 5.3 we retrieve the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}.$$

To use the Variation of Constants Formula (5.19) we calculate

$$\Phi^{-1}(t) = \begin{pmatrix} \frac{2}{5}e^{t} & \frac{1}{5}e^{t} \\ \frac{1}{5}e^{6t} & -\frac{2}{5}e^{6t} \end{pmatrix}, \qquad \Phi^{-1}(t)\tilde{q}(t) = \begin{pmatrix} \frac{2}{5}e^{-t} \\ \frac{1}{5}e^{4t} \end{pmatrix}$$

and

$$\int^{t} \Phi^{-1}(s)\tilde{q}(s)dt = \begin{pmatrix} \int^{t} \frac{2}{5}e^{-s}ds \\ \int^{t} \frac{1}{5}e^{4s}ds \end{pmatrix} = \begin{pmatrix} -\frac{2}{5}e^{-t} \\ \frac{1}{20}e^{4t} \end{pmatrix}$$

$$\Phi(t)\int^{t} \Phi^{-1}(s)\tilde{q}(s)ds = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} -\frac{2}{5}e^{-t} \\ \frac{1}{20}e^{4t} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}e^{-2t} \\ -\frac{1}{2}e^{-2t} \end{pmatrix}.$$

The general solution is

$$\begin{aligned} \tilde{x}\,(t) &= \Phi(t)\tilde{c} + \Phi(t)\int^{t} \Phi^{-1}(s)\tilde{q}(s)ds \\ &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{3}{4}e^{-2t} \\ -\frac{1}{2}e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} 2c_1e^{-t} + c_2e^{-6t} - \frac{3}{4}e^{-2t} \\ c_1e^{-t} - 2c_2e^{-6t} - \frac{1}{2}e^{-2t} \end{pmatrix}. \end{aligned}$$

Example 5.10 *The second order, linear nonhomogeneous equation*

$$x'' + x = \cos 2t$$

is an example of the "forced harmonic oscillator". In the equivalent homogeneous linear system

$$A\,(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \textit{and} \quad \tilde{q}(t) = \begin{pmatrix} 0 \\ \cos 2t \end{pmatrix}$$

From Example 5.4 we retrieve the fundamental solution matrix and its inverse

$$\Phi\,(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \textit{and} \quad \Phi^{-1}\,(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

from which we make the following calculations

$$\begin{aligned} \Phi\,(t)\int^{t} \Phi^{-1}(s)\tilde{q}(s)ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\int^{t} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}\begin{pmatrix} 0 \\ \cos 2s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} -\int^{t} \sin s\cos 2s ds \\ \int^{t} \cos s\cos 2s ds \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3}\cos 2t \\ \frac{2}{3}\sin 2t \end{pmatrix}. \end{aligned}$$

(A table of integrals and trigonometric identities is handy here, or an online integrator.) Using these ingredients in the Variation of Constants Formula (5.19) we calculate

$$\begin{aligned} \tilde{x}\,(t) &= \Phi(t)\tilde{c} + \Phi(t)\int^{t} \Phi^{-1}(s)\tilde{q}(s)ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}\cos 2t \\ \frac{2}{3}\sin 2t \end{pmatrix} \\ &= \begin{pmatrix} c_1\cos t + c_2\sin t - \frac{1}{3}\cos 2t \\ -c_1\sin t + c_2\cos t + \frac{2}{3}\sin 2t \end{pmatrix}. \end{aligned}$$

The general solution of the second order equation is the first component of this vector solution:

$$x(t) = c_1 \cos t + c_2 \sin t - \frac{1}{3}\cos 2t. \tag{5.20}$$

(The second component of the vector solutions is the derivative $x'(t)$, which follows from the manner in which we convert second order equations to systems.)

To solve an initial value problem $\tilde{x}(t_0) = \tilde{x}_0$ for a linear nonhomogeneous system is a matter of appropriately choosing the vector $\tilde{c}$ of arbitrary constants in the general solution. One can either calculate the general solution by means of the Variation of Constants Formula (5.19) and then calculate $\tilde{c}$ from the initial condition. Or a more general approach is to derive a specialized Variation of Constants Formula that directly calculates the unique solution of the initial value problem. This formula is most conveniently derived from (5.19) by choosing a specific integral (anti-derivative) in the formula. (See Remark 8.) Specifically, we choose the integral that equals $\tilde{0}$ at $t = t_0$, namely, we use the definite integral

$$\int_{t_0}^{t} \Phi^{-1}(s)\,\tilde{q}(s)\,ds$$

in (5.19) to obtain

$$\tilde{x}(t) = \Phi(t)\tilde{c} + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\tilde{q}(s)ds.$$

Now we look for the constant vector $\tilde{c}$ that will give us the solution to the initial value problem $\tilde{x}(t_0) = \tilde{x}_0$. Note that

$$\tilde{x}(t_0) = \Phi(t_0)\tilde{c}$$

(because we chose the specific integral that we did!) and hence we calculate $\tilde{c}$ by solving $\Phi(t_0)\tilde{c} = \tilde{x}_0$ for

$$\tilde{c} = \Phi^{-1}(t_0)\tilde{x}_0.$$

Theorem 5.7 ***(Variation of Constants Formula for Initial Value Problems) Assume the coefficient matrix*** $A(t)$ ***and the nonhomogeneous term*** $\tilde{q}(t)$ ***are continuous on an interval*** $\alpha < t < \beta$***. If*** $\Phi(t)$ ***is a fundamental solution matrix of the homogeneous system*** $\tilde{x}' = A(t)\tilde{x}$***, then the unique solution of the initial value problem (with*** $\alpha < t_0 < \beta$***)***

$$\begin{aligned} \tilde{x}' &= A(t)\tilde{x} + \tilde{q}(t) \\ \tilde{x}(t_0) &= \tilde{x}_0 \end{aligned}$$

is given by the formula

$$\tilde{x}(t) = \Phi(t)\Phi^{-1}(t_0)\tilde{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\tilde{q}(s)ds. \tag{5.21}$$

Example 5.11 *Consider the initial value problem*

$$\begin{aligned} x' &= -2x + 2y + e^{-2t} \\ y' &= 2x - 5y \\ x(0) &= \frac{1}{10}, \quad y(0) = \frac{21}{20} \end{aligned}$$

As in 5.9 we have

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}$$

$$\Phi^{-1}(t) = \begin{pmatrix} \frac{2}{5}e^{t} & \frac{1}{5}e^{t} \\ \frac{1}{5}e^{6t} & -\frac{2}{5}e^{6t} \end{pmatrix}, \qquad \Phi^{-1}(t)\tilde{q}(t) = \begin{pmatrix} \frac{2}{5}e^{-t} \\ \frac{1}{5}e^{4t} \end{pmatrix}.$$

In the Variation of Constants Formula (5.21) we have (using the Fundamental Theorem of Calculus)

$$\int_0^t \Phi^{-1}(s)\tilde{q}(s)dt = \begin{pmatrix} \int_0^t \frac{2}{5}e^{-s}ds \\ \int_0^t \frac{1}{5}e^{4s}ds \end{pmatrix} = \begin{pmatrix} \frac{2}{5} - \frac{2}{5}e^{-t} \\ -\frac{1}{20} + \frac{1}{20}e^{4t} \end{pmatrix}$$

$$\begin{aligned} \Phi(t)\int_0^t \Phi^{-1}(s)\tilde{q}(s)ds &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} \frac{2}{5} - \frac{2}{5}e^{-t} \\ -\frac{1}{20} + \frac{1}{20}e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{5}e^{-t} - \frac{3}{4}e^{-2t} - \frac{1}{20}e^{-6t} \\ \frac{2}{5}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{10}e^{-6t} \end{pmatrix}. \end{aligned}$$

The unique solution of the initial value problem is

$$\begin{aligned} \tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\tilde{q}(s)ds \\ &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} \frac{9}{4} \\ -\frac{3}{2} \end{pmatrix} + \begin{pmatrix} \frac{4}{5}e^{-t} - \frac{3}{4}e^{-2t} - \frac{1}{20}e^{-6t} \\ \frac{2}{5}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{10}e^{-6t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - \frac{3}{4}e^{-2t} + e^{-6t} \\ e^{-t} - \frac{1}{2}e^{-2t} - 2e^{-6t} \end{pmatrix}. \end{aligned}$$

Example 5.12 *Consider the initial value problem*

$$\begin{aligned} x'' + x &= \cos 2t \\ x(0) = 0, &\quad y(0) = 0. \end{aligned}$$

As pointed out in Example 5.10 the equivalent linear system to the the second order, linear nonhomogeneous equation has coefficient matrix and nonhomogeneous term

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \textit{and} \quad \tilde{q}(t) = \begin{pmatrix} 0 \\ \cos 2t \end{pmatrix}$$

for which

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \textit{and} \quad \Phi^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

from which we make the following calculations

$$\begin{aligned}
\Phi(t)\int_0^t \Phi^{-1}(s)\tilde{q}(s)ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ \cos 2s \end{pmatrix} ds \\
&= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\int_0^t \sin s \cos 2s ds \\ \int_0^t \cos s \cos 2s ds \end{pmatrix} \\
&= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \frac{1}{6}\cos 3t - \frac{1}{2}\cos t + \frac{1}{3} \\ \frac{1}{2}\sin t + \frac{1}{6}\sin 3t \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}\cos t - \frac{1}{3}\cos 2t \\ -\frac{1}{3}\sin t + \frac{2}{3}\sin 2t \end{pmatrix}.
\end{aligned}$$

(A table of integrals and trigonometric identities is handy here, or an online integrator.) Using these ingredients in the Variation of Constants Formula (5.19) we calculate

$$\begin{aligned}
\tilde{x}(t) &= \Phi(t)\tilde{c} + \Phi(t)\int^t \Phi^{-1}(s)\tilde{q}(s)ds \\
&= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3}\cos t - \frac{1}{3}\cos 2t \\ -\frac{1}{3}\sin t + \frac{2}{3}\sin 2t \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}\cos t - \frac{1}{3}\cos 2t \\ -\frac{1}{3}\sin t + \frac{2}{3}\sin 2t \end{pmatrix}.
\end{aligned}$$

The solution of the initial value problem associated with the second order equation is the first component of this vector solution:

$$x(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t. \tag{5.22}$$

(The second component of the vector solutions is the derivative $x'(t)$, which follows from the manner in which we convert second order equations to systems.)

Remark 9. Just as we learned when studying single first order equations, shortcut methods there are sometimes available that we can use to calculate $x_p(t)$. One such method, for example, was the Method of Undetermined Coefficients that we used in in Section 2.2.1 of Chapter 2. We will re-visit this method in the next section where we will apply it particularly to second order equations. In fact, the general solution (5.20) in Example 5.10 and the solution (5.22) of the initial value problem in Example 5.12 can by calculated quickly by this shortcut method, with much less effort (and no integration or trigonometric identities) than by using the Variation of Constants Formula as is done in these in Examples. Shortcuts apply, however, only to specialized kinds of equations, whereas the Variation of Constants Formula can always be used.

We have focussed on two dimensional systems in this chapter, i.e., two linear differential equations in two unknowns. One advantage of matrix notation is that our results, derivations, and proofs are (usually) independent of the dimension of the system and the size of the matrices. For example, we can extend Definition 5.2 of a fundamental solution matrix $\Phi(t)$ to n-dimensional systems in a straight forward manner as follows.

An $n \times n$ matrix $\Phi(t)$ is a fundamental solution matrix if and only if it is nonsingular (invertible) on the interval $\alpha < t < \beta$ and satisfies the "matrix" differential equation

$$\Phi'(t) = A(t)\Phi(t). \tag{5.23}$$

Then the Variation of Constants Formulas (5.19) and (5.21) remain valid formulas for systems of n linear differential equations.

Example 5.13 *The three dimensional system*

$$\begin{aligned} x' &= x - 2y + 2z + \frac{1}{3}\cos t - \frac{1}{3}\sin t \\ y' &= -x + 6y - 4z + 2\cos t \\ z' &= -x + 7y - 5z + 2\cos t - \frac{2}{3}\sin t \end{aligned} \tag{5.24a}$$

can be written in matrix form $\tilde{x}' = A(t)\,\tilde{x} + \tilde{q}(t)$ with

$$\tilde{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 6 & -4 \\ -1 & 7 & -5 \end{pmatrix}, \qquad q(t) = \begin{pmatrix} \frac{1}{3}\cos t - \frac{1}{3}\sin t \\ 2\cos t \\ 2\cos t - \frac{2}{3}\sin t \end{pmatrix}.$$

A fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} e^{t} & e^{-t} & 0 \\ e^{t} & -e^{-t} & e^{2t} \\ e^{t} & -2e^{-t} & e^{2t} \end{pmatrix}.$$

You will learn how to calculate this fundamental solution matrix in the next chapter. In the meantime, you can verify that it is a fundamental solution matrix by showing

$$\Phi'(t) = A(t)\,\Phi(t)$$

and

$$\det \Phi(0) = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix} = 1 \neq 0.$$

The general solution of the associated homogeneous system is

$$\begin{aligned} \tilde{x}_h(t) &= \Phi(t)\,\tilde{c} = \begin{pmatrix} e^{t} & e^{-t} & 0 \\ e^{t} & -e^{-t} & e^{2t} \\ e^{t} & -2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^{t} + c_2 e^{-t} \\ c_1 e^{t} - c_2 e^{-t} + c_3 e^{2t} \\ c_1 e^{t} - 2c_2 e^{-t} + c_3 e^{2t} \end{pmatrix}. \end{aligned}$$

To find the general solution of the nonhomogeneous system by using the Variation of Constants Formula (5.19), we calculate the inverse

$$\Phi^{-1}(t) = \begin{pmatrix} e^{-t} & -e^{-t} & e^{-t} \\ 0 & e^{t} & -e^{t} \\ -e^{-2t} & 3e^{-2t} & -2e^{-2t} \end{pmatrix}$$

and

$$\Phi^{-1}(t)\,\tilde{q}(t) = \begin{pmatrix} e^{-t} & -e^{-t} & e^{-t} \\ 0 & e^{t} & -e^{t} \\ -e^{-2t} & 3e^{-2t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} \frac{1}{3}\cos t - \frac{1}{3}\sin t \\ 2\cos t \\ 2\cos t - \frac{2}{3}\sin t \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^{-t}(\cos t - 3\sin t) \\ \frac{2}{3}e^{t}\sin t \\ \frac{5}{3}e^{-2t}(\cos t + \sin t) \end{pmatrix}$$

$$\int^{t} \Phi^{-1}(s)\tilde{q}(s)ds = \begin{pmatrix} \int^{t} \frac{1}{3}e^{-s}(\cos s - 3\sin s)\,ds \\ \int^{t} \frac{2}{3}e^{s}\sin s ds \\ \int^{t} \frac{5}{3}e^{-2s}(\cos s + \sin s)\,ds \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^{-t}(\cos t + 2\sin t) \\ \frac{1}{3}e^{t}(-\cos t + \sin t) \\ -\frac{1}{3}e^{-2t}(3\cos t + \sin t) \end{pmatrix}$$

$$\begin{aligned} \tilde{x}_p(t) &= \Phi(t)\int^{t} \Phi^{-1}(s)\tilde{q}(s)ds = \begin{pmatrix} e^{t} & e^{-t} & 0 \\ e^{t} & -e^{-t} & e^{2t} \\ e^{t} & -2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{1}{3}e^{-t}(\cos t + 2\sin t) \\ \frac{1}{3}e^{t}(-\cos t + \sin t) \\ -\frac{1}{3}e^{-2t}(3\cos t + \sin t) \end{pmatrix} \\ &= \begin{pmatrix} \sin t \\ -\frac{1}{3}\cos t \\ -\frac{1}{3}\sin t \end{pmatrix} \end{aligned}$$

The general solution is of (5.24) is

$$\begin{aligned} \tilde{x}(t) &= \tilde{x}_h(t) + \tilde{x}_p(t) = \begin{pmatrix} e^{t} & e^{-t} & 0 \\ e^{t} & -e^{-t} & e^{2t} \\ e^{t} & -2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} \sin t \\ -\frac{1}{3}\cos t \\ -\frac{1}{3}\sin t \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^{t} + c_2 e^{-t} + \sin t \\ c_1 e^{t} - c_2 e^{-t} + c_3 e^{2t} - \frac{1}{3}\cos t \\ c_1 e^{t} - 2c_2 e^{-t} + c_3 e^{2t} - \frac{1}{3}\sin t \end{pmatrix}. \end{aligned}$$

5.4 Chapter Summary

If the coefficient matrix $A(t)$ and nonhomogeneous term $\tilde{q}(t)$ are continuous on an interval $\alpha < t < \beta$, then the Extended Fundamental Existence and Uniqueness Theorem for a two dimensional linear systems $\tilde{x}' = A(t)\,\tilde{x} + \tilde{q}(t)$ guarantees the existence and uniqueness of a solution on the entire interval $\alpha < t < \beta$. The general solution has the additive decomposition $\tilde{x} = \tilde{x}_h + \tilde{x}_p$ where $\tilde{x}_h$ is the general solution of the associated homogeneous system $\tilde{x}' = A(t)\,\tilde{x}$ and $\tilde{x}_p$ is any particular solution of the nonhomogeneous system. The general solution $\tilde{x}_h$ of a two dimensional homogeneous linear system is a two dimensional linear vector space and, therefore, is the span of any two independent solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, a fact that can be written $\tilde{x}_h(t) = \Phi(t)\,\tilde{c}$ where $\Phi(t) = \text{col}(\ \tilde{x}_1(t)\ \ \tilde{x}_2(t)\ )$ is a fundamental solution matrix and $\tilde{c}$ is a vector of arbitrary constants. Two solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are independent if and only if $\Phi(t)$ is invertible on $\alpha < t < \beta$. A formula for the general solution of a nonhomogeneous linear system is given by the Variation of Constants Formula (5.19). Another version of the Variation of Constants Formula (5.21) gives a formula for the solution of any initial value problem $\tilde{x}(t_0) = \tilde{x}_0$.

5.5 Exercises

Which of the following systems are linear and which are nonlinear? If the system is linear, is it homogeneous or nonhomogeneous?

Exercise 5.1 $\begin{cases} x' = -3x + y \\ y' = x + 5y \end{cases}$

Exercise 5.2 $\begin{cases} x' = (\sin t)\, x + y \\ y' = x + (\cos t)\, y \end{cases}$

Exercise 5.3 $\begin{cases} x' = 2(1 - x) + 3y \\ y' = -x + y^2 \end{cases}$

Exercise 5.4 $\begin{cases} x' = x(1 - x) - xy \\ y' = -x + xy \end{cases}$

Exercise 5.5 $\begin{cases} x' = x - \sin t + y \\ y' = 2x - y \end{cases}$

Exercise 5.6 $\begin{cases} x' = -1 + y - 3x \\ y' = 1.5x - y + 4.2 \end{cases}$

Exercise 5.7 $\begin{cases} x' = rx - 14y \\ y' = x - sy \end{cases}$ *r, s are constants*

Exercise 5.8 $\begin{cases} x' = \pi y - 3.2(x - 2) + e^{-t} \\ y' = k(x + y) + t \end{cases}$ *k is a constant*

For the linear systems below find the coefficient matrix $A(t)$ and the nonhomogeneous term $\tilde{q}(t)$. Write the system in matrix form.

Exercise 5.9 $\begin{cases} x' = 3(x - y) + 2(x - y) \\ y' = -7 - x - y \end{cases}$

Exercise 5.10 $\begin{cases} x' = 2(0.5 - y) + x - 6 \\ y' = 3(2x - y) \end{cases}$

Exercise 5.11 $\begin{cases} x' = c(x + 3y) - 4x + t^2 \\ y' = -x + (d - 1)y - t \end{cases}$ *c, d are constants*

Exercise 5.12 $\begin{cases} x' = (k^2 - 1)\,(x - y) + \sin t \\ y' = x + ky - \cos t \end{cases}$ *k is a constant*

Consider the linear homogeneous systems with the coefficient matrices $A(t)$ and nonhomogeneous (forcing) terms $\tilde{q}\,(t)$ below. Use the Extended Existence and Uniqueness Theorem 5.1 to answer the following question: on what (maximal) interval do the solutions of the indicated initial value problems exist?

Exercise 5.13 $A(t) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad \tilde{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercise 5.14 $A(t) = \begin{pmatrix} -0.35 & \pi \\ 3 & \sqrt{17} \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \quad \tilde{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Exercise 5.15 $A(t) = \begin{pmatrix} (1-t^2)^{-1} & 1 \\ -1 & (1+t^2)^{-1} \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} (t-3)^{-1} \\ (t+3)^{-1} \end{pmatrix}, \quad \tilde{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Exercise 5.16 $A(t) = \begin{pmatrix} (1-t^2)^{-1} & 1 \\ -1 & (1+t^2)^{-1} \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} (t-3)^{-1} \\ (t+3)^{-1} \end{pmatrix}, \quad \tilde{x}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Exercise 5.17 $A(t) = \begin{pmatrix} \ln t & (t^2-2)^{-1} \\ \sin t & \cos \pi t \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} (t+1)^{-1} \\ t^{-2} \end{pmatrix}, \quad \tilde{x}(1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Exercise 5.18 $A(t) = \begin{pmatrix} \ln t & (t^2-2)^{-1} \\ \sin t & \cos \pi t \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} (t+1)^{-1} \\ t^{-2} \end{pmatrix}, \quad \tilde{x}(2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Exercise 5.19 *The general solutions $\tilde{x}$ and $\tilde{x}_h$ of (5.6) and (5.7) are sets of solutions. Define the set*

$$\tilde{x}_h + \tilde{x}_p = \left\{ \tilde{z}(t) + \tilde{x}_p(t) | \tilde{z} \in \tilde{x}_h \right\}.$$

Prove Theorem 5.2 by showing the sets $\tilde{x}$ and $\tilde{x}_h + \tilde{x}_p$ are identical using the following method from set theory: two sets A and B and identical if and only if $A \subseteq B$ and $B \subseteq A$.

Exercise 5.20 *Consider the linear homogeneous system with coefficient matrix*

$$A(t) = \begin{pmatrix} \frac{3}{2t} & -\frac{1}{2} \\ -\frac{1}{2t^2} & \frac{1}{2t} \end{pmatrix}$$

on the interval $t > 0$.

(a) Verify that

$$\tilde{x}_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad \textit{and} \quad \tilde{x}_2(t) = \begin{pmatrix} t^2 \\ -t \end{pmatrix}$$

are both solutions for $t > 0$.

(b) Prove $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are independent on the interval $t > 0$ and write down a fundamental solution matrix $\Phi(t)$. Obtain a formula for the general solution $\tilde{x}_h$.

(c) Find a formula for the solution of the initial value problem

$$\tilde{x}(1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Exercise 5.21 *Consider the linear homogeneous system with coefficient matrix*

$$A(t) = \begin{pmatrix} 0 & 2t \\ -2t & 0 \end{pmatrix}.$$

(a) Verify that

$$\tilde{x}_1(t) = \begin{pmatrix} \cos(t^2) \\ -\sin(t^2) \end{pmatrix} \quad \text{and} \quad \tilde{x}_2(t) = \begin{pmatrix} \sin(t^2) \\ \cos(t^2) \end{pmatrix}$$

are both solutions on the interval $-\infty < t < +\infty$.

(b) Prove $\tilde{x}_1(t)$ *and* $\tilde{x}_2(t)$ *are independent on the interval* $-\infty < t < +\infty$ *and write down a fundamental solution matrix* $\Phi(t)$. *Obtain a formula for the general solution* $\tilde{x}_h$.

(c) Find a formula for the solution of the initial value problem

$$\tilde{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Exercise 5.22 *(a) Verify that*

$$\tilde{x}_1(t) = \begin{pmatrix} e^{4t} \\ 2e^{4t} \end{pmatrix} \quad \text{and} \quad \tilde{x}_2(t) = \begin{pmatrix} -3e^{-3t} \\ e^{-3t} \end{pmatrix}$$

are both solutions of the linear homogeneous system with coefficient matrix

$$A(t) = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix}$$

on the interval $-\infty < t < +\infty$.

(b) Prove $\tilde{x}_1(t)$ *and* $\tilde{x}_2(t)$ *are independent on the interval* $-\infty < t < +\infty$ *and write down a fundamental solution matrix* $\Phi(t)$. *Obtain a formula for the general solution* $\tilde{x}_h$.

(c) Verify that

$$\tilde{x}(t) = \begin{pmatrix} 2e^{4t} + 3e^{-3t} \\ 4e^{4t} - e^{-3t} \end{pmatrix}$$

is also a solution of the system on the interval $-\infty < t < +\infty$.

(d) Write the solution in (c) as a linear combination of the independent solutions in (a).

Exercise 5.23 *(a) Verify that*

$$\tilde{x}_1(t) = \begin{pmatrix} 2e^{-t}\cos 3t \\ e^{-t}(\cos 3t - \sin 3t) \end{pmatrix} \quad \text{and} \quad \tilde{x}_2(t) = \begin{pmatrix} 2e^{-t}\sin 3t \\ e^{-t}(\cos 3t + \sin 3t) \end{pmatrix}$$

are both solutions of the linear homogeneous system with coefficient matrix

$$A(t) = \begin{pmatrix} -4 & 6 \\ -3 & 2 \end{pmatrix}$$

on the interval $-\infty < t < +\infty$.

(b) Prove $\tilde{x}_1(t)$ *and* $\tilde{x}_2(t)$ *are independent on on the interval* $-\infty < t < +\infty$ *and write down a fundamental solution matrix* $\Phi(t)$. *Obtain a formula for the general solution* $\tilde{x}_h$.

(c) Verify that

$$\tilde{x}(t) = \begin{pmatrix} -2e^{-t}(2\cos 3t + \sin 3t) \\ e^{-t}(-3\cos 3t + \sin 3t) \end{pmatrix}$$

is also a solution of the system on the interval $-\infty < t < +\infty$.

(d) Write the solution in (d) as a linear combination of the independent solutions in (a).

Exercise 5.24 *Given that*

$$\tilde{x}_1(t) = \begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix} \quad \text{and} \quad \tilde{x}_2(t) = \begin{pmatrix} e^{5t} \\ 2e^{5t} \end{pmatrix}$$

are both solutions of the linear homogeneous system with coefficient matrix

$$A(t) = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

use them to solve the following initial value problems.

(a) $\tilde{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ *(b)* $\tilde{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Exercise 5.25 *Given that*

$$\tilde{x}_1(t) = \begin{pmatrix} 7\cos\left(\sqrt{3}t\right) \\ 2\cos\left(\sqrt{3}t\right) - \sqrt{3}\sin\left(\sqrt{3}t\right) \end{pmatrix} \quad \text{and} \quad \tilde{x}_2(t) = \begin{pmatrix} 2\cos\left(\sqrt{3}t\right) + \sqrt{3}\sin\left(\sqrt{3}t\right) \\ \cos\left(\sqrt{3}t\right) \end{pmatrix}$$

are both solutions of the linear homogeneous system with coefficient matrix

$$A(t) = \begin{pmatrix} -2 & 7 \\ -1 & 2 \end{pmatrix}$$

use them to solve the following initial value problems.

(a) $\tilde{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *(b)* $\tilde{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Exercise 5.26 *Suppose* $\tilde{x}_1(t)$ *and* $\tilde{x}_2(t)$ *are dependent solutions of a linear homogeneous system* $\tilde{x}' = A(t)\,\tilde{x}$ *with a coefficient matrix* $A(t)$ *that is continuous on an interval* $\alpha < t < \beta$*. Prove that*

$$\det \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} = 0$$

on the interval $\alpha < t < \beta$*. It follows that if*

$$\det \begin{pmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{pmatrix} \neq 0$$

for some t_0 *in the interval, then the solution pairs are independent.*

Exercise 5.27 *Suppose* $\tilde{x}_1(t)$ *and* $\tilde{x}_2(t)$ *are solutions of a linear homogeneous system* $\tilde{x}' = A(t)\,\tilde{x}$ *with a coefficient matrix* $A(t)$ *that is continuous on an interval* $\alpha < t < \beta$*. Prove that the determinant*

$$z(t) = \det \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}$$

is either never equal to 0 *for* $\alpha < t < \beta$ *or else it is identically (i.e., always) equal to* 0 *on* $\alpha < t < \beta$*. (Hint: by direct calculation show the determinant satisfies the first order, linear homogeneous equation* $z' = p(t)z$ *where* $p(t) = a(t) + d(t)$ *and solve the initial value problem for* $z(t)$*.)*

Exercise 5.28 *Use the following two facts about matrix multiplication*

$$M(\tilde{x}+\tilde{y}) = M\tilde{x} + M\tilde{y}$$
$$M(k\tilde{x}) = k(M\tilde{x}) \text{ for any real number } k$$

to prove that a linear combination $k_1\tilde{x}_1 + k_2\tilde{x}_2 + \cdots + k_n\tilde{x}_n$ *of any number of solutions* $\tilde{x}_1$, $\tilde{x}_2$, ... , $\tilde{x}_n$ *of* $\tilde{x}' = A(t)\tilde{x}$ *is also a solution.*

Exercise 5.29 *Find solution formulas for the initial value problems below associated with the linear homogenous system with coefficient matrix*

$$A(t) = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}.$$

in Example 5.3.

(a) $\tilde{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *(b)* $\tilde{x}(0) = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}$

(c) $\tilde{x}(0) = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$ *(d)* $\tilde{x}(0) = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

Exercise 5.30 *Find solution formulas for the following initial value problems associated with the simple harmonic oscillator equation* $x'' + x = 0$.

(a) $x(0) = -1$, $x'(0) = 1$ *(b)* $x(0) = 2$, $x'(0) = -3$

(c) $x(\pi) = -1$, $x'(\pi) = 1$ *(d)* $x(\pi) = 2$, $x'(\pi) = -3$

Exercise 5.31 *(a) Use the Variation of Constants Formula to find a formula for the general solution of* $\tilde{x}' = A\tilde{x} + \tilde{q}(t)$ *with*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Use the Variation of Constants Formula (5.21) to find a formula for the solution of the initial value problem

$$\tilde{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exercise 5.32 *(a) Use the Variation of Constants Formula to find a formula for the general solution of* $\tilde{x}' = A\tilde{x} + \tilde{q}(t)$ *with*

$$A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} 2e^{t} \\ 2e^{t} \end{pmatrix}, \quad \Phi(t) = \begin{pmatrix} e^{-4t} & e^{-t} \\ -2e^{-4t} & e^{-t} \end{pmatrix}.$$

(b) Use the Variation of Constants Formula (5.21) to find a formula for the solution of the initial value problem

$$x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Exercise 5.33 *Consider the nonhomogeneous system*

$$x' = -3x + 4y + e^{-t}$$
$$y' = -6x + 7y + 1.$$

and the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}.$$

(a) Use the Variation of Constants Formula (5.19) to find the general solution.
(b) Solve the initial value problem $x(0) = 0$, $y(0) = 0$.
(c) Solve the initial value problem $x(0) = 2$, $y(0) = 1$.

Exercise 5.34 *Consider the nonhomogeneous system*

$$x' = -3x + 5y - 1$$
$$y' = -2x + 3y + 1$$

and the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 3\cos t - \sin t & \cos t + 3\sin t \end{pmatrix}.$$

(a) Use the Variation of Constants Formula (5.19) to find the general solution.
(b) Solve the initial value problem $x(0) = 0$, $y(0) = 0$.
(c) Solve the initial value problem $x(0) = -2$, $y(0) = 1$.

Exercise 5.35 *Consider the nonhomogeneous system $\tilde{x}' = A(t)\tilde{x} + \tilde{q}(t)$ for $t > 0$ when*

$$A(t) = \begin{pmatrix} \frac{3}{2t} & -\frac{1}{2} \\ -\frac{1}{2t^2} & \frac{1}{2t} \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}.$$

(a) Verify that

$$\Phi(t) = \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix}$$

is a fundamental solution matrix for the associated homogeneous system $\tilde{x}' = A(t)\tilde{x}$.
(b) Use the Variation of Constants Formula (5.19) to find the general solution.
(c) Use the Variation of Constants Formula (5.21) to solve the initial value problem

$$\tilde{x}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exercise 5.36 *Consider the second order equation $x'' + x = \tan t$.*

(a) Use the Variation of Constants formula (5.19) to find a formula for the general solution.

(b) Use the Variation of Constants formula (5.21) to find a formula for the solution of the initial value problem $x(0) = 0$, $x'(0) = 0$.

(c) Use the Variation of Constants formula (5.21) to find a formula for the solution of the initial value problem $x(0) = 1$, $x'(0) = 0$.

Exercise 5.37 *Use the given fundamental solution matrix $\Phi(t)$ and the Variation of Constants Formula (5.19) to find a formula for the general solution of the following systems*

(a)

$$\begin{aligned} x' &= -\tfrac{3}{2}x + y - \tfrac{1}{2} \\ y' &= -2x + \tfrac{3}{2}y + 1 \end{aligned} \quad \textit{and } \Phi(t) = \begin{pmatrix} e^{-t/2} & e^{t/2} \\ e^{-t/2} & 2e^{t/2} \end{pmatrix}$$

(b)

$$\begin{aligned} x' &= -\tfrac{3}{2}x + y + 2 \\ y' &= -2x + \tfrac{3}{2}y - \tfrac{1}{4} \end{aligned} \quad \textit{and } \Phi(t) = \begin{pmatrix} e^{-t/2} & e^{t/2} \\ e^{-t/2} & 2e^{t/2} \end{pmatrix}$$

(c)

$$\begin{aligned} x' &= 3x + 5y - 2 \\ y' &= -2x - 3y + 1 \end{aligned} \quad \textit{and } \Phi(t) = \begin{pmatrix} 3\cos t - \sin t & \cos t + 3\sin t \\ -2\cos t & -2\sin t \end{pmatrix}$$

(d)

$$\begin{aligned} x' &= 3x + 5y - 13e^{-5t} \\ y' &= -2x - 3y \end{aligned} \quad \textit{and } \Phi(t) = \begin{pmatrix} 3\cos t - \sin t & \cos t + 3\sin t \\ -2\cos t & -2\sin t \end{pmatrix}$$

(e)

$$\begin{aligned} x' &= 5x + 8y + r \\ y' &= -3x - 5y \end{aligned} \quad \textit{and } \Phi(t) = \begin{pmatrix} 4e^{-t} & 2e^{t} \\ -3e^{-t} & -e^{t} \end{pmatrix}$$

where r is a constant.

(f)

$$\begin{aligned} x' &= x' = 5x + 8y + rt \\ y' &= -3x - 5y + 2 \end{aligned} \quad \textit{and } \Phi(t) = \begin{pmatrix} 4e^{-t} & 2e^{t} \\ -3e^{-t} & -e^{t} \end{pmatrix}$$

where r is a constant.

Exercise 5.38 *Find a formula for the solution of the initial value problem $x(0) = 1$, $y(0) = -1$ for each system in Exercises 5.37.*

Exercise 5.39 *The strengths of two opposing armies are $x = x(t)$ and $y = y(t)$ (as measured, for example, by the number of troops or armaments). In a battle an army's strength is reduced. Assume the rate of reduction is proportional to the strength of the opposing army. During the battle reinforcements arrive at rates $h_1(t)$ and $h_2(t)$. Thus, x and y satisfy the nonhomogeneous linear system*

$$\begin{aligned} x' &= -ay + h_1(t) \\ y' &= -bx + h_2(t). \end{aligned}$$

Assume the armies are "evenly matched", by which we mean $a = b$. Let us assume $a = b = 1$. Finally, suppose reinforcements arrive at exponentially decreasing rates $h_1(t) = e^{-ct}$ and $h_2(t) = e^{-dt}$ where c and d are positive constants. Under these conditions we have the system

$$\begin{aligned} x' &= -y + e^{-ct} \\ y' &= -x + e^{-dt} \\ x(0) &= x_0 > 0, \qquad y(0) = y_0. \end{aligned}$$

The constants c and d measure how fast the reinforcement rates decrease.

(a) Use a computer to study the solutions of the initial value problem when both armies are reinforced at the same (decreasing) rate, i.e., $c = d$. Organize your explorations in the following way. Choose a value for $c = d$ and have a computer draw graphs of x and y for a selection of initial conditions $x_0 \neq y_0$. Repeat this for several choices of $c = d$. Draw conclusions about the winner of the battle. (Note: an army loses if its strength equals 0 at some time t.)

(b) Given the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{pmatrix}$$

of the associated homogeneous system, find a formula for the solution of the initial value problem when $c = d$. (Note: be careful. You should have two different formulas, one for $c \neq 1$ and one for $c = 1$.)

(c) Use your answer in (b) to verify (or disprove) your answers in (a).

Exercise 5.40 *Suppose the rates at which two distinct groups move into and out of a city are proportional to the numbers $x = x(t)$ and $y = y(t)$ present. Specifically, members of group x are attracted to each other and so their numbers increase at a rate proportional to their numbers. However, members of group x do not like members of group y and they leave the city at a rate proportional to the number of y group members present. The fundamental in-flow/out-flow (compartmental) rule implies $x' = ax - by$ where a and b are positive constants. Assume group y feels the same way about its own members and those of group x so that $y' = -cx + dy$ for positive constants c and d. Suppose $a = d$ (i.e., both populations grow at the same exponential rate in the absence of the other). In fact, take $a = d = 1$ to arrive at the initial value problem*

$$\begin{aligned} x' &= x - by \\ y' &= -cx + y \\ x(0) = x_0 > 0, &\qquad y(0) = y_0 > 0. \end{aligned}$$

(a) Use a computer to study solutions of the initial value problem. Do this for $b = c = 1$. In the long run can both groups live together in the city? Under what conditions is a group eventually gone from the city (i.e., is the city totally segregated)? Repeat for $b = 4$, $c = 1$.

(b) Given the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \sqrt{bc}e^{\lambda_1 t} & \sqrt{bc}e^{\lambda_2 t} \\ -ce^{\lambda_1 t} & ce^{\lambda_2 t} \end{pmatrix}$$
$$\lambda_1 = 1 + \sqrt{bc}, \quad \lambda_2 = 1 - \sqrt{bc}$$

for the associated homogeneous system, find a formula for the solution of the initial value problem

(c) Use your answer in (b) to verify your answers in (a).

Exercise 5.41 *The system*

$$\begin{aligned} x' &= x - by \\ y' &= -cx + y \\ x(0) = x_0 > 0, &\qquad y(0) = y_0 > 0. \end{aligned}$$

models the numbers of two groups moving into and out of a city. Here b and c are positive constants. (See Exercise 5.40). Suppose initially there are no members of population x in the city. Suppose individuals of group x are added to (immigrate into) the city at a constant rate $r > 0$. Then we have the initial value problem

$$\begin{aligned} x' &= x - by + r \\ y' &= -cx + y \\ x(0) = 0, &\qquad y(0) = y_0 > 0. \end{aligned}$$

(a) Use a computer to study solutions of this initial value problem. Organize your exploration as follows. Choose and fix an initial condition $y_0 > 0$ and graph x and y for an increasing sequence of immigration rates r. Repeat this for several choices of $y_0 > 0$. What do you conclude about the long term group composition of the city?

(b) Find a formula for the solution of this initial value problem.

(c) Use your answer in (b) to verify your answers in (a).

Chapter 6

Autonomous Linear Homogeneous Systems

We learned in Chapter 5 that a formula for the solutions of general linear systems is available (namely, the Variation of Constants Formula) provided one can calculate a fundamental solution matrix $\Phi(t)$ for the associated homogeneous system. Unfortunately, in general there is no method that will allow us to calculate a fundamental solution matrix for a linear homogeneous system. (An exception is the case of a single linear equation, i.e., a system of dimension one, as we saw in Chapter 2).

There is an important special case, however, when methods do exist for calculating a fundamental solution matrix of a homogeneous system. This is the special case of *autonomous* homogeneous linear systems , that is, for homogeneous linear systems that have a constant coefficient matrix A.

We have two main goals in this chapter. The first is to learn how to calculate a fundamental solution matrix for an autonomous linear homogeneous system

$$\tilde{x}' = A\tilde{x}.$$

We will focus on the two dimensional case (in which case A is a 2×2 constant matrix), i.e. on systems of two linear equations in two unknowns. (We will point out, at appropriate times, when the methods we learn are applicable to systems of any dimension.) Don't forget that whenever we learn something about systems of first order differential equations we automatically learn something about higher order equations. We will take a close look at second order equations in Chapter 7.

Our second main goal in this chapter is to study the phase plane portraits of two dimensional autonomous linear systems.

6.1 Review of Eigenvalues

Recall from linear algebra that an *eigenvalue* of an $n \times n$ matrix A is a (real or complex) number λ such that $A\tilde{v} = \lambda\tilde{v}$, or equivalently

$$(A - \lambda I)\,\tilde{v} = \tilde{0}, \tag{6.1}$$

for some vector $\tilde{v} \neq \tilde{0}$, which is called an *eigenvector* associated with λ (I is then $n \times n$ identity matrix). We'll refer to λ and $\tilde{v}$ as an *eigen-pair*.

In order that the linear algebraic equation (6.1) to have a nontrivial solution $\tilde{v} \neq \tilde{0}$, it is necessary that the matrix $A - \lambda I$ be singular (non-invertible), i.e., that

$$\det(A - \lambda I) = 0. \tag{6.2}$$

The determinant $\det(A - \lambda I)$ is an n-degree polynomial in λ, called the *characteristic polynomial* of A.

For 2×2 matrices A the characteristic polynomial is quadratic and therefore the eigenvalues can be calculated by the quadratic formula. Specifically, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the *characteristic equation* (6.2) is

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

An easy way to find the characteristic equation is to notice that $a + d$ is the trace of A and $ad - bc$ is the determinant of A :

$$a + d = \operatorname{tr}(A), \qquad ad - bc = \det A$$

Then the characteristic equation can be written as

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det A = 0.$$

It follows that the eigenvalues of A are

$$\lambda_{\pm} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det A}\right).$$

or in terms of the entries in A

$$\lambda_{\pm} = \frac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right).$$

Once an eigenvalue λ has been calculated, we find an associated eigenvector by solving the linear algebraic system (6.1) for $\tilde{v} \neq \tilde{0}$, which is possible because λ is an eigenvalue (which makes the coefficient matrix $A - \lambda I$ singular). For the 2×2 matrix case the linear algebraic system (6.1) consists of two (linear) equations in two unknowns, namely, the components of the vector

$$\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Specifically, we must solve the system

$$\begin{aligned} (a - \lambda)v_1 + bv_2 &= 0 \\ cv_1 + (d - \lambda)v_2 &= 0 \end{aligned}$$

for v_1 and v_2 not both equal to 0. As a practical matter, note that one only needs to solve one of the equations. (The equations are made dependent by choosing λ to be an eigenvalue.) Thus, one can for example solve the first equation by choosing v_2 (or v_1) to be any convenient number and let the equation determine v_1 (or v_2). Or one can do a similar thing with the second equation.

Recall that there is not a unique eigenvector associated with an eigenvalue λ. For example, any nonzero constant multiple of an eigenvector is also an eigenvector.

For matrices of higher dimension than $n = 2$ the procedure for calculating eigen-pairs is the same. The main difficulty for higher dimensional matrices is that the characteristic polynomial becomes a high degree polynomial and find its roots becomes more difficult.

6.2 The Putzer Algorithm

The Putzer Algorithm is a method for calculation a fundamental solution matrix for an autonomous linear homogeneous system

$$\tilde{x}' = A\tilde{x}.$$

of an dimension n. Here the coefficient matrix is an $n \times n$ matrix. Although our focus in this course is on the two dimensional case $n = 2$, the Putzer Algorithm for n dimensional systems is just as easy to state.

Recall that there is not a unique fundamental matrix associated with a linear system. The columns of a fundamental solution matrix form a basis for the linear vector space of solutions and, as you know from linear algebra, there is not a unique basis for a vector space.

Theoretically, by means of the Fundamental Existence and Uniqueness Theorem 5.1, one can calculate a fundamental solution matrix $\Phi(t)$ by using for its columns the unique solutions of the initial value problem

$$\begin{aligned} \tilde{x}' &= A\tilde{x} \\ \tilde{x}(t_0) &= \tilde{e}_i \end{aligned}$$

where

$$\tilde{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad , \quad \tilde{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

is the canonical basis of the n-dimensional Euclidean vector space. This is exactly what we did in Section 5.2.2 Chapter 5 for the $n = 2$ dimensional case when we established the general existence of independent solutions. The resulting fundamental solution matrix satisfies

$$\Phi(t_0) = I$$

where

$$I = \operatorname{col}\begin{pmatrix} \tilde{e}_1 & \tilde{e}_2 & \cdots & \tilde{e}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is the $n \times n$ identity matrix.

Definition 6.1 *A fundamental solution matrix that satisfies $\Phi(t_0) = I$ is called a* ***normalized fundamental solution matrix*** *(normalized at t_0).*

One advantage of working for a fundamental solution matrix normalized at $t = t_0$ is that the formula for solutions of initial value problems at $t = t_0$

$$\tilde{x}(t) = \Phi(t)\,\Phi^{-1}(t_0)\,\tilde{x}_0$$

is simplifies to

$$\tilde{x}(t) = \Phi(t)\,\tilde{x}_0.$$

This simplification also occurs in the Variation of Constants Formula for the solution of nonhomogeneous systems.

The Putzer Algorithm for autonomous linear homogenous system calculates the normalized fundamental solution matrix under the assumption that one can calculate the eigenvalues of the coefficient matrix A.

Theorem 6.1 ***(Putzer Algorithm) Let*** $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots\ ,\ \lambda_n$ ***be the eigenvalues of the*** $n \times n$ ***coefficient matrix*** A ***with multiplicities included. The fundamental solution matrix of the autonomous linear homogeneous system*** $\tilde{x}' = A\tilde{x}$ ***normalized at*** $t_0 = 0$ ***is***

$$\Phi(t) = r_1(t)P_1 + r_2(t)P_2 + \cdots + r_n(t)P_n \tag{6.3}$$

where the $n \times n$ ***matrices*** P_i ***are calculated sequentially as follows:***

$$\begin{aligned} P_1 &= I \\ P_2 &= A - \lambda_1 I \\ P_3 &= (A - \lambda_2 I)\,P_2 \\ &\vdots \\ P_n &= (A - \lambda_{n-1} I)\,P_{n-1}, \end{aligned}$$

where $r_1(t) = e^{\lambda_1 t}$ ***and where the remaining*** $r_i(t)$ ***are the (unique) solutions of the initial value problems***

$$\begin{aligned} r_2' &= \lambda_2 r_2 + r_1(t), \qquad & r_2(0) &= 0 \\ r_3' &= \lambda_3 r_3 + r_2(t), \qquad & r_3(0) &= 0 \\ &\vdots \\ r_n' &= \lambda_n r_n + r_{n-1}(t), \qquad & r_n(0) &= 0. \end{aligned}$$

Remark 1. The initial value problems for $r_i(t)$ for $i = 2, 3, \cdots, n$ involve first order, linear nonhomogeneous differential equations, which we studied in Chapter 2. Their solutions can be found by either the Variation of Constants Formula

$$r_i(t) = e^{\lambda_i t} \int_0^t e^{-\lambda_i s} r_{i-1}(s)\,ds$$

or, since the coefficient λ_i is a constant, by the Method of Undetermined Coefficients.

Remark 2. If an eigenvalue λ_i is complex, then the procedure is to carry out the solution of the initial value problem for $r_i(t)$ as usual, accepting the fact that the answer will be a complex valued function. Also, the matrix P_{i+1} will have complex entries. Nonetheless, the fundamental solution matrix given by the formula (6.3) will, in the end, have no complex entries in it. To carry out this calculation you will need to recall how to do complex arithmetic and use the formulas

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta. \tag{6.4}$$

As an example, if we apply the algorithm in Theorem 6.1 to the $n = 2$ dimensional case, then the normalized fundamental solution matrix is

$$\Phi(t) = e^{\lambda_1 t} I + r_2(t)(A - \lambda_1 I)$$

where $r_2(t)$ is the unique solution of the initial value problem

$$r_2' = \lambda_2 r_2 + e^{\lambda_1 t}, \qquad r_2(0) = 0.$$

By the Variation of Constants Formula (or the Method of Undetermined Coefficients) we obtain

$$r_2(t) = e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds = \begin{cases} \frac{1}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) & \text{if } \lambda_1 \neq \lambda_2 \\ te^{\lambda t} & \text{if } \lambda_1 = \lambda_2 = \lambda \end{cases}$$

and, as a result,

$$\Phi(t) = e^{\lambda_1 t} I + r_2(t)(A - \lambda_1 I) = \begin{cases} e^{\lambda_1 t} I + \frac{1}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)(A - \lambda_1 I) & \text{if } \lambda_1 \neq \lambda_2 \\ e^{\lambda t} I + te^{\lambda t}(A - \lambda I) & \text{if } \lambda_1 = \lambda_2 = \lambda. \end{cases}$$

In terms of the entries in the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the Putzer Algorithm yields the fundamental solution matrix

$$\Phi(t) = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} (a-\lambda_2)e^{\lambda_1 t} - (a-\lambda_1)e^{\lambda_2 t} & be^{\lambda_1 t} - be^{\lambda_2 t} \\ ce^{\lambda_1 t} - ce^{\lambda_2 t} & (d-\lambda_2)e^{\lambda_1 t} - (d-\lambda_1)e^{\lambda_2 t} \end{pmatrix} \quad \text{if} \quad \lambda_1 \neq \lambda_2 \tag{6.5}$$

$$\Phi(t) = e^{\lambda t}\begin{pmatrix} 1 + (a-\lambda)t & bt \\ ct & 1 + (d-\lambda)t \end{pmatrix} \quad \text{if} \quad \lambda_1 = \lambda_2 = \lambda. \tag{6.6}$$

As a matter of practice, when calculating the fundamental solution matrix for a specific $n = 2$ dimension homogeneous system, one can either use these formulas or step through the algorithm given in Theorem 6.1.

Remark 3. A matrix $\Phi(t)$ is a solution matrix of a first order homogeneous system

$$\tilde{x}' = A\tilde{x}.$$

if all of its its columns are solutions of the system, that is to say if

$$\Phi'(t) = A\Phi(t)$$

(recall Definition (5.2) in Chapter 5). The matrix is a fundamental solution matrix normalized at t_0 if it satisfies the (matrix) initial value problem

$$\Phi'(t) = A\Phi(t), \quad \Phi(t_0) = I.$$

The Putzer Algorithm gives a formula for the solution of the initial value problem

$$\Phi'(t) = A\Phi(t), \quad \Phi(0) = I. \tag{6.7}$$

Example 6.1 *(Distinct real eigenvalues) The linear homogeneous system*

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned}$$

has the constant coefficient matrix

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = -1 \quad \textit{and} \quad \lambda_2 = -6.$$

By Theorem 6.1, the fundamental solution matrix normalized at $t_0 = 0$ is

$$\Phi(t) = e^{-t}I + r_2(t)(A + I)$$

where $r_2(t)$ is the unique solution of the initial value problem

$$r_2' = -6r_2 + e^{-t}, \qquad r_2(0) = 0,$$

namely

$$\begin{aligned} r_2(t) &= e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds = e^{-6t}\int_0^t e^{5s}ds \\ &= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-6t}. \end{aligned}$$

Thus,

$$\begin{aligned} \Phi(t) &= e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{1}{5}e^{-t} - \frac{1}{5}e^{-6t}\right)\left(\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \frac{1}{5}\begin{pmatrix} 4e^{-t} + e^{-6t} & 2e^{-t} - 2e^{-6t} \\ 2e^{-t} - 2e^{-6t} & e^{-t} + 4e^{-6t} \end{pmatrix}. \end{aligned}$$

(The student should check this answer by showing that it satisfies the initial value problem (6.7)). Alternatively, one cab arrive at the same answer by using formula (6.5).

A formula for the solution of an initial value problem

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4e^{-t}+e^{-6t} & 2e^{-t}-2e^{-6t} \\ 2e^{-t}-2e^{-6t} & e^{-t}+4e^{-6t} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Example 6.2 *(Complex eigenvalues) The linear homogeneous system (the system equivalent of the second order equation $x''+x=0$)*

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

has the constant coefficient matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = i \quad \textit{and} \quad \lambda_2 = -i.$$

By Theorem 6.1, the fundamental solution matrix normalized at $t_0 = 0$ is

$$\Phi(t) = e^{it}I + r_2(t)(A - iI)$$

where $r_2(t)$ is the unique solution of the initial value problem

$$r_2' = -ir_2 + e^{it}, \qquad r_2(0) = 0,$$

namely

$$\begin{aligned} r_2(t) &= e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds = e^{-it}\int_0^t e^{2is}ds \\ &= e^{-it}\left(\frac{1}{2i}e^{2is}\right)\Big|_{s=0}^{s=t} = \frac{1}{2i}e^{-it}\left(e^{2it}-1\right) \\ &= \frac{1}{2i}\left(e^{it}-e^{-it}\right). \end{aligned}$$

Recalling the trigonometric identities (6.4) we obtain

$$r_2(t) = \sin t$$

and the fundamental solution matrix

$$\begin{aligned} \Phi(t) &= (\cos t + i\sin t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\sin t)\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

The student should check this answer by showing that it satisfies the initial value problem (6.7). (Alternatively, one can arrive at the answer by using formula (6.5)).

A formula for the solution of an initial value problem

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Example 6.3 *(Double eigenvalue) The linear homogeneous system (the equivalent system to the second order equation $x'' + x = 0$)*

$$\begin{aligned} x' &= x + y \\ y' &= -4x + 5y \end{aligned}$$

has the constant coefficient matrix

$$A = \begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix}.$$

which has the double eigenvalue $\lambda = 3$.

By Theorem 6.1, the fundamental solution matrix normalized at $t_0 = 0$ is

$$\Phi(t) = e^{3t} I + r_2(t)(A - 3I)$$

where $r_2(t)$ is the unique solution of the initial value problem

$$r_2' = 3r_2 + e^{3t}, \qquad r_2(0) = 0,$$

namely

$$\begin{aligned} r_2(t) &= e^{\lambda_2 t} \int_0^t e^{(\lambda_1 - \lambda_2)s} ds = e^{3t} \int_0^t ds \\ &= te^{3t}. \end{aligned}$$

Thus,

$$\begin{aligned} \Phi(t) &= e^{3t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + te^{3t} \left(\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} (1-2t)e^{3t} & te^{3t} \\ -4te^{3t} & (1+2t)e^{3t} \end{pmatrix}. \end{aligned}$$

(The student should check this answer by showing that it satisfies the initial value problem (6.7).) Alternatively, one can arrive at the answer by using formula (6.5).

A formula for the solution of an initial value problem

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (1-2t)e^{3t} & te^{3t} \\ -4te^{3t} & (1+2t)e^{3t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Example 6.4 *The $n = 3$ dimensional linear homogeneous system*

$$\begin{aligned} x' &= -x - 2y + 2z \\ y' &= 3x + 5y - 3z \\ z' &= 3x + 4y - 2z \end{aligned}$$

has the constant coefficient matrix

$$A = \begin{pmatrix} -1 & -2 & 2 \\ 3 & 5 & -3 \\ 3 & 4 & -2 \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

By Theorem 6.1 the normalized fundamental solution matrix at $t_0 = 0$ is

$$\Phi(t) = r_1(t)P_1 + r_2(t)P_2 + r_3(t)P_3$$

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} -1 & -2 & 2 \\ 3 & 5 & -3 \\ 3 & 4 & -2 \end{pmatrix} - (-1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ 3 & 6 & -3 \\ 3 & 4 & -1 \end{pmatrix}$$

$$P_3 = \left(\begin{pmatrix} -1 & -2 & 2 \\ 3 & 5 & -3 \\ 3 & 4 & -2 \end{pmatrix} - (1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 0 & -2 & 2 \\ 3 & 6 & -3 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 6 & -3 \\ 3 & 6 & -3 \end{pmatrix}$$

and

$$r_1(t) = e^{-t}.$$

The initial value problems for $r_2(t)$ and $r_3(t)$, and their solutions, are

$$\left.\begin{array}{l} r_2' = 1 \cdot r_2 + e^{-t} \\ r_2(0) = 0 \end{array}\right\} \Longrightarrow r_2(t) = \frac{1}{2}\left(e^t - e^{-t}\right)$$

$$\left.\begin{array}{l} r_3' = 2r_3 + \left(\frac{1}{2}e^t - \frac{1}{2}e^{-t}\right) \\ r_3(0) = 0 \end{array}\right\} \Longrightarrow r_3(t) = \frac{1}{3}e^{2t} - \frac{1}{2}e^t + \frac{1}{6}e^{-t}.$$

Thus

$$\Phi(t) = e^{-t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\left(e^t - e^{-t}\right) \begin{pmatrix} 0 & -2 & 2 \\ 3 & 6 & -3 \\ 3 & 4 & -1 \end{pmatrix} + \left(\frac{1}{3}e^{2t} - \frac{1}{2}e^t + \frac{1}{6}e^{-t}\right) \begin{pmatrix} 0 & 0 & 0 \\ 3 & 6 & -3 \\ 3 & 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-t} - e^t & e^t - e^{-t} \\ e^{2t} - e^{-t} & 2e^{2t} - e^{-t} & e^{-t} - e^{2t} \\ e^{2t} - e^{-t} & 2e^{2t} - e^{-t} - e^t & e^t + e^{-t} - e^{2t} \end{pmatrix}.$$

The student should check this answer by showing that it satisfies the initial value problem (6.7).

The solution of initial value problem

$$\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{-t}-e^{t} & e^{t}-e^{-t} \\ e^{2t}-e^{-t} & 2e^{2t}-e^{-t} & e^{-t}-e^{2t} \\ e^{2t}-e^{-t} & 2e^{2t}-e^{-t}-e^{t} & e^{t}+e^{-t}-e^{2t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

Example 6.5 *The system linear homogeneous*

$$\begin{aligned} x' &= -\alpha x - \beta y \\ y' &= \gamma x - \delta y \end{aligned} \tag{6.8}$$

is a model of the glucose/insulin regulation system in the bloodstream. In these equations x and y are, the difference in the concentrations of glucose and insulin from their equilibrium levels respectively. Thus, a negative value of x (or y) is a deficiency in the glucose (or insulin) and a positive value is an excess of glucose (or insulin) in the bloodstream. The rate constant $\alpha > 0$ is related to the efficiency that the liver absorbs glucose, $\beta > 0$ to the rate at which glucose is absorbed by muscle, $\gamma > 0$ to the rate that insulin is produced by the pancreas and $\delta > 0$ the rate at which insulin is degraded by the liver.

Some typical values of the rate constants (per hour) are[1]

$$\alpha = 2.92, \quad \beta = 4.34, \quad \gamma = 0.208, \quad \delta = 0.780. \tag{6.9}$$

The eigenvalues of the coefficient matrix

$$A = \begin{pmatrix} -2.92 & -4.34 \\ 0.208 & -0.780 \end{pmatrix}$$

are (to three significant digits)

$$\lambda_1 = -2.34, \quad \lambda_2 = -1.36.$$

Using these in the formula (6.5) yields the normalized fundamental solution matrix (to four decimals digits)

$$\Phi(t) = \begin{pmatrix} 1.5918e^{-2.34t} - 0.5918e^{-1.36t} & 4.4286e^{-2.34t} - 4.4286e^{-1.36t} \\ 0.2122e^{-1.36t} - 0.2122e^{-2.34t} & 1.5918e^{-1.36t} - 0.5918e^{-2.34t} \end{pmatrix} \tag{6.10}$$

A solution formula for an initial value problem

$$\tilde{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

[1] V. W. Bolie, Journal of Applied Physiology 16 (1960), p. 783.

is

$$\Phi(t)\,\tilde{x}(0) = \begin{pmatrix} 1.5918e^{-2.34t} - 0.5918e^{-1.36t} & 4.4286e^{-2.34t} - 4.4286e^{-1.36t} \\ 0.2122e^{-1.36t} - 0.2122e^{-2.34t} & 1.5918e^{-1.36t} - 0.5918e^{-2.34t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

As an application of this solution formula, consider that a dose $x_0 > 0$ of glucose is introduced into the bloodstream at $t = 0$, prior to which the system is at equilibrium levels, so that $y(0) = 0$. To find the effect of this disturbance from equilibrium investigate the solution formula

$$\begin{aligned} \Phi(t)\,\tilde{x}(0) &= \begin{pmatrix} 1.5918e^{-2.34t} - 0.5918e^{-1.36t} & 4.4286e^{-2.34t} - 4.4286e^{-1.36t} \\ 0.2122e^{-1.36t} - 0.2122e^{-2.34t} & 1.5918e^{-1.36t} - 0.5918e^{-2.34t} \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \\ &= x_0 \begin{pmatrix} 1.5918e^{-2.34t} - 0.5918e^{-1.36t} \\ -0.2122e^{-2.34t} + 0.2122e^{-1.36t} \end{pmatrix}. \end{aligned}$$

or in terms of the component state variables of the glucose/insulin regulation model

$$\begin{aligned} x(t) &= x_0\left(1.5918e^{-2.34t} - 0.5918e^{-1.36t}\right) \\ y(t) &= x_0\left(-0.2122e^{-2.34t} + 0.2122e^{-1.36t}\right). \end{aligned} \tag{6.11}$$

Notice

$$\lim_{t\to+\infty} x(t) = 0 \quad \textit{and} \quad \lim_{t\to+\infty} y(t) = 0$$

which means that this model (with the estimated parameters) predicts that the glucose and insulin levels in the bloodstream will, in the long run, return to their equilibrium values after the introduction of glucose dose into the bloodstream. Figure 6.1 shows plots of $x(t)$ and $y(t)$ for initial condition $x_0 = 1$. Notice that $x(t)$ decreases to a negative minimum before increasing to 0, i.e., the glucose concentration in the blood stream drops below and then increases up to the equilibrium level. Also notice y increases to a maximum and then decreases to 0, i.e., after the glucose dose is administered the insulin concentration in the blood stream increases above, and then returns to, the equilibrium level.

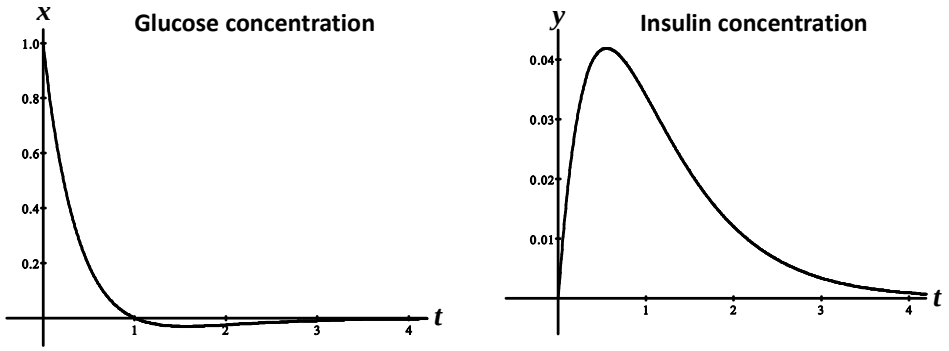


Figure 6.1. The graphs of the components (6.11) of the solution of the glucose/insulin model (6.8) with parameters (6.9) and an initial dose $x_0 = 1$ of glucose (with insulin at equilibrium $y_0 = 0$).

Remark 4. In Example 6.5 both components of the solution (6.11) investigated tend to 0 as $t \to +\infty$. The reason for this is that both exponentials appearing in the fundamental solution matrix (6.10), and hence in the solution components $x(t)$ and $y(t)$, tend to 0. Those exponentials tend to 0 because both eigenvalues of the coefficient matrix are negative. Since this is true of every entry in the fundamental solution matrix, all initial value problems have solutions that tend to 0 as $t \to +\infty$. That this occurs because of negative eigenvalues is a point worth remembering, for it will return later when we study phase plane portraits and stability theory (Section 6.4).

6.3 Eigenvectors and Fundamental Solution Matrices

Another method for calculating fundamental solution matrices is based on the eigenvectors associated with the eigenvalues of the coefficient matrix A. The connection between solutions and the eigenvectors of A is often important. This will be the case in the following Section 6.4. Therefore, we will briefly consider this method in this section.

Suppose we look for a nontrivial solution of a linear homogeneous system

$$\tilde{x}' = A\tilde{x}$$

of the form

$$\tilde{x}(t) = \tilde{v}e^{\lambda t}$$

by using the Method of Undetermined Coefficients. Then, after substituting this guess into the differential system and cancelling $e^{\lambda t}$ from both sides, we discover that this exponential is a nontrivial solution if and only if $\tilde{v}$ and λ satisfy the equation

$$(A - \lambda I)\,\tilde{v} = \tilde{0}, \quad \tilde{v} \neq \tilde{0}.$$

Thus, $\tilde{v}e^{\lambda t}$ *is a nontrivial solution if and only if* $\tilde{v}$ *and* λ *is an eigen-pair of the coefficient matrix* A.

If we can find n independent solutions of this form, then we can use them as columns in a fundamental solution matrix. In the $n = 2$ dimensional case, we need two independent

solutions of the form $\tilde{v}e^{\lambda t}$ in order to build a fundamental solution matrix. If it turns out that A has two different eigenvalues, then this will be possible. The reason is that from linear algebra we know that eigenvectors corresponding to different eigenvalues are necessarily independent.

Suppose the eigenvalues $\lambda_1 \neq \lambda_2$ have eigenvectors $\tilde{v}$ and $\tilde{w}$ respectively. The solution matrix

$$\Phi(t) = \begin{pmatrix} \tilde{v}e^{\lambda_1 t} & \tilde{w}e^{\lambda_2 t} \end{pmatrix} \tag{6.12}$$

satisfies

$$\det \Phi(0) = \det = \begin{pmatrix} \tilde{v} & \tilde{w} \end{pmatrix} \neq 0$$

(because, as pointed out, $\tilde{v}$ and $\tilde{w}$ are necessarily independent vectors). By Theorem 5.4 in Chapter 5 the solution matrix (6.12) is a fundamental solution matrix.

Example 6.6 *The linear homogeneous system*

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned}$$

in Example 6.1 has the constant coefficient matrix

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Eigenvectors associated with these eigenvalues are respectively

$$\tilde{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad \tilde{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

These give rise to the independent solutions

$$\begin{aligned} \tilde{x}_1(t) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} \\ \tilde{x}_2(t) &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t} = \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix} \end{aligned}$$

and the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}.$$

This fundamental solution matrix is not normalized at $t_0 = 0$ and therefore is not the same fundamental solution matrix we calculated in Example 6.1 by means of the Putzer Algorithm.

Indeed, there are infinitely many other fundamental solution matrices that can be calculated by selecting other eigenvectors associated with the eigenvalues $\lambda_1 = -1$ and/or $\lambda_2 = -6$.

We can calculate the fundamental solution matrix normalized at $t_0 = 0$ as follows:

$$\begin{aligned}\Phi(t)\Phi^{-1}(0) &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{5}e^{-t} + \frac{1}{5}e^{-6t} & \frac{2}{5}e^{-t} - \frac{2}{5}e^{-6t} \\ \frac{2}{5}e^{-t} - \frac{2}{5}e^{-6t} & \frac{1}{5}e^{-t} + \frac{4}{5}e^{-6t} \end{pmatrix}\end{aligned}$$

which is the same as found in Example 6.1 by means of the Putzer Algorithm.

The fundamental solution matrix normalized at an arbitrary t_0 is

$$\begin{aligned}\Phi(t)\Phi^{-1}(t_0) &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\begin{pmatrix} 2e^{-t_0} & e^{-6t_0} \\ e^{-t_0} & -2e^{-6t_0} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}\frac{1}{5}\begin{pmatrix} 2e^{t_0} & e^{t_0} \\ e^{6t_0} & -2e^{6t_0} \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} e^{-6(t-t_0)} + 4e^{-(t-t_0)} & 2e^{-(t-t_0)} - 2e^{-6(t-t_0)} \\ 2e^{-(t-t_0)} - 2e^{-6(t-t_0)} & 4e^{-6(t-t_0)} + e^{-(t-t_0)} \end{pmatrix}.\end{aligned}$$

This is suitable for finding solution formulas for initial value problems

$$\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

posed at any t_0 :

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} e^{-6(t-t_0)} + 4e^{-(t-t_0)} & 2e^{-(t-t_0)} - 2e^{-6(t-t_0)} \\ 2e^{-(t-t_0)} - 2e^{-6(t-t_0)} & 4e^{-6(t-t_0)} + e^{-(t-t_0)} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

If A has a complex eigenvalue, then we can still construct the (complex values) solution $\tilde{v}e^{\lambda t}$ using an associated eigenvector $\tilde{v}$ (which is now also complex). To obtain real valued solutions we can make use of the following fact:

> If λ is a complex eigenvalue and $\tilde{v}$ is an associated eigenvector then the real and imaginary parts of $\tilde{v}e^{\lambda t}$
>
> $$\tilde{x}_1(t) = \operatorname{Re}\tilde{v}e^{\lambda t} \quad \text{and} \quad \tilde{x}_1(t) = \operatorname{Im}\tilde{v}e^{\lambda t}$$
>
> are two independent real solutions:

More explicitly, we can write down formulas for the two independent real solutions obtain by extracting the real and imaginary parts from a complex valued solution from the complex forms of the eigen-pair

$$\lambda = \alpha + i\beta, \quad \tilde{v} = \tilde{u} + i\tilde{w}$$

where α and β are respectively the real and imaginary parts of λ and $\tilde{u}$ and $\tilde{w}$ are respectively the real and imaginary parts of $\tilde{v}$. Then, noting that

$$e^{\lambda t} = e^{(\alpha+i\beta)t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t$$

we have

$$\begin{aligned}\tilde{v}e^{\lambda t} &= (\tilde{u}+i\tilde{w})\left(e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t\right)\\ &= \left(\tilde{u}e^{\alpha t}\cos\beta t - \tilde{w}e^{\alpha t}\sin\beta t\right) + i\left(\tilde{u}e^{\alpha t}\sin\beta t + \tilde{w}e^{\alpha t}\cos\beta t\right)\end{aligned}$$

and

$$\tilde{x}_1(t) = \operatorname{Re}\tilde{v}e^{\lambda t} = \tilde{u}e^{\alpha t}\cos\beta t - \tilde{w}e^{\alpha t}\sin\beta t \quad \text{and} \quad \tilde{x}_2(t) = \operatorname{Im}\tilde{v}e^{\lambda t} = \tilde{u}e^{\alpha t}\sin\beta t + \tilde{w}e^{\alpha t}\cos\beta t$$

are two independent real solutions. These give us a fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \tilde{u}e^{\alpha t}\cos\beta t - \tilde{w}e^{\alpha t}\sin\beta t & \tilde{u}e^{\alpha t}\sin\beta t + \tilde{w}e^{\alpha t}\cos\beta t \end{pmatrix}.$$

Note: If $\lambda = \alpha + i\beta$ is a complex eigenvalue, then so is its complex conjugate $\lambda = \alpha - i\beta$. You might wonder why the procedure above is not repeated for the conjugate eigenvalue. The answer is that you can repeat the procedure, but you will obtain solutions that are not independent from those calculated for $\lambda = \alpha + i\beta$. If you think about, you know this is true because we learned that the general solution is a two dimensional linear vector space and that, therefore, there cannot be more than two independent solutions.

As a practical matter this procedure for complex eigenvalues, while rather straight forward (if you're comfortable with complex numbers and their algebra), is usually fairly tedious to perform in specific examples. Therefore, in this course we will generally use the Putzer Algorithm when the eigenvalues are complex.

The case of a double eigenvalue can also be handled using eigenvector theory, although its more complicated and involves "generalized" eigenvectors, something you not doubt did not encounter in your first linear algebra course. Therefore, in this course, we will use the Putzer Algorithm in this case.

The eigenvalue-eigenvector method for constructing fundamental solution matrices $\Phi(t)$ works for systems of any dimension. For higher dimensional coefficient matrices A, one not only has the challenge of calculating the eigenvalues but other challenges arise as well, such as the possibility of repeated complex eigenvalues, multiple eigenvalues of order greater than 2, a mix of complex and real eigenvalues, and so on. Mathematicians have thoroughly worked out the construction of $\Phi(t)$ in a general setting for any dimension n, but this procedure involves linear algebraic topics beyond the prerequisites of this course (in particular, the so-called Jordan Canonical Form of a matrix). The following example illustrates the method in the simplest case when A has n distinct real eigenvalues and eigenvectors which then produce a basis of exponent solutions $\tilde{v}e^{\lambda t}$.

Example 6.7 *Ecosystem modelers often use compartmental models to account for quantities transferring into and out of subsystems. Since ecosystems can involve a large number of subsystems, such compartmental models often involve a large number (even hundreds) of differential equations. Here we consider an example involving three compartments.*

Scientists can use a radioactive isotope to trace the flow of nutrients in food chains. Suppose a radioactive isotope is placed into the water of an aquarium in order to trace the flow of nutrients in an aquatic food chain consisting of zooplankton and phytoplankton. Let x_1, x_2 and x_3 denote the concentration of the isotope in the water, phytoplankton, and

zooplankton respectively. The compartment diagram in Figure 6.2 shows the (linear) transfer rates (in microcuries per hour) between these subsystems of the food chain.

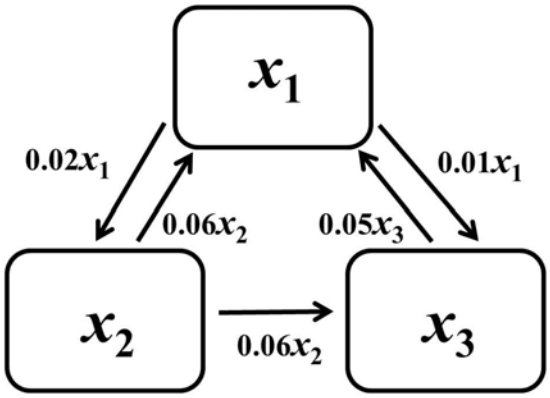


Figure 6.2

The differential equations for this compartmental model are

$$\begin{aligned} x_1' &= (-0.02 - 0.01)x_1 + 0.06x_2 + 0.05x_3 \\ x_2' &= 0.02x_1 + (-0.06 - 0.06)x_2 \\ x_3' &= 0.01x_1 + 0.06x_2 - 0.05x_3. \end{aligned} \tag{6.13}$$

This is a linear homogeneous system of equations with constant coefficients. The coefficient matrix

$$A = \begin{pmatrix} -0.03 & 0.06 & 0.05 \\ 0.02 & -0.12 & 0.00 \\ 0.01 & 0.06 & -0.05 \end{pmatrix}$$

of the aquatic food chain model (6.13) has eigenvalues and eigenvectors (to 4 significant figures)

$$\lambda_1 = -0.07551, \quad \tilde{v}_1 = \begin{pmatrix} 0.5169 \\ 0.2324 \\ -0.7493 \end{pmatrix}$$

$$\lambda_2 = -0.1245, \quad \tilde{v}_2 = \begin{pmatrix} 0.2677 \\ -1.191 \\ 0.9234 \end{pmatrix}$$

$$\lambda_3 = 0, \quad \tilde{v}_3 = \begin{pmatrix} 0.9415 \\ 0.1569 \\ 0.3766 \end{pmatrix}.$$

These eigen-pairs yield exponential solutions $\tilde{v}_i e^{\lambda_i t}$ which are independent (since the eigenvectors $\tilde{v}_i$, being associated with different eigenvalues, are independent). These exponential

solutions yield the fundamental solution matrix

$$\begin{aligned}\Phi(t) &= \left(\begin{array}{ccc} \tilde{v}_1 e^{\lambda_1 t} & \tilde{v}_2 e^{\lambda_2 t} & \tilde{v}_3 e^{\lambda_3 t} \end{array} \right) \\ &= \left(\begin{array}{ccc} 0.5169e^{-0.07551t} & 0.2677e^{-0.1245t} & 0.9415 \\ 0.2324e^{-0.07551t} & -1.191e^{-0.1245t} & 0.1569 \\ -0.7493e^{-0.07551t} & 0.9234e^{-0.1245t} & 0.3766 \end{array} \right)\end{aligned}$$

and the general solution (rounded to two significant digits, since the original coefficients have this accuracy)

$$\tilde{x}\,(t) = \Phi(t)\tilde{c}$$

$$\left(\begin{array}{c} x_1\,(t) \\ x_2\,(t) \\ x_3\,(t) \end{array} \right) = \left(\begin{array}{c} 0.52c_1e^{-0.076t} + 0.27c_2e^{-0.12t} + 0.94c_3 \\ 0.23c_1e^{-0.076t} - 1.20c_2e^{-0.12t} + 0.16c_3 \\ -0.75c_1e^{-0.076t} + 0.92c_2e^{-0.12t} + 0.38c_3 \end{array} \right).$$

Suppose 100 *microcuries of tracer are introduced into the water of the aquatic food chain modeled by Figure 6.2. Suppose no tracer is initially present in either the phytoplankton or the zooplankton. This provides the initial conditions*

$$\tilde{x}_0 = \left(\begin{array}{c} 100 \\ 0 \\ 0 \end{array} \right) \tag{6.14}$$

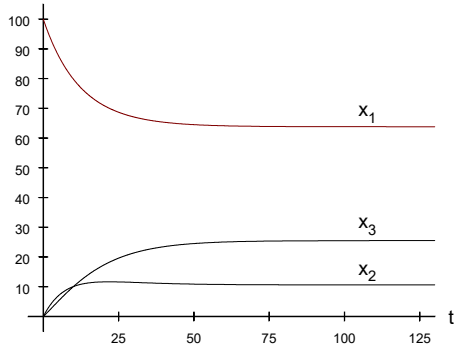
for the differential system (6.13). From $\tilde{x}\,(t) = \Phi(t)\Phi^{-1}(0)\tilde{x}_0$ *we calculate the formula*

$$x(t) \approx \left(\begin{array}{c} 31e^{-0.076t} + 5.5e^{-0.12t} + 64 \\ 14e^{-0.076t} - 25e^{-0.12t} + 11 \\ -44e^{-0.076t} + 19e^{-0.12t} + 25 \end{array} \right).$$

for the unique solution of this initial value problem.

In conclusion, note this solution formula implies that the tracer amounts in each compartment tend to constant levels as $t \to +\infty$, *namely* 64, 11 *and* 25 *(microcuries) for the water, phytoplankton and zooplankton respectively. The graphs of each solution component appear in Figure 6.3.*

Figure 6.3 Shown are computer generated graphs of the solutions of the initial value problem (6.14) for the aquatic food chain system (6.13).



6.4 Phase Plane Portraits

We studied phase plane portraits for two dimensional systems in Chapter 4. In this section we will take up a detailed study of the phase plane portraits of autonomous linear systems of dimension two:

$$\begin{gathered} \tilde{x}' = A\tilde{x} \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{gathered} \tag{6.15}$$

Our goal is to classify all possible types of phase portraits, understand the geometry of each type, and learn how to determine and sketch the portrait of such systems.

Recall that an *orbit* is the curve in the (x, y)-plane described by the coordinates $(x(t), y(t))$ where

$$\tilde{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is a solution of (6.15). (Technically, the orbit is the range of the vector-valued function $\tilde{x} = \tilde{x}(t)$ that maps the interval $-\infty < t < +\infty$ into the (x, y)-plane.)

Geometrically, the simplest orbits are points in the (x, y)-plane obtain from constant solutions, i.e., from equilibria. A solution is an equilibrium if and only if its derivative equals 0 and, therefore, a solution is an equilibrium of (6.15) if and only if it is a solution of the linear algebraic equation

$$A\tilde{x} = \tilde{0} \tag{6.16}$$

or, in component form, the equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$

Therefore, every homogeneous system (6.15) has the *trivial equilibrium*

$$\tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which means that the origin is always an orbit in the phase plane portrait of (6.15).

There can be other equilibria, i.e. nontrivial solutions of (6.16). We know from linear algebra that this can only occur if A is singular, i.e., if $\det A = 0$. In this section we will not consider this possibility. We will only investigate the case when the origin is the only equilibrium and therefore we will assume throughout this section that

$$\det A = ad - bc \neq 0.$$

Such systems (generic in the sense they are not frequently occurring) we call *simple*. For example of phase portraits for non-simple cases, see Exercise 6.62.

Our goal in this section is to describe the non-equilibrium orbits and to classify the phase plane portraits of simple systems 6.15.

As we have learned from our study of homogeneous systems (6.15), the algebraic characteristics of the general solution fall into three categories that depend on the type of eigenvalues the coefficient matrix A has. Eigenvalues are the roots of the characteristic equation

$$\lambda^2 - (\mathrm{tr}A)\,\lambda + \det A = 0 \tag{6.17}$$

or equivalently

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

From the quadratic formula, we have the formula

$$\lambda = \frac{1}{2}\left(\mathrm{tr}A \pm \sqrt{\Delta}\right)$$

for the eigenvalues where

$$\Delta \triangleq (\mathrm{tr}A)^2 - 4\det A$$

is the discriminant of the characteristic polynomial. Algebraically the two eigenvalues λ_1 and λ_2 fall naturally into three categories, depending on the sign of Δ :

Case 1: $\Delta > 0$ and λ_1, λ_2 are real and different
Case 2: $\Delta < 0$ and λ_1, λ_2 are complex conjugates
Case 3: $\Delta = 0$ and $\lambda_1 = \lambda_2 = \lambda$ is a double root

As we saw in Sections 6.2 and 6.3 the fundamental solution matrices, and hence the general solutions, differ from each other in significant ways in the cases. For example, Case 2 involves sine and cosine functions whereas Case 1 does not. These differences mean that the phase plane portraits in these cases will have different geometries.

In the following sections we will make use of solution formulas to determine the orbits and the phase plane portraits for the homogeneous system (6.15). However, in the end we will see that one can (rather easily) construct phase portraits without having to find solution formulas. All one needs know are the eigenvalues (and in Cases 1 and 3 the associated eigenvectors).

Remark 4. From the characteristic equation (6.17) we see that $\det A = 0$ if and only if $\lambda = 0$ is an eigenvalue. Therefore, by restricting our attention to simple systems in the following sections, we will not be concerned with an eigenvalue $\lambda = 0$.

6.4.1 Case 1: two different real eigenvalues

In this case, the fundamental solution matrix is (6.12)

$$\Phi(t) = \begin{pmatrix} \tilde{v}e^{\lambda_1 t} & \tilde{w}e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} v_1 e^{\lambda_1 t} & w_1 e^{\lambda_1 t} \\ v_2 e^{\lambda_1 t} & w_2 e^{\lambda_1 t} \end{pmatrix}$$

where

$$\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad \tilde{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

are eigenvectors associated with different eigenvalues λ_1 and λ_2 respectively.

Since the eigenvalues are different, one is larger than the other. We will always choose our notation so that

$$\lambda_2 < \lambda_1.$$

The components of the general solution

$$\tilde{x}(t) = \Phi(t)\,\tilde{c}$$

are

$$\begin{aligned} x(t) &= c_1 v_1 e^{\lambda_1 t} + c_2 w_1 e^{\lambda_2 t} \\ y(t) &= c_1 v_2 e^{\lambda_1 t} + c_2 w_2 e^{\lambda_2 t} \end{aligned} \tag{6.18}$$

and our challenge is to determine the nature of the orbits in the (x, y)-plane described by these formulas.

Let's begin with the *eigen-solutions*, and their *eigen-orbits*, themselves. The orbit of

$$\begin{aligned} x(t) &= c_1 v_1 e^{\lambda_1 t} \\ y(t) &= c_1 v_2 e^{\lambda_1 t} \end{aligned}$$

can be ascertain by noting that the ratio of $x(t)$ and $y(t)$ is constant, i.e.

$$\frac{y(t)}{x(t)} = \frac{v_2}{v_1}$$

which means, for each t, the orbit lies on the straight line through the origin with the slope of the eigenvector $\tilde{v}$:

$$y = \frac{v_2}{v_1} x. \tag{6.19}$$

Since clearly both $x(t)$ and $y(t)$ have one sign for all values of t, we conclude that the eigen-orbits are half lines.

Similar reasoning applies to the other eigen-solutions

$$\begin{aligned} x(t) &= c_2 w_1 e^{\lambda_2 t} \\ y(t) &= c_2 w_2 e^{\lambda_2 t} \end{aligned}$$

which are half-lines lying on the line

$$y = \frac{w_2}{w_1} x. \tag{6.20}$$

We conclude that the eigen-solutions produce for half line orbits lying on the lines (through the origin) with slopes determined by the eigenvectors $\tilde{v}$ and $\tilde{w}$.

Remark 5. If $v_1 = 0$, then the orbit of the eigen-solutions

$$\begin{aligned} x(t) &= 0 \\ y(t) &= c_2 v_2 e^{\lambda_1 t} \end{aligned}$$

is a half-line lying on the y-axis. Similarly, if $w_1 = 0$, then the orbit of the eigen-solutions

$$\begin{aligned} x(t) &= 0 \\ y(t) &= c_2 w_2 e^{\lambda_2 t} \end{aligned}$$

is a half-line lying on the y-axis.

Recall that orbits have orientation arrows indicating the direction that the point $(x(t), y(t))$ moves as t increases. With regard to the half line eigen-orbits, we observe that as $t \to +\infty$

either the exponential $e^{\lambda_i t}$ approaches 0 or $+\infty$. Thus, either the half line eigen-orbit either approaches the origin and the orientation arrow points toward the origin, if $\lambda_i < 0$. See Figure 6.4. Or the orientation arrow points away from the origin if $\lambda_i > 0$.

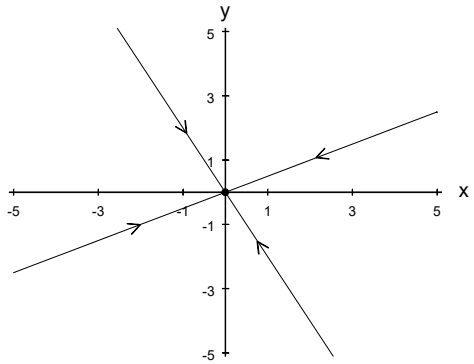


Figure 6.4

What remains to figure out is the geometry of the orbits of the remaining solutions, i.e., the solutions (6.18) when

$$c_1 \neq 0 \quad \text{and} \quad c_2 \neq 0$$

are both nonzero. Taking a hint from the analysis above of the eigen-solutions and their orbits, we consider the ratio of

$$\begin{aligned} \frac{y(t)}{x(t)} &= \frac{c_1 v_2 e^{\lambda_1 t} + c_2 w_2 e^{\lambda_2 t}}{c_1 v_1 e^{\lambda_1 t} + c_2 w_1 e^{\lambda_2 t}} \\ &= \frac{v_2 + \frac{c_2}{c_1} w_2 e^{(\lambda_2 - \lambda_1)t}}{v_1 + \frac{c_2}{c_1} w_1 e^{(\lambda_2 - \lambda_1)t}} \end{aligned}$$

and notice that, because $\lambda_2 < \lambda_1$,

$$\lim_{t \to +\infty} \frac{y(t)}{x(t)} = \frac{v_2}{v_1}.$$

This means that these orbits, while the do not lie on the line (6.19), they do approach this line as an asymptote.

With these facts in hand, our final determination of the phase portrait's geometry relies on the signs of the two real eigenvalues λ_1 and λ_2, for the solutions involve the exponentials $e^{\lambda_i t}$ whose asymptotic behavior as $t \to +\infty$ depends crucially on the sign of λ_i.

If both λ_i are negative, then clearly all orbits tend to the origin as $t \to +\infty$. If both are positive, then all orbits move away from the origin as $t \to +\infty$ (or, one can say, they move toward the origin as $t \to -\infty$). These two cases give rise to phase portraits as shown in Figures 6.4 and 6.5.

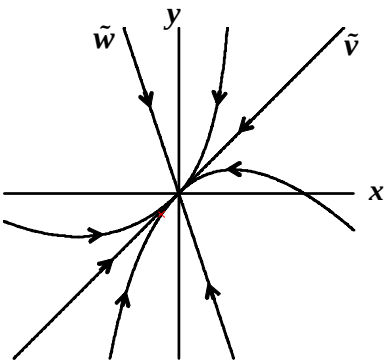


Figure 6.5 A stable node

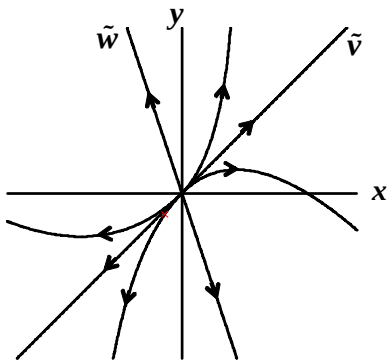


Figure 6.5 An unstable node

Note the orbital tangency to the eigenvector direction $\tilde{v}$ in both nodes. The eigenvector $\tilde{v}$ corresponds to the eigenvalue of least absolute value.

Definition 6.2 *When eigenvalues of the coefficient matrix A are real, different and negative, the phase plane portrait of (6.15) is called a* ***stable node*** *and origin is called an* ***attractor*** *(Figure 6.5)).*

When the eigenvalues are real, different and positive, the phase plane portrait is called an ***unstable node*** *and the origin is called a* ***repeller*** *(Figure 6.6)*

Remark 6. Note that the asymptotic tangency of orbits near the origin is to the direction determined by the eigenvector associated with the eigenvalue of smallest magnitude ($\tilde{v}$ and λ_1 in our presentation).

Remark 7. The reason the word "stable" is used for the phase portrait in Figure 6.5 is the following. If one starts on the origin (i.e., chooses the origin as an initial condition at, say, $t_0 = 0$) then, of course, you remain at the origin for all t. This is what it means to be an equilibrium. If you move or are bumped off the origin (i.e. pick any initial condition other than the origin), then you will always return to the origin as $t \to +\infty$. In this sense, the origin is stable against perturbations away from it. Of course, a return to the origin does not occur in the phase portrait Figure 6.6 and this is why that portrait is called unstable. More will be said about stability in Section 6.5.

The final case occurs when the eigenvalues have opposite signs

$$\lambda_1 < 0 < \lambda_2.$$

In this case, the half-line orbits associated with λ_1 and its eigenvector $\tilde{v}$ point towards the origin while that associated with λ_2 and is eigenvector $\tilde{w}$ point away from the origin. The former half-line orbits form what is called the *stable manifold* of the portrait while the latter form what is called the *unstable manifold* of the portrait. The question remains: what do all the other orbits do?

We write the ratio of the solution components as

$$\frac{y(t)}{x(t)} = \frac{c_1 v_2 e^{\lambda_1 t} + c_2 w_2 e^{\lambda_2 t}}{c_1 v_1 e^{\lambda_1 t} + c_2 w_1 e^{\lambda_2 t}} = \frac{c_1 v_2 e^{(\lambda_1 - \lambda_2)t} + c_2 w_2}{c_1 v_1 e^{(\lambda_1 - \lambda_2)t} + c_2 w_1}$$

and see that

$$\lim_{t \to +\infty} \frac{y(t)}{x(t)} = \frac{w_2}{w_1}.$$

(The exponential $e^{(\lambda_1-\lambda_2)t}$ tends to 0 as $t \to +\infty$ since $\lambda_1 - \lambda_2 < 0$). Thus, the half-line orbits associated with $\lambda_2 > 0$ (in the direction of its eigenvector $\tilde{w}$) are asymptotes of orbits as $t \to +\infty$.

On the other hand, writing the ratio as

$$\frac{y(t)}{x(t)} = \frac{c_1 v_2 e^{\lambda_1 t} + c_2 w_2 e^{\lambda_2 t}}{c_1 v_1 e^{\lambda_1 t} + c_2 w_1 e^{\lambda_2 t}} = \frac{c_1 v_2 + c_2 w_2 e^{(\lambda_2-\lambda_1)t}}{c_1 v_1 + c_2 w_1 e^{(\lambda_2-\lambda_1)t}}$$

and see that

$$\lim_{t\to-\infty} \frac{y(t)}{x(t)} = \frac{v_2}{v_1}.$$

(The exponential $e^{(\lambda_2-\lambda_1)t}$ tends to $0\, t \to -\infty$ since $\lambda_2 - \lambda_1 > 0$). Thus, the half-line orbits associated with $\lambda_1 > 0$ (in the direction of its eigenvector $\tilde{v}$) are asymptotes of orbits as $t \to -\infty$.

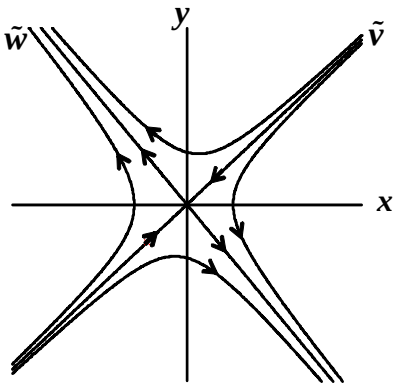


Figure 6.7. A saddle node.

Putting all this information together, we conclude that all orbits (other than the eigen-orbits), "travel" asymptotically from the stable manifold to the unstable manifold as t increases from $-\infty$ to $+\infty$. See Figure 6.7.

Definition 6.3 *When the eigenvalues of the coefficient matrix A are real and of opposite signs, the phase plane portrait of (6.15) is called a* ***saddle node*** *(Figure 6.7).*

Remark 8. When asked to sketch a phase plane portrait of a node or a saddle in exercises, what is expected is that the sketch include the two straight lines (through the origin) determined by eigenvectors, with proper orientation arrows shown, and at least one typical orbit within each of the four sectors formed by the eigen-orbits. In the case of a node, it is expected that the proper tangency of orbits at the origin be drawn.

Example 6.8 *The coefficient matrix*

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

associated with the system

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned} \tag{6.21}$$

has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. We know immediately from this that the phase plane portrait is a stable node.

To understand the geometry of this node we need to calculate eigenvectors associated with these eigenvalues. Here are eigenvectors associated with $\lambda_1 = -1$ and $\lambda_2 = -6$ respectively:

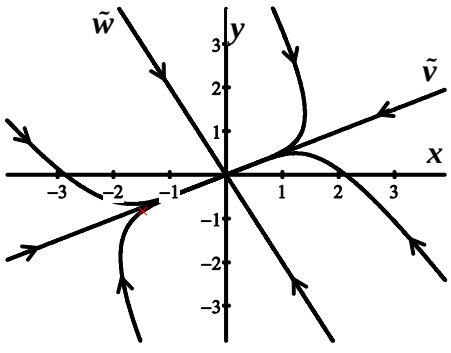


Figure 6.8. Phase portrait of (6.21).

$$\tilde{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \textit{and} \quad \tilde{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

These determine the framework of the node, the straight lines through the origin that constitute the half-line eigen-orbits. Since $\lambda_1 = -1$ has the smallest absolute value, all orbits (other than the half-line eigen-orbits associated with $\lambda_2 = -6$) tend to the origin tangentially to the vector $\tilde{v}$ as $t \to +\infty$. See Figure 6.8.

Example 6.9 *The coefficient matrix*

$$A = \begin{pmatrix} 3 & 5 \\ -1 & 9 \end{pmatrix}$$

associated with the system

$$\begin{aligned} x' &= 3x + 5y \\ y' &= -x + 9y \end{aligned} \tag{6.22}$$

has positive, unequal roots $\lambda_1 = 4$ and $\lambda_2 = 8$. and the phase portrait is an unstable node. We know immediately from this that the phase plane portrait is an unstable node.

To understand the geometry of this node we need to calculate eigenvectors associated with these eigenvalues. Here are eigenvectors associated with $\lambda_1 = 4$ and $\lambda_2 = 8$ respectively:

$$\tilde{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \textit{and} \quad \tilde{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These determine the framework of the node, the straight lines through the origin that constitute the half-line eigen-orbits. Since $\lambda_1 = 4$ has the smallest absolute value, all orbits (other than the half-line eigen-orbits associated with $\lambda_2 = 8$) tend to the origin tangentially to the vector $\tilde{v}$ as $t \to -\infty$. See Figure 6.9.

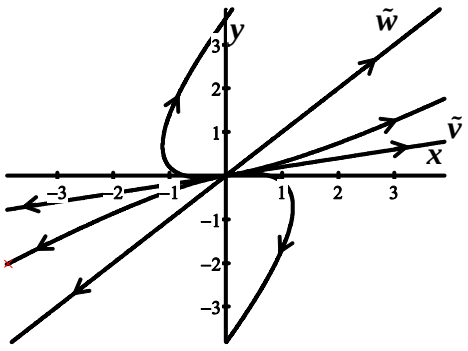


Figure 6.9 Phase portrait of (6.22).

Example 6.10 *The system*

$$\begin{aligned} x' &= by, \qquad b > 0 \\ y' &= cx, \qquad c > 0 \end{aligned} \tag{6.23}$$

has been used as a simple starting model for the study of armament race between two countries in which x and y are armament budgets of two opposing countries. The model assumes that each country increases its budget spending rate in direct proportion the budget amount of the other country. Consider the case $b = c = 1$ (which means both countries respond, with budget increase rates, in the same way to the other country's armament budget amount).

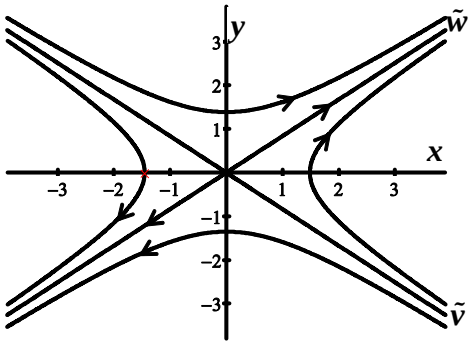


Figure 6.10. The portrait of (6.23).

The coefficient matrix

$$A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda_1 = -\sqrt{bc} < 0 \quad \text{and} \quad \lambda_2 = \sqrt{bc} > 0.$$

We know immediately from this that the phase plane portrait is an saddle node. To understand the geometry of this node we need to calculate eigenvectors associated with these eigenvalues. Here are eigenvectors associated with $\lambda_1 = -\sqrt{bc}$ and $\lambda_2 = \sqrt{bc}$ respectively:

$$\tilde{v} = \begin{pmatrix} \sqrt{b} \\ -\sqrt{c} \end{pmatrix} \quad \textit{and} \quad \tilde{w} = \begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix}.$$

These determine the framework of the saddle, the straight lines through the origin that constitute the half-line eigen-orbits. The direction $\tilde{v}$, which (no matter the numerical value of b and c) points into the southeast (fourth) quadrant, indicates the stable manifold. The direction $\tilde{w}$,which (no matter the numerical value of b and c) points into the northeast (first) quadrant, indicates the unstable manifold. All orbits asymptotically run from the stable to the unstable manifold. See Figure 6.10 for an example with $b = c = 1$.
In this application, only positive values of x and y are meaningful. Observing the northeast phase plan portrait, we see that all meaningful orbits are unbounded, i.e. the budgets of both countries grow without bound, and an "arms race" ensues.

6.4.2 Case 2: complex eigenvalues

If the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has a complex eigenvalue

$$\lambda = \alpha + i\beta, \quad \beta \neq 0$$

we saw in Section 6.3 that two independent solutions are

$$\begin{aligned} \tilde{x}_1(t) &= \operatorname{Re} \tilde{v}e^{\lambda t} = \tilde{u}e^{\alpha t}\cos\beta t - \tilde{w}e^{\alpha t}\sin\beta t \\ \tilde{x}_2(t) &= \operatorname{Im} \tilde{v}e^{\lambda t} = \tilde{u}e^{\alpha t}\sin\beta t + \tilde{w}e^{\alpha t}\cos\beta t \end{aligned}$$

where

$$\tilde{v} = \tilde{u} + i\tilde{w}$$

is an associated (complex) eigenvector. Here

$$\tilde{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \tilde{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

are the real and imaginary parts of $\tilde{v}$. In component form, these two solutions are

$$\tilde{x}_1(t) = e^{\alpha t} \begin{pmatrix} u_1 \cos \beta t - w_1 \sin \beta t \\ u_2 \cos \beta t - w_2 \sin \beta t \end{pmatrix}, \qquad \tilde{x}_2(t) = e^{\alpha t} \begin{pmatrix} u_1 \sin \beta t + w_1 \cos \beta t \\ u_2 \sin \beta t + w_2 \cos \beta t \end{pmatrix}.$$

The general solution is a linear combination of these two independent solutions

$$\tilde{x}(t) = \Phi(t)\tilde{c} = e^{\alpha t} \begin{pmatrix} (c_1 u_1 + c_2 w_1) \cos \beta t + (-c_1 w_1 + c_2 u_1) \sin \beta t \\ (c_1 u_2 + c_2 w_2) \cos \beta t + (-c_1 w_2 + c_2 u_2) \sin \beta t \end{pmatrix}$$

or component-wise

$$\begin{aligned} x(t) &= e^{\alpha t} \left[(c_1 u_1 + c_2 w_1) \cos \beta t + (-c_1 w_1 + c_2 u_1) \sin \beta t \right] \\ y(t) &= e^{\alpha t} \left[(c_1 u_2 + c_2 w_2) \cos \beta t + (-c_1 w_2 + c_2 u_2) \sin \beta t \right]. \end{aligned} \tag{6.24}$$

These are the coordinates of orbits in the phase plane.

Note: Neither b nor c can equal 0 in this case. If either equals 0, then coefficient matrix becomes triangular and the eigenvalues (which as a result appear along the diagonal) are real (specially, a and d).

First consider the orbit of $c_1 \tilde{x}_1(t)$ when $\alpha = 0$. Then

$$a + d = 0, \qquad \beta = \sqrt{bc + a^2}$$

some algebra shows that the components of this solution

$$\begin{aligned} x(t) &= c_1 u_1 \cos \beta t - c_1 w_1 \sin \beta t \\ y(t) &= c_1 u_2 \cos \beta t - c_1 w_2 \sin \beta t \end{aligned}$$

satisfy the quadratic equation

$$\left(2a^2 + bc\right) x^2 + 2abxy + b^2 y^2 = c_1^2 b^2 \left(a^2 + bc\right) \tag{6.25}$$

for all values of t. Therefore, the orbit lies on the curve described by this equation in the (x, y)-plane. From analytic geometry we know that the graph associated with a quadratic equation of the form

$$p_1 x^2 + p_2 xy + p_3 y^2 = p_4$$

is an ellipse if $p_2^2 - 4p_1 p_3$ is negative. For the quadratic equation (6.25) associated with the orbit, we calculate

$$p_2^2 - 4p_1 p_3 = (2ab)^2 - 4\left(2a^2 + bc\right) b^2 = -4b^2 \left(a^2 + bc\right) < 0$$

and conclude that the orbit of $c_1 \tilde{x}_1(t)$

Similarly, one can show that the orbit of $c_2\tilde{x}_1(t)$ is also an ellipse. In fact, in this case when $\alpha = 0$ one can the general solution is also an ellipse.

If $\alpha \neq 0$ the orbits are clearly no longer ellipses. The result is that each component $x(t)$ and $y(t)$ in the general solution (6.24) have the exponential factor $e^{\alpha t}$. Therefore, if $\alpha < 0$ and the elliptical motion of the orbit now shrinks as both components tend to 0 as $t \to +\infty$. We conclude in this case that the orbit is a spiral that tends to the origin as $t \to +\infty$. In contrast, if $\alpha > 0$ the orbit is a spiral that tends away from the origin as as $t \to +\infty$.

Definition 6.4 *When the coefficient matrix A has a complex eigenvalue $\lambda = \alpha + i\beta$, $\beta \neq 0$, then the phase plane portrait of (6.15) is called a* ***stable spiral*** *when $\alpha < 0$, a* ***center*** *when $\alpha = 0$, and an* ***unstable spiral*** *when $\alpha > 0$. See Figure 6.11, 6.12 and 6.13.*

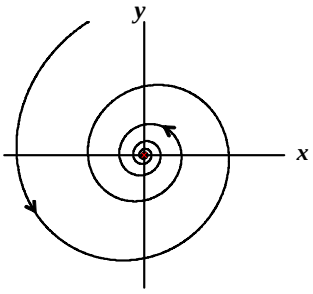


Figure 6.11. A stable spiral.

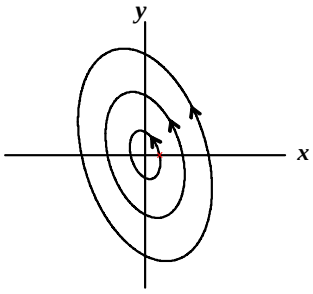


Figure 6.12. A center.

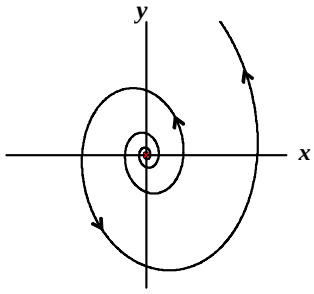


Figure 6.13. An unstable spiral.

Remark 9. When asked to sketch phase portraits of spirals and centers in exercises, what is expected is that the sketch a spiral from the origin that includes an arrow with the proper clockwise/counterclockwise orientation. This orientation can be readily determined by choosing a convenient test point in the plane and determining the direction field arrow at that point. For example, a simple choice is to use the test point $(x, y) = (1, 0)$. At this point the direction field is in the direction of the vector

$$\begin{pmatrix} a \\ c \end{pmatrix}.$$

Example 6.11 *Suppose the glucose/insulin model in Example 6.5 we set $\alpha = 0$ to describe the failure of the liver to process glucose. Then, keeping the other coefficients in the model at the estimated values in (6.9), the coefficient matrix*

$$A = \begin{pmatrix} 0 & -4.34 \\ 0.208 & -0.078 \end{pmatrix}$$

of the model

$$\begin{aligned} x' &= -4.34y \\ y' &= 0.208x - 0.078y \end{aligned} \tag{6.26}$$

has a complex eigenvalue $\lambda = -0.039 + 0.949i$ with negative real part $\alpha = -0.039$. The phase plane portrait is therefore a stable spiral. See Figure 6.14.

With the failure of this level function, the glucose and insulin levels still return to their equilibrium state, after a dosage of glucose is given to the bloodstream, but now they oscillate between positive and negative values, i.e., the patient experiences oscillatory episodes of high and low glucose levels in the bloodstream. This is in contrast to the non-oscillatory return to equilibrium when the liver is healthy, as shown by the stable node phase portrait in Example 6.5.

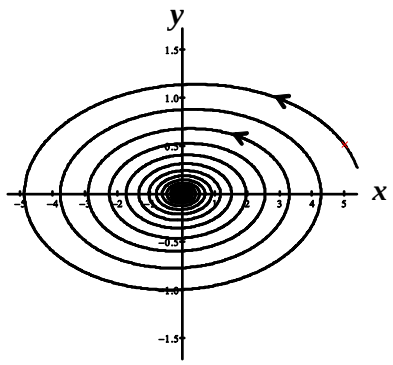


Figure 6.14. Phase portrait of (6.26)

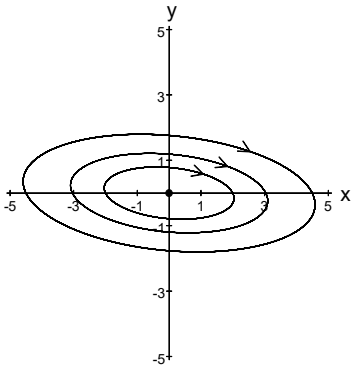
Example 6.12 *The coefficient matrix*

$$A = \begin{pmatrix} 1 & 13 \\ -2 & -1 \end{pmatrix}$$

of the system

$$\begin{aligned} x' &= x + 13y \\ y' &= -2x - y \end{aligned} \tag{6.27}$$

are complex $\lambda = \pm 5i$ with real part $\alpha = 0$. The phase plane portrait is therefore a center. Several orbits are shown Figure 6.15.



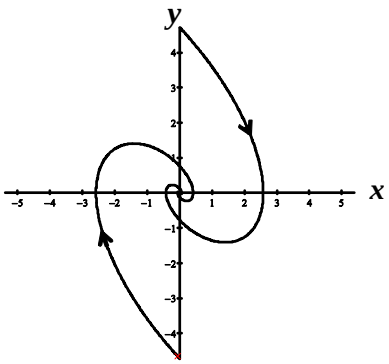


Figure 6.15. The phase portrait of (6.27).Figure 6.16. Phase portrait of (6.28).

Example 6.13 *The first order system equivalent to the second order equation*

$$x'' + x' + x = 0 \tag{6.28}$$

has coefficient matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

whose eigenvalues are $\lambda = -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$ with negative real part $\alpha = -\frac{1}{2}$. Therefore, the phase plane portrait for this second order equation is a stable spiral. See Figure 6.16.

6.4.3 Case 3: double eigenvalue

When the coefficient matrix A has a double eigenvalue λ (i.e., its characteristic polynomial has a double root), then there is one eigen-orbit corresponding to this eigenvalue whose half-line orbits are determined by the direction of the associated eigenvectors $\tilde{v}$. This is no different from Case 1. What is different in this case from Case 1 is that there is not necessarily a second eigen-orbit.

There are two possibilities. If the coefficient matrix is a multiple of the identity matrix

$$A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then all orbits obtain from the general solution

$$\begin{aligned} x(t) &= x(0) e^{\lambda t} \\ y(t) &= y(0) e^{\lambda t} \end{aligned}$$

are half-line orbits. Each orbit has its own slope $y(0)/x(0)$.

If A is not a multiple of the identity matrix, then we see from the general solution

$$\begin{aligned} x(t) &= [c_1 v_1 + c_2 (w_1 + v_1 t)] e^{\lambda t} \\ y(t) &= [c_1 v_2 + c_2 (w_2 + v_2 t)] e^{\lambda t} \end{aligned}$$

that

$$\lim_{t \to \pm\infty} \frac{y(t)}{x(t)} = \lim_{t \to +\infty} \frac{c_1 v_2 + c_2 (w_2 + v_2 t)}{c_1 v_1 + c_2 (w_1 + v_1 t)} = \frac{v_2}{v_1}$$

and hence every orbit is asymptotically tangent to the half-line eigen-orbit as $t \to +\infty$ and as $t \to -\infty$.

Definition 6.5 *Suppose the coefficient matrix A has a double eigenvalue λ. Then the phase portrait of (6.15) is called a* ***degenerate node****. If A is a multiple of the identity matrix, then the degenerate node is called a* ***star point****. If the coefficient matrix A is not a multiple of the identity matrix, then the degenerate node is called a* ***improper node****. These phase portraits are stable if $\lambda < 0$ and unstable if $\lambda > 0$. See Figures 6.17-6.20.*

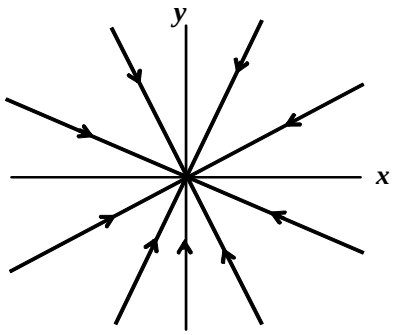


Figure 6.17. Stable star point.

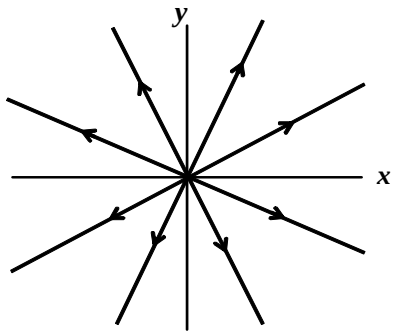


Figure 6.18.Unstable start point

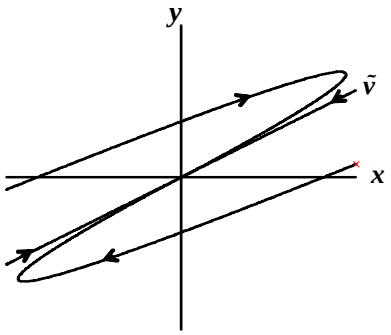


Figure 6.19. Stable improper node

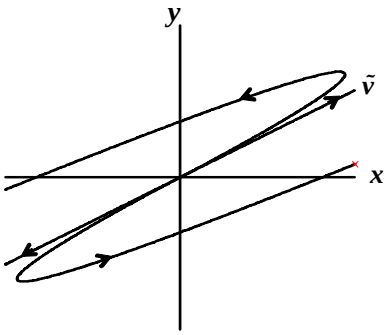


Figure 6.20. Unstable improper node

Remark 10. When asked to sketch a phase plane portrait of an improper node in exercises, what is expected is that the sketch include the straight line (through the origin) determined by eigenvector, with proper orientation arrows shown, and at least one typical orbit within each of the half planes on either side of the line. It is expected that the proper tangency of orbits at the origin be drawn.

Example 6.14 *The coefficient matrix*

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

of the system

$$\begin{aligned} x' &= -x + y \\ y' &= -y \end{aligned} \tag{6.29}$$

has a double eigenvalue $\lambda = -1$ *with an associated eigenvector*

$$\tilde{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The phase portrait is therefore a stable improper node with orbits that approach the origin tangentially to the x*-axis. See Figure 6.21.*

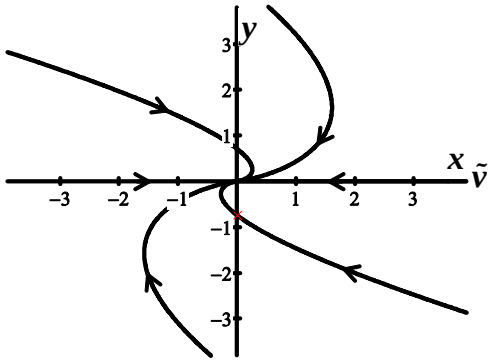


Figure 6.21. Phase portrait for (6.29)

6.5 Remarks on Stability

In Section 6.4 we classified the possible phase plane portraits of a 2×2 linear homogeneous system

$$\tilde{x}' = A\tilde{x} \tag{6.30}$$

with a nonsingular, constant coefficient matrix A. In making this classification, we used to words "stable" and "unstable" in a mathematically informal way. We will also make use of the notions of stability and instability in the next Chapter 8.

It turns out that stability theory is an immense subject. There are numerous concepts and mathematical definitions of stability and instability. For linear systems we spoke, in Section 6.4, about the stability or instability of a phase plane portrait. We can, for linear systems, equivalently speak about the stability or instability of the origin as an equilibrium of the system. This is a global notion of stability in which all orbits in the plane either tend to the origin (the one and only one equilibrium) or move away from the origin. (The center is the only exception.) In Section 6.4, where we will study stability theory for nonlinear systems, these global phenomenon do not always occur. Instead, it is convenient to define stability as a property that orbits possess in a neighborhood of an equilibrium and not necessarily throughout the whole phase plane. Thus, we will be concerned with "local" stability and instability.

The following definition applies to the origin as an equilibrium of a linear homogeneous system. In preparation for Chapter 8 we give the following definitions for any equilibrium (constant solution) for a system of first order differential equations. As usual, after a conversion to a first order system, the definitions apply to higher order differential equations as well.

Definition 6.6 *An equilibrium $\tilde{x}_e$ is **Lyapunov stable** if any orbit that starts close to $\tilde{x}_e$ will stay close to $\tilde{x}_e$ in the following sense. Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|\tilde{x}(t_0) - \tilde{x}_e| < \delta \text{ implies } |\tilde{x}(t) - \tilde{x}_e| \text{ for all } t \geq t_0.$$

*An equilibrium $\tilde{x}_e$ is a **local attractor** if any orbit that starts close to $\tilde{x}_e$ will tend to $\tilde{x}_e$ as $t \to +\infty$, that is, there exists a $\delta > 0$ such that*

$$|\tilde{x}(t_0) - \tilde{x}_e| < \delta \text{ implies } \lim_{t \to +\infty} |\tilde{x}(t) - \tilde{x}_e| = 0.$$

*An equilibrium $\tilde{x}_e$ is a **locally asymptotically stable** if it is both Lyapunov stable and a local attractor.*

*An equilibrium $\tilde{x}_e$ is a **unstable** if it is not Lyapunov stable.*

Clearly if an equilibrium is Lyapunov stable then it is not necessarily a local attractor. The center phase portrait for a linear system (6.30) is an example.

For the linear system (6.30) if the origin is an attractor then it is Lyapunov stable and hence locally asymptotically stable. It turns out that, in general for nonlinear systems, this is not necessarily true. Surprisingly as it may seem, there are examples of (nonlinear) systems that possess a local attractor that is *not* Lyapunov stable. Therefore, in general these "Lyapunov stability" and "local attractor" are independent properties, i.e. neither implies the other.

Remark. "Locally asymptotically stable" is a bit of a mouthful, so people often abbreviate this notion to simply "stable". *This is what we did in Section 6.4 in our classification of linear phase portraits.* Do not, however, confuse "stable" with "Lyapunov stable". The center phase portrait in Section 6.4 is not stable by the accepted convention that stable means locally asymptotically stable. A center is Lyapunov stable, however. Sometimes it is called *neutrally stable*.

A review of Section 6.4 shows the following result about stable phase plane portraits of linear homogeneous systems (6.30). Note that the real part of a real number is simply the real number itself.

Theorem 6.2 ***The phase plane portrait of a (simple) autonomous linear homogeneous system***

$$\tilde{x}' = A\tilde{x}, \quad \det A \neq 0$$

is (locally asymptotically) stable if and only if the real parts of all eigenvalues of A ***are negative.***

If at least one eigenvalue has a positive real part, then the phase plane portrait is unstable.

A knowledge of the eigenvalues of A identifies the type of the phase plane portrait and a knowledge of the sign of the real parts indicates the stability property of the portrait. Eigenvectors are only needed if detail about the geometry of the phase portrait is sought.

Example 6.15 *Consider the system*

$$\begin{aligned} x' &= px - y \\ y' &= -x + py \end{aligned}$$

where p is a (numerically unspecified) constant (called a system "parameter"). The coefficient matrix

$$A = \begin{pmatrix} p & -1 \\ -1 & p \end{pmatrix}$$

has two real and different eigenvalues $\lambda_1 = p + 1$ and $\lambda_2 = p - 1$. Referring to Theorem 6.2 we conclude the following:

	Eigenvalues	*Phase Plane Portrait*
$p < -1$	$\lambda_2 < \lambda_1 < 0$	*stable (node)*
$-1 < p < 1$	$\lambda_2 < 0 < \lambda_1$	*unstable (saddle)*
$1 < p$	$0 < \lambda_2 < \lambda_1$	*unstable (node)*

As p increases through -1 we see an example of a bifurcation, as the phase plane portrait (i.e. the equilibrium at the origin) is destabilizes.

Example 6.16 *The second order differential equation*

$$\begin{gathered} mx'' + cx' + kx = 0 \\ m, c, k > 0 \end{gathered}$$

arises in many applications that involve Newton's Laws of Motion. The coefficient matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$$

of the equivalent first order system has eigenvalues

$$\lambda_1 = \frac{1}{2}\left(-c + \sqrt{c^2 - 4mk}\right) \quad \text{and} \quad \lambda_2 = \frac{1}{2}\left(-c - \sqrt{c^2 - 4mk}\right)$$

To determine the stability properties of the equilibrium $x = 0$, $y = x' = 0$ by means of Theorem 6.2, we investigate the real parts of these eigenvalues.

If $c < 2\sqrt{mk}$ then the eigenvalues are complex with real part $\alpha = -c/2 < 0$. In this case (when c is small enough) the phase portrait is stable (a spiral).

If $c = 2\sqrt{mk}$ then the eigenvalues $\lambda = -c/2$ is real and double. Since it is negative the phase portrait is stable (an improper node).

If $c > 2\sqrt{mk}$ then the eigenvalues are real. Clearly $\lambda_2 < 0$. To see that λ_1 is also negative note that

$$\lambda_1 = \frac{1}{2}\left(-c + \sqrt{c^2 - 4mk}\right) < \frac{1}{2}\left(-c + \sqrt{c^2}\right) = 0.$$

Therefore, in this case the phase portrait is stable (a node).

6.6 The Trace-Determinant Criteria

The phase portrait classification of a linear homogeneous system with a constant coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

depends on the eigenvalues of A. Therefore, the phase portrait type depends on the numerical values of the four coefficients a, b, c and d. There is not really "four degrees of freedom" in determining the phase portrait, however, as the formula

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\mathrm{det}A}}{2} \tag{6.31}$$

for the eigenvalues shows. The eigenvalues, and hence the phase portrait type, depends on only two numbers, the trace $\mathrm{tr}A$ and the determinant $\det A$ of the coefficient matrix A.

One way to summarize how the phase portrait depends on the trace $\mathrm{tr}A$ and the determinant $\det A$ is as follows. Consider the two quantities $\mathrm{tr}A$ and $\mathrm{det}A$ as an ordered pair ($\mathrm{tr}A$, $\mathrm{det}A$) and hence as the coordinates of a point in a "trace-determine" plane. Since knowing $\mathrm{tr}A$ and $\mathrm{det}A$ is sufficient to determine the system's phase portrait, we can associate a unique phase portrait with each point in this plane.

For example, the point ($\mathrm{tr}A$,$\mathrm{det}A$) = $(2, 2)$ is associated with an unstable spiral because the roots of the quadratic

$$\lambda^2 - (\mathrm{tr}A)\lambda + \mathrm{det}A = \lambda^2 - 2\lambda + 2$$

are the complex conjugates $\lambda = 1 \pm i$ with positive real part.

Another example is the point ($\mathrm{tr}A$,$\mathrm{det}A$) = $(2, -2)$, which is associated with a saddle because the roots of the quadratic

$$\lambda^2 - (\mathrm{tr}A)\lambda + \mathrm{det}A = \lambda^2 - 2\lambda - 2$$

are $\lambda_1 = 1 - \sqrt{3} < 0$ and $\lambda_2 = 1 + \sqrt{3} > 0$ are real and of opposite signs.

By associating phase portraits with points in the ($\mathrm{tr}A$,$\mathrm{det}A$)-plane we obtain a "map" that locates of phase portraits.

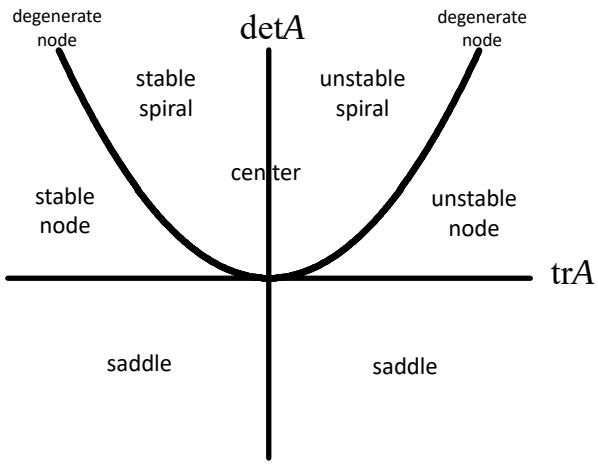


Figure 6.22 The Trace-Determinant Map

For example, the set of points (tr$A$,det$A$) associated with unstable spirals form a certain region in the plane; the set of points associated with saddles form another region in the plane; and so on. This map is shown in Figure 6.22. The region of points in Figure 6.22 associated with spirals is separated from the region of points associated with nodes by a parabola. This parabola

$$\det A = \frac{1}{4}(\mathrm{tr}A)^2 \tag{6.32}$$

is obtained by setting the discriminant under the radical sign in (6.31) equal to 0. Incidentally, points lying on this parabola are associated with improper nodes because for such points the quadratic formula (6.31) gives a repeated root. A point (tr$A$, det$A$) lies above the parabola in Figure 6.20 if $\det A > (\mathrm{tr}A)^2/4$ and lies below the parabola if $\det A < (\mathrm{tr}A)^2/4$. Thus, for points above the parabola the roots (6.31) are complex and below the parabola are real.

Above the parabola in Figure 6.20 are stable spirals if the real part of the eigenvalue is negative, i.e., if $\mathrm{tr}A < 0$. This is the region above the parabola and in the left half plane. Similarly, the region above the parabola and in the right half plane corresponds to unstable spirals. Points in between these two regions, i.e. lying on the vertical axis where $\mathrm{tr}A = 0$, correspond to centers.

Similar reasoning determines the phase portrait types in the regions lying below the parabola, as shown in Figure 6.22.

The main use of the tr-det map is in studying systems in which there appears an coefficient (sometimes called a parameter) that is numerically unspecified. The goal is then to determine how the phase plane portrait depends on the value of the parameter. This should sound familiar. It is exactly the problem that gives rise to bifurcation theory, a topic we studied for single first order equations in Section 3.1.4 of Chapter 3.

Example 6.17 *In Example 6.15 we investigated that stability of the phase portrait associated with the system*

$$\begin{aligned} x' &= px - y \\ y' &= -x + py \end{aligned}$$

in which p is a (numerically unspecified) parameter. We did that by calculating and studying the eigenvalues of the coefficient matrix

$$A = \begin{pmatrix} p & -1 \\ -1 & p \end{pmatrix}.$$

We can also study this system by using the tr-det map in Figure 6.20 and noting that

$$trA = 2p \quad and \quad \det A = p^2 - 1.$$

Where does the point $(trA, \det A)$ *lie in the tr-det map. In particular, is it above*

$$detA > \frac{1}{4}(trA)^2$$

or below

$$detA < \frac{1}{4}(trA)^2$$

the parabola 6.32)?. Since

$$\frac{1}{4}(trA)^2 = p^2$$

we see that the point is always below the parabola. Which region below the parabola?

The tr-det point is in the lower half plane if $\det A = p^2 - 1 < 0$, *or if* $-1 < p < 1$, *in which case the portrait is a saddle.*

If the tr-det point is in the upper half plane $(p^2 - 1 > 0)$, *it will lie in the left half plane if* $trA = 2p < 0$ *or* $p < 0$ *and in the right half plane if* $trA = 2p > 0$ *or* $p > 0$.

Thus, we have reached the same conclusions as in Example 6.15.

Some easy to see consequences of tr-det map in Figure 6.22 are the following stability and instability tests.

Theorem 6.3 ***The phase plane portrait of a*** 2×2 ***linear homogeneous with a constant coefficient matrix*** A ***is***

(a) (locally asymptotically) stable if

$$trA < 0 \quad \textbf{\textit{and}} \quad \det A > 0.$$

(b) unstable if

$$\det A < 0$$

(c) unstable if

$$trA > 0 \quad \textbf{\textit{and}} \quad \det A \neq 0.$$

Example 6.18 *The linear homogeneous system associated with the coefficient matrix*

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

is stable since tr $A = -7 < 0 \quad and \quad \det A = 6 > 0$.

The linear homogeneous system associated with the coefficient matrix

$$A = \begin{pmatrix} -1 & 3 \\ -\frac{1}{3} & 4 \end{pmatrix}$$

is unstable since tr $A = 3 > 0$.

The linear homogeneous system associated with the coefficient matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is not covered by Theorem 6.3 because $trA = 0$ *and neither statement in the Theorem is applicable.*

Example 6.19 *In Example 6.16 we considered the second order differential equation*

$$mx'' + cx' + kx = 0$$
$$m, c, k > 0$$

by investigating the eigenvalues of the coefficient matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$$

of its equivalent system. We can also deduce the stability of the phase plane portrait from Theorem 6.3 by noticing that $trA = -c/m < 0$ *and* $\det A = k/m > 0$.

6.7 Chapter Summary

In this chapter we learned two methods for calculating a fundamental solution matrix of a two dimensional, linear homogeneous first order system $\tilde{x}' = A\tilde{x}$ with a constant coefficient matrix A: the Putzer Algorithm and the eigenvalue-eigenvector method. Using the general solution formulas made available by these methods, we were able to classify the phase plane portraits of such systems. (We assumed a unique equilibrium at the origin, i.e., that $\det A \neq 0$). We found (via the three algebraic cases that naturally arise from the roots of a quadratic polynomial) that all phase portraits can be placed into one of three categories: node (including saddles), spirals (and centers), and improper nodes. These correspond respectively to the cases of two real and different eigenvalues, complex conjugate eigenvalues or a double eigenvalue. Nodes and spirals are further classified as stable or unstable. The classification of a system's phase portrait can be made from its eigenvalues alone or, alternatively, from the trace and determinant of the coefficient matrix A. Eigenvectors are also needed if the geometric nature of nodes and saddles is desired.

6.8 Exercises

For the first order homogeneous systems below:

(a) calculate a fundamental solution matrix $\Phi(t)$ using the Putzer Algorithm formulas (6.5)-(6.6),

(b) calculate a formula for the solution of the initial value problem $x(0) = 1$, $y(0) = -1$,

(c) calculate a formula for the solution of the initial value problem $x(0) = 2$, $y(0) = 3$.

Exercise 6.1 $\begin{cases} x' = 4x - 2y \\ y' = 7x - 5y \end{cases}$

Exercise 6.2 $\begin{cases} x' = -2x + y \\ y' = x - 2y \end{cases}$

Exercise 6.3 $\begin{cases} x' = \frac{1}{2}x - \frac{3}{2}y \\ y' = \frac{3}{2}x + \frac{1}{2}y \end{cases}$

Exercise 6.4 $\begin{cases} x' = x + y \\ y' = x - y \end{cases}$

Exercise 6.5 $\begin{cases} x' = -0.012x - 0.45y \\ y' = 2.31x - 3.15y \end{cases}$

Exercise 6.6 $\begin{cases} x' = 0.51x - 0.74y \\ y' = 1.42x + 2.67y \end{cases}$

Exercise 6.7 $\begin{cases} x' = 2x + y \\ y' = -2x \end{cases}$

Exercise 6.8 $\begin{cases} x' = -x + y \\ y' = -x - 2y \end{cases}$

Exercise 6.9 $\begin{cases} x' = -x + y \\ y' = 4x + 2y \end{cases}$

Exercise 6.10 $\begin{cases} x' = 3x + 2y \\ y' = -4x - y \end{cases}$

Exercise 6.11 $\begin{cases} x' = -6.1x + 0.2y \\ y' = -1.1x - 1.5y \end{cases}$

Exercise 6.12 $\begin{cases} x' = 8.3x + 1.2y \\ y' = -1.8x + 0.3y \end{cases}$

Exercise 6.13 $\begin{cases} x' = x + 13y \\ y' = -2x - y \end{cases}$

Exercise 6.14 $\begin{cases} x' = x + 3y \\ y' = 4x + 2y \end{cases}$

Exercise 6.15 $\begin{cases} x' = -\frac{1}{2}x + \frac{3}{4}y \\ y' = -3x + \frac{5}{2}y \end{cases}$

Exercise 6.16 $\begin{cases} x' = -5x + 8y \\ y' = -2x + 3y \end{cases}$

Exercise 6.17 *Find a formula for the solution of the chemical pesticide problem*

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \\ x(0) &= d, \quad y(0) = 0 \end{aligned}$$

in which an initial dose of pesticide is sprayed in the trees, but none is initially present in the soil.

Exercise 6.18 *Suppose* $\tilde{x}(t) = \tilde{v}e^{\lambda t}$, *where* $\tilde{v} = \tilde{u} + i\tilde{w} \neq \tilde{0}$ *and* $\lambda = \alpha + i\beta$, $\beta \neq 0$, *is a complex solution* $\tilde{x}(t) = \tilde{v}e^{\lambda t}$ *of an autonomous linear homogeneous system* $\tilde{x}' = A\tilde{x}$.

(a) Show the real and imaginary parts $\tilde{x}_1(t) = \operatorname{Re}\tilde{v}e^{\lambda t}$ *and* $\tilde{x}_2(t) = \operatorname{Im}\tilde{v}e^{\lambda t}$ *are both solutions. Therefore,* $\Phi(t) = \det(\ \tilde{x}_1(t) \quad \tilde{x}_2(t)\ )$ *is a solution matrix.*

(b) Show the real and imaginary parts $\tilde{x}_1(t) = \operatorname{Re}\tilde{v}e^{\lambda t}$ *and* $\tilde{x}_2(t) = \operatorname{Im}\tilde{v}e^{\lambda t}$ *are independent. Therefore,* $\Phi(t) = \det(\ \tilde{x}_1(t) \quad \tilde{x}_2(t)\ )$ *is a fundamental solution matrix.*

For the systems below, calculate a fundamental solution matrix $\Phi(t)$ using the eigenvalue-eigenvector method.

Exercise 6.19 $\begin{cases} x' = 4x - 2y \\ y' = 7x - 5y \end{cases}$

Exercise 6.20 $\begin{cases} x' = -2x + y \\ y' = x - 2y \end{cases}$

Exercise 6.21 $\begin{cases} x' = \frac{1}{2}x - \frac{3}{2}y \\ y' = \frac{3}{2}x + \frac{1}{2}y \end{cases}$

Exercise 6.22 $\begin{cases} x' = x + y \\ y' = x - y \end{cases}$

Exercise 6.23 $\begin{cases} x' = -0.012x - 0.45y \\ y' = 2.31x - 3.15y \end{cases}$

Exercise 6.24 $\begin{cases} x' = 0.51x - 0.74y \\ y' = 1.42x + 2.67y \end{cases}$

Exercise 6.25 $\begin{cases} x' = 2x + y \\ y' = -2x \end{cases}$

Exercise 6.26 $\begin{cases} x' = -x + y \\ y' = -x - 2y \end{cases}$

Exercise 6.27 $\begin{cases} x' = -x + y \\ y' = 4x + 2y \end{cases}$

Exercise 6.28 $\begin{cases} x' = 3x + 2y \\ y' = -4x - y \end{cases}$

Exercise 6.29 $\begin{cases} x' = -6.1x + 0.2y \\ y' = -1.1x - 1.5y \end{cases}$

Exercise 6.30 $\begin{cases} x' = 8.3x + 1.2y \\ y' = -1.8x + 0.3y \end{cases}$

Exercise 6.31 $\begin{cases} x' = x + 13y \\ y' = -2x - y \end{cases}$

Exercise 6.32 $\begin{cases} x' = x + 3y \\ y' = 4x + 2y \end{cases}$

For each of the homogeneous systems or second order equations below identify the type of its phase plane portrait and hand sketch it. An adequate hand sketch for a node must include the straight line eigen-orbits and one typical orbit in each of the four sectors formed by the eigen-orbits, along with the correct tangency at the origin. An adequate hand sketch for saddles must include the straight line eigen-orbits and one typical orbit in each of the four sectors form by the eigen-orbits with the correct asymptotes as $t \to -\infty$ and $t \to +\infty$. An adequate hand sketch for a spiral or center must include at least two orbits with the correct orientation (clock-wise or counter clock-wise). In all cases be sure to draw orientation arrows on each orbit.

Exercise 6.33 $\begin{cases} x' = 4x - 2y \\ y' = 7x - 5y \end{cases}$

Exercise 6.34 $\begin{cases} x' = -2x + y \\ y' = x - 2y \end{cases}$

Exercise 6.35 $\begin{cases} x' = \frac{1}{2}x - \frac{3}{2}y \\ y' = \frac{3}{2}x + \frac{1}{2}y \end{cases}$

Exercise 6.36 $\begin{cases} x' = x + y \\ y' = x - y \end{cases}$

Exercise 6.37 $\begin{cases} x' = -0.012x - 0.45y \\ y' = 2.31x - 3.15y \end{cases}$

Exercise 6.38 $\begin{cases} x' = 0.51x - 0.74y \\ y' = 1.42x + 2.67y \end{cases}$

Exercise 6.39 $\begin{cases} x' = 2x + y \\ y' = -2x \end{cases}$

Exercise 6.40 $\begin{cases} x' = -x + y \\ y' = -x - 2y \end{cases}$

Exercise 6.41 $\begin{cases} x' = -x + y \\ y' = 4x + 2y \end{cases}$

Exercise 6.42 $\begin{cases} x' = 3x + 2y \\ y' = -4x - y \end{cases}$

Exercise 6.43 $\begin{cases} x' = -6.1x + 0.2y \\ y' = -1.1x - 1.5y \end{cases}$

Exercise 6.44 $\begin{cases} x' = 8.3x + 1.2y \\ y' = -1.8x + 0.3y \end{cases}$

Exercise 6.45 $\begin{cases} x' = x + 13y \\ y' = -2x - y \end{cases}$

Exercise 6.46 $\begin{cases} x' = x + 3y \\ y' = 4x + 2y \end{cases}$

Exercise 6.47 $\begin{cases} x' = -\frac{1}{2}x + \frac{3}{4}y \\ y' = -3x + \frac{5}{2}y \end{cases}$

Exercise 6.48 $\begin{cases} x' = 9x - 17y \\ y' = 5x - 9y \end{cases}$

Exercise 6.49 $x'' + x' + x = 0$

Exercise 6.50 $x'' - x' + x = 0$

Exercise 6.51 $2x'' - x = 0$

Exercise 6.52 $x'' + 2x' + x = 0$

Exercise 6.53 $x'' - 5x' + 4x = 0$

Exercise 6.54 $x'' + 3x' - 4x = 0$

Exercise 6.55 $x'' - 6x' + 9x = 0$

Exercise 6.56 $x'' + 5x = 0$

Exercise 6.57 $x'' + 3.8x' + 3.45x = 0$

Exercise 6.58 $2x'' + 3x' + x = 0$

Exercise 6.59 *The system*

$$\begin{aligned} x' &= -(0.2 + 0.1)\, x + 0.5y \\ y' &= 0.2x - (0.5 + a)\, y \end{aligned}$$

is a model for the amount of drug in the blood stream x and the amount of drug in the tissues y of a patient. The coefficient $a > 0$ is positive.

(a) What type of phase plane portrait does this system have? Show it is same type for all a and show both $x(t)$ and $y(t)$ tend to 0 in the long run.

(b) Sketch a typical phase plane portrait.

(c) Suppose $x(0) \geq 0$ and $y(0) \geq 0$ are not both equal to 0. By referring just to the phase plane portrait argue that the ratio $y(t)/x(t)$ of the amount of drug in the tissues to that in the blood stream approaches a positive limit as $t \to \infty$.

Find a formula for the limit $L = \lim_{t\to\infty} y(t)/x(t)$. This limit $L = L(a)$ will depend on a. What is the maximum and what is the minimum value of this $L(a)$ and at which values of a do they occur?

Exercise 6.60 *The system*

$$\begin{aligned} x' &= -r_1 x + r_2 y \\ y' &= r_1 x - (r_2 + r_3)\, y \end{aligned}$$

models the amount of pesticide in a stand of trees, x, and its soil bed, y. Use the tr-det map to show this system has a stable node. All coefficients r_1, r_2, and r_3 are positive.

Exercise 6.61 *In the homogeneous linear systems below δ is a real number (positive, negative or zero). Determine the type and stability properties of the phase portrait as they depend on δ.*

(a) $\begin{cases} x' = -x + (1-\delta)\, y \\ y' = x - y \end{cases}$ *(b)* $\begin{cases} x' = -\delta x + y \\ y' = -x - \delta y \end{cases}$

Exercise 6.62 *The coefficient matrices of the systems below not simple. Find the general solution of each system and use it to sketch the phase portrait of the system. It turns out that the phase portraits any non-simple linear systems is one of these types (up to a linear change of variables, i.e. to rotations, reflections, and /or re-scalings of the coordinates axes).*

(a) $\begin{cases} x' = -x \\ y' = 0 \end{cases}$ *(b)* $\begin{cases} x' = x \\ y' = 0 \end{cases}$

(c) $\begin{cases} x' = 0 \\ y' = x \end{cases}$ *(d)* $\begin{cases} x' = 0 \\ y' = 0 \end{cases}$

Exercise 6.63 *Consider the initial value problem*

$$\begin{aligned} Lx'' + Rx' + \tfrac{1}{C}x &= 0 \\ x(0) = x_0, \qquad x'(0) &= 0 \end{aligned}$$

for the charge $x = x(t)$ on an electric circuit with a resistor (of R Ohms), inductor (of L Henrys) and capacitor (of C Farads). The circuit has an initial charge of x_0 and no initial current $x'(0)$. Determine the phase plane portrait type for the circuits with the following parameter values and solve solve the initial value problem.

(a) $L = 0.1$, $R = 250$, $C = 10^{-5}$
(b) $L = 0.2$, $R = 200$, $C = 10^{-5}$
(c) $L = 0.1$, $R = 0$, $C = 10^{-5}$
(d) $L = 0.1$, $R = 200$, $C = 10^{-5}$

Find the eigenvalues and eigenvectors of the matrices A below. Then find a fundamental solution matrix $\Phi(t)$ for the homogeneous linear system with the coefficient matrix A.

Exercise 6.64

$$A = \begin{pmatrix} 3 & -6 & -2 \\ 2 & 3 & 2 \\ -2 & 6 & 3 \end{pmatrix}$$

Exercise 6.65

$$A = \begin{pmatrix} -4 & 1 & 2 \\ -22 & 9 & 12 \\ 8 & -4 & -5 \end{pmatrix}$$

Exercise 6.66

$$A = \begin{pmatrix} 7 & 4 & 6 \\ -5 & -3 & -4 \\ -5 & -2 & -5 \end{pmatrix}$$

Exercise 6.67

$$A = \begin{pmatrix} -5 & -10 & 1 \\ 2 & 6 & -2 \\ 6 & 10 & 0 \end{pmatrix}$$

Exercise 6.68

$$A = \begin{pmatrix} -5 & 6 & -3 & -2 \\ -5 & 6 & -1 & -2 \\ 1 & -1 & 2 & 0 \\ -5 & 5 & 0 & -1 \end{pmatrix}$$

Exercise 6.69

$$A = \begin{pmatrix} -13 & -21 & 24 & -15 \\ 11 & 16 & -15 & 11 \\ 4 & 7 & -7 & 4 \\ 5 & 8 & -9 & 7 \end{pmatrix}$$

Exercise 6.70 *Find a formula for the solution of the initial value problem*

$$\widetilde{x}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

for the systems with coefficient matrices given in Exercises 6.64-6.67. (Use a computer as an aid in performing the necessary matrix algebra.)

Exercise 6.71 *Solve the initial value problem*

$$\widetilde{x}(0) = \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix}$$

for the systems with coefficient matrices given in Exercises 6.68 and 6.69. (Use a computer as an aid in performing the necessary matrix algebra.)

Exercise 6.72 *The system below arises from a compartmental model for biomass transfer in a pine-oak forest. The compartments are vegetation (x), litter (y) and humus (z). The unit of time is one year.*

$$\begin{aligned} x' &= -\frac{7}{10}x \\ y' &= \frac{7}{10}x - \frac{3}{10}y \\ z' &= \frac{3}{10}y - \frac{1}{10}z. \end{aligned}$$

Suppose the forest is initially free of litter and humus and starts with x_0 units of biomass in vegetation.

(a) Use a computer to explore the solution of the resulting initial value problem.
  (i) How long does it take for the vegetation to decrease by 90%?
  (ii) At what time will the litter biomass be maximum?
  (iii) At what time will the humus biomass be maximum?
  (iv) How do your answers depend on x_0?

(b) Find a fundamental solution matrix and the general solution.

(c) Find a formula for the solution of the initial value problem.

(d) Use your answer from (c) to corroborate your answers in Exercise (a).

Use the tr-det stability criteria in Theorem 6.3 to determine whether the phase plane portraits of the linear homogeneous systems with the coefficient matrices below is stable or unstable. Then use the tr-det map to determine the phase portrait type.

Exercise 6.73 $A = \begin{pmatrix} -2 & 2 \\ -3 & 1 \end{pmatrix}$

Exercise 6.74 $A = \begin{pmatrix} -5 & 3 \\ 4 & -4 \end{pmatrix}$

Exercise 6.75 $A = \begin{pmatrix} \frac{6}{7} & \frac{2}{3} \\ -3 & 1 \end{pmatrix}$

Exercise 6.76 $A = \begin{pmatrix} -3 & 2 \\ -12 & 7 \end{pmatrix}$

Exercise 6.77 $A = \begin{pmatrix} \frac{3}{4} & \frac{4}{5} \\ -\frac{1}{2} & -\frac{2}{3} \end{pmatrix}$

Exercise 6.78 $A = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$

Exercise 6.79 $A = \begin{pmatrix} -1 & -1 \\ 6 & -6 \end{pmatrix}$

Exercise 6.80 $A = \begin{pmatrix} 3 & -1 \\ -1 & -4 \end{pmatrix}$

Exercise 6.81 $A = \begin{pmatrix} 3 & -1 \\ -15 & -4 \end{pmatrix}$

Use the tr-det stability criteria in Theorem 6.3 to determine those values of p for which the phase plane portraits of the linear homogeneous systems with the coefficient matrices below are stable and those values of p for which they are unstable.

Exercise 6.82 $A = \begin{pmatrix} p-1 & -2 \\ 1 & 1 \end{pmatrix}$

Exercise 6.83 $A = \begin{pmatrix} p+1 & 1 \\ -2 & -2 \end{pmatrix}$

Exercise 6.84 $A = \begin{pmatrix} p+1 & -1 \\ 1 & p-1 \end{pmatrix}$

Exercise 6.85 $A = \begin{pmatrix} p+1 & 2 \\ p & p+1 \end{pmatrix}$

Exercise 6.86 $A = \begin{pmatrix} p & 2 \\ 1 & -2p \end{pmatrix}$

Exercise 6.87 $A = \begin{pmatrix} p^2 - 11 & p^3 \\ p & p^2 \end{pmatrix}$

Use the tr-det map in Section 6.3 to determine the phase plane types for the linear homogeneous systems with the coefficients matrices below.

Exercise 6.88 $A = \begin{pmatrix} p & -p^2 \\ 1 & 0 \end{pmatrix}$

Exercise 6.89 $A = \begin{pmatrix} 0 & \frac{1}{2}p \\ -\frac{1}{2}p & -1 \end{pmatrix}$

Exercise 6.90 $A = \begin{pmatrix} -5 & -1 \\ 5p & 2p \end{pmatrix}$

Exercise 6.91 $A = \begin{pmatrix} p & 1-p \\ -p & p \end{pmatrix}$

Find a formula for the general solution of each system below.

Exercise 6.92 $\begin{cases} x' = -5x \\ y' = -5y \end{cases}$

Exercise 6.93 $\begin{cases} x' = 3x - y \\ y' = x + y \end{cases}$

Exercise 6.94 $\begin{cases} x' = -3x - y \\ y' = 7x + y \end{cases}$

Exercise 6.95 $\begin{cases} x' = 1.33x - 4.31y \\ y' = 8.97x - 1.33y \end{cases}$

Exercise 6.96 $\begin{cases} x' = -4.1x - 5.2y \\ y' = 10.1x + 4.1y \end{cases}$

Exercise 6.97 $\begin{cases} x' = \sqrt{5}x - y \\ y' = 7x + y \end{cases}$

Exercise 6.98 $\begin{cases} x' = -\frac{1}{2}y \\ y' = 5x - \frac{1}{4}\pi y \end{cases}$

Exercise 6.99 $\begin{cases} x' = (\sin\theta)\, x + (\cos\theta)\, y \\ y' = (\cos\theta)\, x - (\sin\theta)\, y \end{cases}$ *where θ is a real number*

Exercise 6.100 $\begin{cases} x' = 3e^{-a}x - 2e^{-2a}y \\ y' = 2x - e^{-a}y \end{cases}$ *where a is a real number*

Exercise 6.101 $\begin{cases} x' = x + y \\ y' = x - ay \end{cases}$ *where a is a constant satisfying $-1 < a < 1$*

Exercise 6.102 $\begin{cases} x' = 0.31x + 4.79y \\ y' = -1.84x - 1.73y \end{cases}$

Exercise 6.103 $\begin{cases} x' = -2.3x + .79y \\ y' = 1.84x - 1.73y \end{cases}$

Exercise 6.104 *Find a formula for the solution of the initial value problem $x(0) = 1$, $y(0) = -1$ for the systems in Exercises 6.92-6.103.*

Exercise 6.105 *Find a formula for the solution of the initial value problem $x(0) = 2$, $y(0) = 3$ for the systems in Exercises 6.92-6.103.*

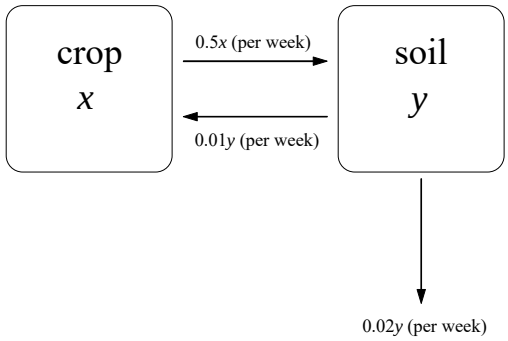
Exercise 6.106 *Find a formula for the general solution of the second order equations below and identify the phase plane portrait.*

(a) $Lx'' + Rx' + \frac{1}{C}x = 0$ where L, R and C are positive constants and $R > 2\sqrt{L/C}$.

(b) $x'' + (c - d)x - (1 + c)x = 0$ where c and d are positive constants.

Exercise 6.107

The compartment diagram for the pesticide DDT in an agricultural crop and its soil appears in the accompanying figure.



(a) Suppose there is initially no DDT in the soil and x_0 units of DDT are sprayed onto the crops. Write an initial value problem linear system for x (the units of DDT in the crop) and y (the units of DDT in the soil).

(b) Use a computer to investigate the orbits in the phase plane of the initial value problem (a) for a variety of initial dosages $x_0 > 0$.

(i) What happens as $t \to +\infty$?

(ii) What happens to the ratio of DDT in the soil to that in the crop as $t \to +\infty$?

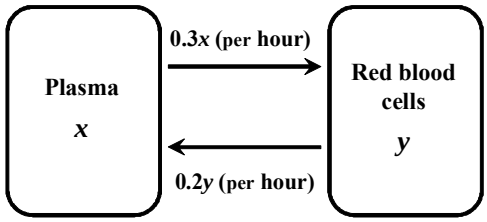
(iii) How do your answers depend on the initial dosage x_0?

(c) Find a formula for the solution of the initial value problem in (a).

(d) Use you answer in (c) to corroborate your answer in (b).

Exercise 6.108

The compartmental diagram for the movement of potassium (using a radioactive isotope of potassium as a tracer) between red blood cells and plasma appears in the accompanying figure..



Suppose there is initially no tracer in the red blood cells and x_0 units of tracer are injected into the plasma.

(a) Write an initial value problem linear system for x (the units of tracer in the plasma) and y (the units of tracer in the red blood cells).

(b) Use a computer to investigate the solution of the initial value problem in (a) for a selection of initial dosages $x_0 > 0$.

(i) What happens as $t \to +\infty$?

(ii) What happens to the ratio of tracer in the red blood cells to that in the plasma as $t \to +\infty$?

(iii) How do your answers depend on the initial dose x_0?

(c) Find a formula for the solution of the initial value problem in (a).

(d) Use you answer in (c) to corroborate your answer in (b).

Exercise 6.109 *To study blood flow through organs tracer dyes are added intravenously and the concentrations in various organs monitored. The compartmental model below describes*

the tracer concentrations (mg/liter) in arterial blood (x_1), venous blood in the liver (x_2) and venous blood in the right atrium (x_3).

$$\begin{aligned} x_1' &= -2x_1 + 2x_3 \\ x_2' &= x_1 - x_2 - R \\ x_3' &= x_1 + x_2 - 2x_3 \end{aligned}$$

R is the removal rate of the dye from the liver (mg/liter/min). Suppose an initial dose of $x_1(0) = 10$ is added to the arterial blood and no dye is initially present in the other compartments. Let $R = 1$.

(a) Use a computer to graph each component of the solution. When is the dye gone from each components?

(b) Find a formula for the solution of the initial value problem.

(c) Use your answer in (b) to determine when the dye is gone from each component.

Exercise 6.110 *Use the Variation of Constants Formula (5.19) to find the general solution of the system $\tilde{x}' = A\tilde{x} + \tilde{q}(t)$ with*

$$A = \begin{pmatrix} -5 & -10 & 14 \\ -4 & -5 & 8 \\ -5 & -8 & 12 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 6.111 *Use the Variation of Constants Formula (5.19) to find the general solution of the system $\tilde{x}' = A\tilde{x} + \tilde{q}(t)$ with*

$$A = \begin{pmatrix} 12 & 19 & -28 \\ 2 & 5 & -4 \\ 6 & 11 & -14 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} 2e^{-t} \\ e^{t} \\ 1 - e^{-t} \end{pmatrix}$$

Chapter 7

2nd Order Linear Differential Equations

In Chapter we saw that a second order differential equation

$$c_2(t)\,x'' + c_1(t)\,x' + c_0(t)\,x = q(t) \tag{7.1}$$

has an equivalent first order system

$$\begin{aligned} x' &= y \\ y' &= -\frac{c_1(t)}{c_2(t)}x - \frac{c_0(t)}{c_2(t)} + \frac{q(t)}{c_2(t)} \end{aligned}$$

or, in matrix notation, $\tilde{x}' = A(t)\,\tilde{x} + \tilde{q}(t)$ with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{c_1(t)}{c_2(t)} & -\frac{c_0(t)}{c_2(t)} \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} 0 \\ \frac{q(t)}{c_2(t)} \end{pmatrix}.$$

This system results from defining the second component $y(t)$ in $\tilde{x}(t) = \operatorname{col}(x(t), y(t))$ to be the derivative $y = x'$. By applying the Extended Fundamental Existence and Uniqueness Theorem 5.1 in 5.2 to this first order system, we obtain the following theorem about the second order equation (7.1.

Theorem 7.1 ***Assume*** $c_0(t), c_1(t), c_2(t)$ ***and*** $q(t)$ ***are continuous and*** $c_0(t)$ ***does not equal*** 0 ***anywhere on the interval*** $\alpha < t < \beta$. ***Then the initial value problem, with*** $\alpha < t_0 < \beta$***,***

$$\begin{gathered} c_2(t)\,x'' + c_1(t)\,x' + c_0(t)\,x = q(t) \\ x(t_0) = x_0, \quad x'(t_0) = x_1 \end{gathered}$$

has a unique solution and it is defined on the entire interval $\alpha < t < \beta$***.***

From the results in Section 5.2 about the structure of the general solution of linear systems, we know that the general solution of the equivalent system to the second order equation (7.1) has the additive decomposition

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t)$$

or, in component notation,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_h(t) \\ y_h(t) \end{pmatrix} + \begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix}.$$

If

$$\Phi(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}$$

is a fundamental solution matrix of the equivalent system, then the general solution has the form

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{pmatrix}. \end{aligned}$$

Remark 10. Note that

$$\Phi(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix}.$$

Since the two columns of $\Phi(t)$ are independent, it follows that the two functions $x_1(t)$ and $x_2(t)$ are independent. Why? Because if they were not, then $c_1 x_1(t) + c_2 x_2(t) \equiv 0$ for some constants c_1, c_2 not both 0. But then, by differentiation, it would follow that $c_1 x_1'(t) + c_2 x_2'(t) \equiv 0$. These two identities imply

$$c_1 \begin{pmatrix} x_1(t) \\ x_1'(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} c_1 x_1(t) + c_2 x_2(t) \\ c_1 x_1'(t) + c_2 x_2'(t) \end{pmatrix} \equiv 0$$

which contradicts the independence of the columns of $\Phi(t)$.

Theorem 7.2 ***Assume*** $c_0(t), c_1(t), c_2(t)$ ***and*** $q(t)$ ***are continuous and*** $c_0(t)$ ***does not equal*** 0 ***anywhere on the interval*** $\alpha < t < \beta$. ***The general solution of the linear second order equation (7.1) has the form***

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$ *is a linear combination of two independent solutions* $x_1(t)$ ***and*** $x_2(t)$ ***of the associated homogeneous equation***

$$c_2(t) x'' + c_1(t) x' + c_0(t) x = 0 \tag{7.2}$$

and $x_p(t)$ ***is any particular solution of the nonhomogeneous equation.***

Example 7.1 *Check that* $x_1(t) = \cos(\ln t)$ *and* $x_2(t) = \sin(\ln t)$ *are, for* $t > 0$, *solutions of the homogeneous equation*

$$t^2 x'' + t x' + x = 0$$

and therefore the general solution is

$$x_h(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t).$$

The general solution of the nonhomogeneous equation

$$t^2x'' + tx' + x = \ln t$$

is $x(t) = c_1\cos(\ln t) + c_2\sin(\ln t) + x_p(t)$ *where* $x_p(t)$ *is any particular solution. A formula for a particular solution can be calculated from the Variation of Constants formula*

$$\begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} = \Phi(t)\int^t \Phi^{-1}(u)\,\tilde{q}(u)\,du$$

for the equivalent system

$$\begin{aligned} x' &= y \\ y' &= --\frac{1}{t^2}x - \frac{1}{t}y + \frac{1}{t^2}\ln t \end{aligned}$$

for which

$$\Phi(t) = \begin{pmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} 0 \\ \frac{1}{t^2}\ln t \end{pmatrix}.$$

A calculation yields

$$\begin{aligned} \begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} &= \begin{pmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{pmatrix}\int^t \begin{pmatrix} \cos(\ln u) & \sin(\ln u) \\ -\frac{\sin(\ln u)}{u} & \frac{\cos(\ln u)}{u} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \frac{1}{u^2}\ln u \end{pmatrix} du \\ &= \begin{pmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{pmatrix}\int^t \begin{pmatrix} \cos(\ln u) & -u\sin(\ln u) \\ \sin(\ln u) & u\cos(\ln u) \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{u^2}\ln u \end{pmatrix} du \\ &= \begin{pmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{pmatrix}\int^t \begin{pmatrix} -\frac{\sin(\ln u)}{u}\ln u \\ \frac{\cos(\ln u)}{u}\ln u \end{pmatrix} du \\ &= \begin{pmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{pmatrix}\begin{pmatrix} (\ln t)\cos(\ln t) - \sin(\ln t) \\ \cos(\ln t) + (\ln t)\sin(\ln t) \end{pmatrix} \\ &= \begin{pmatrix} \ln t \\ \frac{1}{t} \end{pmatrix}. \end{aligned}$$

Hence $x_p(t) = \ln t$ *and*

$$x(t) = c_1\cos(\ln t) + c_2\sin(\ln t) + \ln t.$$

There is no formula or method available to find solution formulas for the general homogeneous equation (7.2). There are methods for special types of coefficients $c_i(t)$, however. The most important case is when all three coefficients are constants. In this case the equivalent 2nd order system is autonomous and formulas for solutions can be found by the methods of Chapter 6. There is a much shorter method, however, which we consider in the next Section 7.1..

Once two independent solutions of the homogeneous equation (7.2) have been found, then a particular solution $x_p(t)$, and hence the general solution, can be found from the equivalent first order system by means of the Variation of Constants Formula, as in Example 7.1. However, for special types of forcing functions $q(t)$, the Method of Undetermined Coefficients provides a significant shortcut, as we see in Section 7.2.

7.1 Homogeneous 2nd Order Equations with Constant Coefficients

For the homogeneous 2nd order differential equation

$$c_2x'' + c_1x' + c_0x = 0 \tag{7.3}$$

with constant coefficients c_i there is a significant shortcut method available for calculating two independent solutions and hence the general solution. The coefficient matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{c_1}{c_2} & -\frac{c_0}{c_2} \end{pmatrix}$$

of the equivalent autonomous homogeneous system has characteristic equation

$$\lambda^2 + \frac{c_0}{c_2}\lambda + \frac{c_1}{c_2}$$

whose roots are the roots of

$$c_2\lambda^2 + c_0\lambda + c_1 = 0. \tag{7.4}$$

We call this the characteristic equation of the second order equation (7.3). Notice that it can be easily determined directly from the equation (7.3) without any need to refer to the matrix A or the equivalent first order system.

Consider the case when the characteristic equation (7.4) as two real and unequal roots λ_1 and λ_2, i.e. the case when the matrix A has two two real and unequal eigenvalues λ_1 and λ_2. The equivalent first order system has the two independent solution pairs $e^{\lambda_1 t}\tilde{v}_1$ and $e^{\lambda_2 t}\tilde{v}_2$ where $\tilde{v}_1$ and $\tilde{v}_2$ are eigenvectors associated with λ_1 and λ_2 respectively. We calculate these eigenvectors by solving

$$(A - \lambda I)\,\tilde{v} = \tilde{0}$$

$$\begin{pmatrix} -\lambda & 1 \\ -\frac{c_1}{c_2} & -\frac{c_0}{c_2} - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -\lambda v_1 + v_2 &= 0 \\ -\frac{c_1}{c_2}v_1 + \left(-\frac{c_0}{c_2} - \lambda\right) v_2 &= 0 \end{aligned}$$

for v_1 and v_2 not both 0. Since λ is an eigenvalue, these two algebraic equations are dependent and we need only solve one of them, say the first one:

$$v_1 = 1, \quad v_2 = \lambda.$$

Thus, two independent solution pairs of the equivalent first order system are

$$\begin{pmatrix} e^{\lambda_1 t} \\ \lambda_1 e^{\lambda_1 t} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{\lambda_2 t} \\ \lambda_2 e^{\lambda_2 t} \end{pmatrix}.$$

This implies that two independent solutions of the second order equation (7.3) are

$$e^{\lambda_1 t} \quad \text{and} \quad e^{\lambda_2 t}$$

and the general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

The shortcut we alluded to above is as follows. Directly from the second order equation (7.3) we write down the characteristic equation (7.4), find its two roots which, if real and unequal, lead to two independent exponential solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. The general solution is a linear combination of these two exponential solutions.

To utilize this shortcut, there is no need to consider the equivalent first order system and its eigenvalues and eigenvectors at all. Similar shortcuts are possible for the other two algebraic situations as well, that is, when the characteristic equation (7.4) has complex roots or has a double real roots. The shortcuts are summarized in Table 8.1. In all cases, there is no need to consider the equivalent first order system and its eigenvalues and eigenvectors.

Roots of the characteristic equation $c_2\lambda^2 + c_1\lambda + c_0 = 0$	General solution of the equation $c_2 x'' + c_1 x' + c_0 x = 0$
Two real, distinct roots: $\lambda_1 \neq \lambda_2$	$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
Complex conjugate roots: $\lambda = \alpha \pm i\beta$	$c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$
A double real roots: λ	$c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

Table 8.1

With a little bit of practice, this table will be committed to memory and you will be able to calculate the general solution of an 2nd order linear homogenous equation as quickly as you can algebraically solve for the roots of its characteristic quadratic polynomial.

Example 7.2 *The characteristic equation associated with*

$$x'' + x = 0$$

is

$$\lambda^2 + 1 = 0.$$

The roots of this equation are complex $\lambda = \pm i$. From Table 8.1 (with $\alpha = 0$ and $\beta = 1$) we obtain the general solution

$$x(t) = c_1 \cos t + c_2 \sin t.$$

More generally, the characteristic equation $m\lambda^2 + k = 0$ associated with the equation

$$mx'' + kx = 0$$

has roots

$$\lambda = \pm\omega i \quad \textit{where} \quad \omega = \sqrt{\frac{k}{m}}.$$

Table 8.1 gives the general solution

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t.$$

Example 7.3 *Consider the equation*

$$x'' + 3x' + 2x = 0.$$

The roots of its characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0$$

are $\lambda = -1$ *and* -2. *Table 8.1 gives the general solution*

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

Example 7.4 *Consider the equation*

$$x'' + 4x' + 4x = 0.$$

The characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

has a double root $\lambda = -2$. *Table 8.1 gives the general solution*

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

7.2 Undetermined Coefficients for 2nd Order Equations

The general solution of the 2nd order equation

$$c_2 x'' + c_1 x' + c_0 x = q(t),$$

with constant coefficients c_i, has the additive decomposition

$$x(t) = x_h(t) + x_p(t).$$

Since the coefficients c_i are constants, $x_h(t)$ can be calculated from Table 8.1. A particular solution $x_p(t)$ can, of course, be calculated from the Variation of Constants formula for the equivalent first order system (as in Example 7.1). However, if the nonhomogeneous (forcing) term $q(t)$ is of an appropriate type, then the same Method of Undetermined Coefficients that we used on first order equations in Chapter 2.2.1 can be used to calculate a particular solution $x_p(t)$. In a nutshell, here is the method:

> *Calculate all independent functions created by repeated differentiations of* $q(t)$. *If no solution of the associated homogeneous equation appears in the list, then use a linear combination of the list for* $x_p(t)$. *If, on the other hand, a solution of the homogeneous equation does appear in the list, then multiple all functions in the list by* t *and use a linear combination of this modified list for* x_p.

Recall that $q(t)$ is of an appropriate type for this method if and only if it generates only a finite number of independent functions upon repeated differentiation. The only change that might occur when applying this method to second order equations is that you might (in rare cases) need to multiply the list generated from differentiating $q(t)$ by t twice (until there are no longer any solutions of the homogeneous equation appearing in the list).

Example 7.5 *As we saw by use of Table 8.1 in Example 7.2, the general solution of the associated homogeneous equation to*

$$x'' + x = \cos 2t$$

is

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The nonhomogeneous term $q(t) = \cos 2t$ generates only two independent functions upon repeated differentiation:

$$\cos 2t, \quad \sin 2t$$

neither of which solves the associated homogeneous equation. Therefore, we search for a particular solution in the form

$$x_p(t) = k_1 \cos 2t + k_1 \sin 2t$$

by using the general Method of Undetermined Coefficients described in Section 2.3.1 in Chapter 2. Namely, we substitute this guess for $x_p(t)$ into the nonhomogeneous equation in order to calculate the undetermined coefficients k_1 and k_2 . The details go as follows. Substituting into the left side of the equation we get

$$x_p''(t) + x_p(t) = -3k_1 \cos 2t - 3k_1 \sin 2t.$$

We want this equal to $\cos 2t$ (the right side of the equation). Clearly this is done (and can only be done) by choosing $k_1 = -1/3$ and $k_2 = 0$. Thus, we arrive at

$$x_p(t) = -\frac{1}{3} \cos 2t$$

and the general solution

$$x(t) = c_1 \cos t + c_2 \sin t - \frac{1}{3} \cos 2t.$$

Example 7.6 *As we saw by use of Table 8.1 in Example 7.3, the general solution of the associated homogeneous equation to*

$$x'' + 3x' + 2x = e^t$$

is

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

No new and independent functions are created by repeated differentiations of $q(t) = e^t$ *and therefore the list consists of this single function (which is not a solution of the associated homogeneous equation). Therefore, we construct*

$$x_p(t) = ke^t$$

and calculate the single undetermined coefficient by substituting this guess into the nonhomogeneous equation. The left side of the equation results in

$$x_p''(t) + 3x_p'(t) + 2x_p(t) = 6ke^t$$

which equals e^t *(the right side of the differential equation) if and only if* $k = 1/6$*. Thus,*

$$x_p(t) = \frac{1}{6}e^t$$

and the general solution is

$$x(t) = c_1e^{-t} + c_2e^{-2t} + \frac{1}{6}e^t.$$

Example 7.7 *As we saw by use of Table 8.1 in Example 7.2, the general solution of the associated homogeneous equation to*

$$mx'' + kx = \alpha \sin \beta t$$

where m, k, α *and* β *are positive constants, is*

$$x_h(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

The nonhomogeneous term $q(t) = \cos 2t$ *generates only two independent functions upon repeated differentiation:*

$$\cos \beta t, \quad \sin \beta t.$$

Case 1: *If* $\beta \neq \omega$ *then the list contains no solution of the homogeneous equation and we construct*

$$x_p(t) = k_1 \cos \beta t + k_2 \sin \beta t.$$

Substituting this into the left side of the nonhomogeneous differential equation, we obtain

$$mx_p''(t) + kx_p(t) = \left(k - m\beta^2\right) k_1 \cos \beta t + \left(k - m\beta^2\right) k_2 \sin \beta t$$

which we want equal to $\alpha \cos \beta t$ *(the right side of the equation. This is possible if and only if we choose* k_1 *and* k_2 *so that*

$$\left(k - m\beta^2\right) k_1 = \alpha, \quad \textit{and} \quad \left(k - m\beta^2\right) k_2 = 0$$

or

$$k_1 = \alpha \frac{1}{k - m\beta^2}, \quad k_2 = 0.$$

Note that the denominator

$$k - m\beta^2 = m\left(\omega^2 - \beta^2\right) \neq 0$$

is nonzero in the case under consideration. Thus, we have obtained

$$x_p(t) = \frac{\alpha}{k - m\beta^2}\cos\beta t$$

and the general solution

$$x(t) = c_1\cos\omega t + c_2\sin\omega t + \frac{\alpha}{k - m\beta^2}\cos\beta t$$

Case 2: *If $\beta = \omega$ then the list generated by differentiating $q(t)$ becomes*

$$\cos\omega t, \quad \sin\omega t$$

both of which are solutions of the homogeneous equation. Therefore, we multiply the list through by t and use the modified list

$$t\cos\omega t, \quad t\sin\omega t$$

(which no longer contains a solution of the homogeneous equation) to construct

$$x_p(t) = k_1 t\cos\beta t + k_2 t\sin\beta t.$$

Substituting this guess into the left side of the nonhomogeneous differential equation, we obtain

$$mx_p'' + kx_p = 2m\omega k_2\cos\omega t - 2m\omega k_1\sin\omega t$$

which we want equal to $\alpha\cos\omega t$ (the right side of the equation, in this case). This is possible if and only if we choose k_1 and k_2 so that

$$2m\omega k_2 = \alpha \quad \textit{and} \quad 2m\omega k_1 = 0$$

or

$$k_1 = 0 \quad \textit{and} \quad k_2 = \frac{\alpha}{2m\omega}.$$

Thus, we have obtained

$$x_p(t) = \frac{\alpha}{2m\omega}t\sin\beta t$$

and the general solution

$$x(t) = c_1\cos\omega t + c_2\sin\omega t + \frac{\alpha}{2m\omega}t\sin\beta t.$$

7.3 Exercises

For each second order equation below, determine those initial conditions t_0, x_0 and y_0 for which Theorem 7.1 applies. Explain your answer. What do you conclude for these initial conditions? What do you conclude for other initial conditions?

Exercise 7.1 $x'' + x = \sin t$ *(a forced simple harmonic oscillator)*

Exercise 7.2 $x'' + x' + x = \cos t$ *(a forced oscillator with friction)*

Exercise 7.3 $t^2x'' + tx' + x = 0$ *(a Legendre equation)*

Exercise 7.4 $x'' + px' + qx^3 = 0$ *where p and q are constants (Duffing equation)*

Exercise 7.5 $x'' + \alpha x' + x = \beta \sin \theta t$ *where α, β and θ are constants (a forced van der Pol equation)*

Exercise 7.6 $mx'' + k \sin x = 0$, *where $m \neq 0$ and k are constants (the frictionless pendulum equation)*

Find a formula for the general solution of the following second order equations.

Exercise 7.7 $x'' + x' + x = 0$

Exercise 7.8 $x'' - x' + x = 0$

Exercise 7.9 $2x'' - x = 0$

Exercise 7.10 $x'' + 2x' + x = 0$

Exercise 7.11 $x'' + 3x' - 4x = 0$

Exercise 7.12 $x'' - 5x' + 4x = 0$

Exercise 7.13 $x'' + 5x = 0$

Exercise 7.14 $x'' + 3.8x' + 3.45x = 0$

Exercise 7.15 $x'' - 6x' + 9x = 0$

Exercise 7.16 $3x'' + 2x = 0$

Exercise 7.17 *When the roots of the characteristic equation are complex $\lambda = \alpha \pm \beta i$, $\beta \neq 0$, prove the general solution of the second order equation $ax'' + bx' + cx = 0$ is $x = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$.*

Exercise 7.18 *When the characteristic equation has double root λ, verify that the general solution of the second order equation $ax'' + bx' + cx = 0$ is $x = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$.*

Exercise 7.19 *Consider the equation $mx'' + cx' + kx = 0$ for $m > 0$, $c > 0$, $k > 0$. Show all solutions tend to 0 as $t \to +\infty$.*

Exercise 7.20 *Consider the initial value problem*

$$\begin{gathered} mx'' + cx' + kx = 0 \\ x(0) = 0, \quad x'(0) = v_0 \neq 0 \end{gathered}$$

for the suspension system of an automobile.

(a) Find a formula for the cut-off value c_0 of the damping constant c such that $c > c_0$ implies no oscillation occurs and $c < c_0$ implies oscillations do occur. Justify your answer using formulas for the solutions in each case.

(b) Find a formula for the solution of the initial value problem when $c = c_0$. Does the car frame oscillate in this case? Explain your answer.

Consider the second order equation $t^2x'' - 2tx' + 2x = t^2$ for $t > 0$.

Exercise 7.21 *Find two independent solutions (for $t > 0$) of the associated homogeneous equation $t^2x'' - 2tx' + 2x = 0$ of the form $x = t^m$ for a constant m.*

Exercise 7.22 *Find the general solution (for $t > 0$) of the nonhomogeneous equation by using the Variation of Constants Formula for the equivalent first order system.*

Exercise 7.23 *Find a formula for the solution of the initial value problem $x(1) = 0$, $x'(1) = 0$.*

Exercise 7.24 *Find a formula for the solution of the initial value problem $x(1) = 1$, $x'(1) = -1$.*

Find a particular solution of the following second order equations using the Method of Undetermined Coefficients.

Exercise 7.25 $x'' + x = e^{-t} \sin t$

Exercise 7.26 $x'' + x = te^{-t}$

Exercise 7.27 $x'' - x = e^t$

Exercise 7.28 $x'' - x = e^{-t}$

Exercise 7.29 *For the second order equations in Exercises 7.25-7.28 solve the initial value problem $x(0) = 0$, $x'(0) = 0$.*

Consider the initial value problem

$$\begin{gathered} x'' + 2x' + 2x = e^{-t} \\ x(0) = 0, \quad x'(0) = 0. \end{gathered}$$

Exercise 7.30 *Use a computer program to plot the solution for* $0 < t < 8$. *How many roots does* $x(t)$ *have in this interval?*

Exercise 7.31 *Solve the initial value problem using the Method of Undetermined Coefficients.*

Exercise 7.32 *Use your answer in Exercise 7.31 to validate or correct your answers in Exercise 7.30.*

Consider the initial value problem

$$\begin{aligned} Lx'' + Rx' + \tfrac{1}{C}x &= E(t) \\ x(0) = 0, \qquad x'(0) &= 0 \end{aligned}$$

for the charge $x = x(t)$ on an electric circuit with a resistor (of R Ohms), inductor (of L Henrys), capacitor (of C Farads) and impressed voltage (e.g., from a battery). The circuit has no initial charge $x\,(0)$ and no initial current $x'(0)$. Suppose $E(t) = E_0$ is constant. Solve the initial value problem for the circuits below.

Exercise 7.33 $L = 0.1,\ R = 250,\ C = 10^{-5}$

Exercise 7.34 $L = 0.2,\ R = 200,\ C = 10$

Exercise 7.35 $L = 0.1,\ R = 0,\ C = 10^{-5}$

Exercise 7.36 $L = 0.1,\ R = 200,\ C = 10^{-5}$

Solve the initial value problem

$$\begin{aligned} Lx'' + Rx' + \tfrac{1}{C}x &= E(t) \\ x(0) = 0, \qquad x'(0) &= 0 \end{aligned}$$

with $L = 1,\ R = 2,\ C = 1$ and the impressed voltages below.

Exercise 7.37 $E(t) = \sin t$

Exercise 7.38 $E(t) = \cos t$

Exercise 7.39 $E(t) = e^{-t}$

Exercise 7.40 $E(t) = te^{-t}$

Consider the second order equation $x'' - x = e^t$.

Exercise 7.41 *Use the Variation of Constants Formula for the equivalent first order system to find the general solution.*

Exercise 7.42 *Solve the initial value problem* $x(0) = 0,\ x'(0) = 0$.

Exercise 7.43 *Solve the initial value problem* $x(0) = 2$, $x'(0) = -2$.

Exercise 7.44 *Consider the nonhomogeneous equation* $mx'' + cx' + kx = \alpha \sin \omega t$ *where all five parameters* m, c, k, α *and* ω *are positive.*

(a) Find a formula for the general solution. Hint: For $x_p(t)$ *by using the Method of Undetermined Coefficients.*

(b) Show the general solution tends to $x_p(t)$ *as* $t \to +\infty$. *(Hint:* $x_h(t) = x(t) - x_p(t)$ *tends to* 0 *as* $t \to +\infty$.*) Thus,* $x_p(t)$ *is called the "steady state" and* $x_h(t)$ *is called the "transient part" of the general solution.*

Find a formula for the general solution of the 2nd order equations below.

Exercise 7.45 $x'' + 2x' - 3x = 6$

Exercise 7.46 $x'' + 2x' - 3x = -2$

Exercise 7.47 $x'' + 2x' - 3x = 2 \sin t$

Exercise 7.48 $x'' + 2x' - 3x = -e^{-t}$

Exercise 7.49 $x'' + 2x' - 3x = kt$, *k is a constant*

Exercise 7.50 $z'' + 2z' - 3z = k \cos t$, *k is a constant*

Exercise 7.51 $x'' + 2x' + 2x = \cos t$

Exercise 7.52 $x'' + 2x' + 2x = e^{-t} \sin t$

Exercise 7.53 $x'' + 2x' + 2x = 2 \cos t - e^{-t} \sin t$

Exercise 7.54 $x'' + 6x' + 5x = e^{-2t}$

Exercise 7.55 $x'' + 6x' + 5x = e^{at}$ *where a is a constant*

Exercise 7.56 $x'' + 6x' + 5x = te^{-t}$

Exercise 7.57 $x'' + 4x' + 4x = t$

Exercise 7.58 $x'' + 4x' + 4x = e^{at}$ *where a is a constant*

Exercise 7.59 $x'' + 4x' + 4x = 3t - e^{-t} + 2e^{-2t}$

Exercise 7.60 $x'' + k^2 x = e^{-t}$

Exercise 7.61 $x'' + k^2 x = \sin t$

Chapter 8

Nonlinear Systems

In Chapters 5 and 6 we studied systems of linear differential equations. In this chapter we turn our attention to systems of nonlinear equations. Since any higher order equation is equivalent to a first order system, our study includes nonlinear higher order equations. Unlike the case of single nonlinear equations, there are virtually no methods available for calculating solution formulas for nonlinear systems. Therefore, we must use other methods to analyze nonlinear systems. In this chapter we focus on methods for analyzing autonomous systems and their phase portraits.

8.1 Introduction

Consider the autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{8.1}$$

of two differential equations. As with linear systems consisting of two equations, the phase portrait of the nonlinear systems (8.1) consists of the orbits drawn in the (x, y)-plane. For this reason (8.1) is called a *planar autonomous system.* Our basic goal in this chapter is to develop methods for analyzing and drawing the phase plane portraits of planar autonomous systems.

We learned in Chapter 3 how to construct phase line portraits for single first order autonomous equations. We found that equilibria play a key role and that all non-equilibrium solutions are monotonic. Equilibria also play a key role in phase plane portraits of systems. However, drawing a phase plane portrait for a system of equations is generally more difficult than drawing a phase line portrait for a single equation. Solutions of autonomous systems are not necessarily monotonic, and the possibility of solution oscillations (and even periodic solutions) introduces a new features to phase portraits. We have already seen this in the case of spirals and centers for linear, planar autonomous systems (Chapter 6). Despite these added complications, mathematicians have developed a nearly complete theory of phase portraits for planar autonomous systems. A full exposition of this theory is, however, beyond the scope of this introductory book. Nonetheless, we will study the basic ingredients of the theory and learn how to use them to obtain accurate sketches of phase plane portraits. Of

course, numerical approximations and computer graphics can also play an important and helpful role in this endeavor.

Systems of three or more autonomous equations also arise in applications. Solutions and orbits of three (or higher) dimensional systems can exhibit extremely complicated dynamic behavior that is not possible for planar autonomous systems. So-called "strange attractors" and "chaos" are examples. For this reason (and because of the difficulty of drawing pictures in more than three dimensions), it is considerably more difficult to study phase portraits for higher dimensional systems. We will take only a brief look at higher dimensional systems in Section 8.6.

Here are some examples of nonlinear planar autonomous systems that arise in applications. The planar autonomous system

$$\begin{aligned} x' &= x - xy \\ y' &= 2y - 2xy \end{aligned} \tag{8.2}$$

is an model of competition between two populations or groups. This system is nonlinear because of the term xy. Planar autonomous systems often have coefficients (or "parameters"). For example, the system

$$\begin{aligned} x' &= r_1 x - axy \\ y' &= r_2 y - bxy \end{aligned} \tag{8.3}$$

is a more general competition system that includes (8.2) as a special case. The system

$$\begin{aligned} x' &= b - dx - c\frac{y}{x+y}x \\ y' &= c\frac{y}{x+y}x - (d+a)\,y \end{aligned} \tag{8.4}$$

is another example of a planar autonomous system, one that arises in an application to AIDS epidemics. This system is nonlinear because of the term $xy/(x+y)$. A third example of a nonlinear planar autonomous system is

$$\begin{aligned} x' &= r\left(1 - \frac{x}{K}\right)x - m\frac{xy}{a+x} \\ y' &= -dy + cm\frac{xy}{a+x}, \end{aligned} \tag{8.5}$$

a system that arises in theoretical ecology as a model of the interaction between a predator y and its prey x.

Planar autonomous systems also arise from autonomous second order equations. For example, the system

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha(x^2 - 1)y \end{aligned} \tag{8.6}$$

is equivalent to the van der Pol equation

$$x'' + \alpha\left(x^2 - 1\right)x' + x = 0 \tag{8.7}$$

which arises in applications involving electric circuits. This equation is nonlinear because of the term x^2x'.

An example of a nonlinear system of more than two equations is the Lorenz system

$$\begin{aligned} x' &= \sigma\,(y - x) \\ y' &= \rho x - y - xz \\ z' &= -\beta z + xy \end{aligned} \tag{8.8}$$

which arises in meteorological studies. It is nonlinear because of the terms xz and xy.

8.2 The Linearization Principle

In Chapter 3 we learned to draw phase line portraits for a single autonomous differential equation $x' = f(x)$. The equilibria (i.e., the roots of $f(x)$) and the signs of $f(x)$ between equilibria are enough to determine the entire phase line portrait. Equilibria also play an important role in the phase portraits of planar autonomous systems (8.1). Unfortunately, it is more difficult to determine the entire (global) phase portrait for systems – even systems of only two equations – than it is for a single equation.

We consider planar autonomous systems

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{8.9}$$

for which initial value problems

$$x(t_0) = x_0, \quad y(t_0) = y_0 \tag{8.10}$$

have unique solutions. To guarantee this we require that f and g and their first order derivatives with respect to x and y to be continuous on some domain of points D in the plane (Theorem 4.1 in Chapter 4)[1]. In some cases D may not be the entire plane. For example, in the AIDS system (8.4) f and g both are undefined when $y = -x$. In this system D can be one of half planes determined by this line.

We begin with some remarks about the basic features of phase plane portraits. Recall that the *orbit* associated with a solution pair $x = x(t)$, $y = y(t)$ is the set of points $\{(x(t), y(t))\}$ in the (x, y)-plane (Definition 4.2 in Chapter 4). In general an orbit is a curve in the plane. An orbit has a direction (or orientation) determined by the motion along the curve as t increases. Since initial value problems have unique solutions, an orbit passes through each point (x_0, y_0) in the domain D. To see this, simply use x_0 and y_0 as the initial condition in (8.10) and apply the Fundamental Existence Theorem 4.1; the orbit of the unique solution of this initial value problem passes through the point (x_0, y_0) at time $t = t_0$. Different initial times t_0 yield different initial value problems whose solutions produce the same orbit, except that they arrive at the point (x_0, y_0) at different times. See Exercise 8.67. This means there are infinitely many solutions associated with an orbit. (Recall this is true for single nonlinear

[1] A domain is an "open" set. This means each point in the domain can be surrounded by a small circular disk all of which lies in the domain. The inside of a circle or a rectangle are examples of domains.

equations and their phase line orbits as well.) Thus, *orbits "fill up" the domain D* in the sense that there is an orbit through every point in D. Furthermore, *different orbits cannot have a point in common and, in particular, cannot cross one another* (Exercise 8.68).

We now turn to the problem of determining the geometry of nonlinear phase portraits. Equilibria are of fundamental importance in solving this problem and we begin with their study. Recall that equilibria are constant solutions :

$$\begin{aligned} x(t) &= x_e = \text{constant} \\ y(t) &= y_e = \text{constant}. \end{aligned}$$

The orbit associated with an equilibrium is just the single point (x_e, y_e) in the (x, y)-phase plane. Since the derivative of a constant is zero, it follows that the equilibrium (x_e, y_e) points of (8.9) satisfy the two equations

$$\begin{aligned} 0 &= f(x_e, y_e) \\ 0 &= g(x_e, y_e). \end{aligned}$$

Thus, the equilibria of the planar autonomous system (8.9) are the roots of the "equilibrium equations"

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0. \end{aligned}$$

For nonlinear systems, these algebraic equations are nonlinear. As a result their solution is likely to be a difficult algebraic problem. In some applications one can solve the equilibrium equations explicitly. In other applications, one can use computers to approximate solutions

Example 8.1 *The equilibrium equations for the competition system (8.2) are*

$$\begin{aligned} x - xy &= 0 \\ 2y - 2xy &= 0 \end{aligned}$$

or

$$\begin{aligned} (1 - y)\, x &= 0 \\ 2\,(1 - x)\, y &= 0. \end{aligned}$$

There are two ways to solve the first equation: either $x = 0$ or $y = 1$. In the first case, the second equation implies $y = 0$. This gives the equilibrium $(x_e, y_e) = (0, 0)$. In the second case $y = 1$, the second equation implies $x = 1$. This gives the equilibrium $(x_e, y_e) = (1, 1)$.

The equilibrium equations for the system (8.3) are

$$\begin{aligned} r_1 x - axy &= 0 \\ r_2 y - bxy &= 0. \end{aligned}$$

Similar reasoning produces the two equilibria

$$(x_e, y_e) = (0, 0) \quad \textit{and} \quad \left(\frac{r_2}{b}, \frac{r_1}{a}\right).$$

Remark 2. Notice in Example 8.1 that the system (8.2) has more than one equilibrium (as does (8.3). This is not an uncommon feature of nonlinear systems.

Example 8.2 *The equilibrium equations for the planar autonomous system*

$$\begin{aligned} x' &= \sin x - y \\ y' &= x - 2y \end{aligned} \tag{8.11}$$

are

$$\begin{aligned} \sin x - y &= 0 \\ x - 2y &= 0. \end{aligned}$$

Solving the second equation for

$$y = \frac{1}{2}x \tag{8.12}$$

and substituting this result into the first equation, we obtain the equation

$$\sin x = \frac{1}{2}x \tag{8.13}$$

for x. Any solution x of this equation, together with (8.12), yields an equilibrium of the system (8.11).

One solution of equation (8.13) is rather easy to observe: $x = 0$. This yields the equilibrium $(x_e, y_e) = (0, 0)$. Are there any other equilibria? That is to say, are there any other solutions of the equation (8.13)?

It is not possible to solve equation (8.13) algebraically for x. However, from the intersection points of the graph of $\sin x$ with the graph of $x/2$ (see Figure 8.1) we see that there are two other solutions, one positive and one negative. Using a computer or calculator to approximate these solutions, we obtain $x \approx 1.8955$ and -1.8955. Thus, $(x_e, y_e) \approx (1.8955, 0.9478)$ and $(-1.895, -0.9478)$ are the remaining two equilibria of the system (8.11).

We can find the equilibria of a higher order differential equation by finding the equilibria of an equivalent first order system. Or, one can proceed more directly and derive an equilibrium equation by setting derivatives of all orders equal to zero in the differential equation.

For example, $x' = 0$ and $x'' = 0$ in the van der Pol equation (8.7) yields the equilibrium equation $x = 0$. Thus, the only equilibrium of this second order equation is $x_e = 0$. In the phase plane, for its equivalent system (8.6), this equilibrium corresponds to the point $(x_e, y_e) = (0, 0)$.

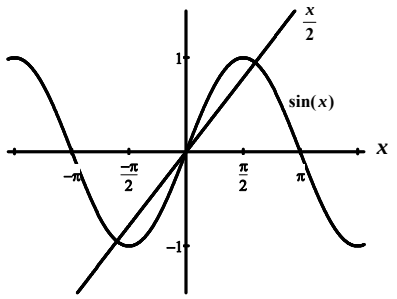


Figure 8.1

Systems of $n \geq 3$ equations can also have equilibrium solutions. The equilibrium equations are obtained from the differential system by setting all derivatives equal to zero. The result is a system of n algebraic equations whose solutions are the equilibria of the differential system.

Example 8.3 *The equilibrium equations for the Lorenz system*

$$\begin{aligned} x' &= \sigma(y-x) \\ y' &= \rho x - y - xz \\ z' &= -\beta z + xy \end{aligned}$$

are

$$\begin{aligned} \sigma(y-x) &= 0 \\ \rho x - y - xz &= 0 \\ -\beta z + xy &= 0. \end{aligned}$$

To find the equilibria of the Lorenz system we must solve these three algebraic equations for x, y and z. One way to do this is as follows. The first equilibrium equation implies $x = y$. Letting $x = y$ in remaining equations yields the two equations

$$\begin{aligned} \rho y - y - yz &= 0 \\ -\beta z + y^2 &= 0 \end{aligned}$$

or

$$\begin{aligned} (\rho - 1 - z)\, y &= 0 \\ -\beta z + y^2 &= 0 \end{aligned}$$

for y and z. The first of these equations leads to two alternatives: either $y = 0$ or $z = \rho - 1$.

Consider the first alternative $y = 0$. The second equation implies $z = 0$ and we obtain the equilibrium $(x_e, y_e, z_e) = (0, 0, 0)$.

Now consider the second alternative $z = \rho - 1$. In this case the second equation implies $y^2 = \beta(\rho - 1)$. If $\rho \geq 1$ this equation has solutions $y = \pm\sqrt{\beta(\rho-1)}$. Consequently, this alternative yields the equilibria

$$(x_e, y_e, z_e) = (\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho - 1)$$

when $\rho \geq 1$. Notice these two points coincide with each other (and equal the origin $(0, 0, 0)$) when $\rho = 1$.

To summarize: the Lorenz system (8.8) has one equilibrium if $\rho \leq 1$ and three equilibria if $\rho > 1$.

Once we have found the equilibria our next step, in the construction of a phase plane portrait, is to determine the properties of non-equilibrium orbits. In general this is a difficult task. The monotonicity property of solutions of single autonomous equations, which is so fundamental in the construction of phase line portraits for single autonomous equations, has no general counterpart in the two dimensional case of planar systems. However, one of the techniques we used for analyzing single equations in Chapter 3 does carry over to planar systems (and also to higher order systems), namely, the Linearization Principle.

By way of review, let's recall the Linearization Principle for single equations (see Section 3.1.3, Chapter 3). The linearization of an autonomous first order equation

$$x' = f(x)$$

centered at (or around) an equilibrium x_e (i.e., a root of the equation $f(x) = 0$) is the linear, homogeneous equation

$$u' = \lambda u, \qquad \lambda = \left.\frac{df}{dx}\right|_{x_e}. \tag{8.14}$$

Recall that the equilibrium x_e is called hyperbolic if $\lambda \neq 0$. The Linearization Principle states that the local phase portrait near an hyperbolic equilibrium x_e is of the same type as the equilibrium $u = 0$ of its linearization (8.14).

Our goal in this section is to extend this linearization principle to systems of autonomous first order equations. To this we first need to understand what the linearization of a system, carried out at an equilibrium, is and how to calculate it and then, secondly, learn how the local phase plane portrait near the equilibrium is related to the phase portrait of its linearization. Presumably, this will involve some requirement analogous to the requirement of hyperbolicity.

In Chapter 3 (section 3.1.3) we derived the linearization (8.14) of the scalar equation $x' = f(x)$ at an equilibrium $x = x_e$ from the linear Taylor polynomial approximation $f(x_e) + \lambda(x - x_e)$ to $f(x)$. We did this as follows. Using $f(x_e) = 0$ and the resulting (first order) Taylor approximation

$$f(x) \approx f(x_e) + \lambda(x - x_e) = \lambda(x - x_e),$$

together with the notation $u = x - x_e$, we obtained the linearization (8.14) of $x' = f(x)$ at the equilibrium x_e.

We proceed in a similar manner for a planar autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y). \end{aligned} \tag{8.15}$$

We first approximate $f(x, y)$ and $g(x, y)$ by linear Taylor series polynomials centered at an equilibrium (x_e, y_e). These polynomials are

$$\begin{aligned} &f(x_e, y_e) + a(x - x_e) + b(y - y_e) \\ &g(x_e, y_e) + c(x - x_e) + d(y - y_e). \end{aligned}$$

where coefficients are calculated from the derivatives of f and g by the formulas[2]

$$\begin{aligned} a &= \left.\frac{df}{dx}\right|_{(x_e, y_e)}, \qquad b = \left.\frac{df}{dy}\right|_{(x_e, y_e)} \\ c &= \left.\frac{dg}{dx}\right|_{(x_e, y_e)}, \qquad d = \left.\frac{dg}{dy}\right|_{(x_e, y_e)}. \end{aligned} \tag{8.16}$$

[2]This notation means the first the derivative is calculated and then the answer is evaluated at $(x, y) = (x_e, y_e)$.

Since $f(x_e, y_e) = 0$, $g(x_e, y_e) = 0$, the Taylor polynomial approximations become

$$f(x, y) \approx a\,(x - x_e) + b\,(y - y_e)$$
$$g(x, y) \approx c\,(x - x_e) + d\,(y - y_e)\,.$$

Using the notation

$$u = x - x_e, \qquad v = y - y_e,$$

we obtain the autonomous, linear homogeneous system

$$\begin{aligned} u' &= au + bv \\ v' &= cu + dv \end{aligned} \tag{8.17}$$

as an approximation to the system (8.15). This autonomous, linear homogeneous system, with coefficients given by the formulas (8.16), is called the *linearization of (8.15) at the equilibrium* (x_e, y_e).

In matrix notation, the linearization (8.17) is the linear homogeneous system $\tilde{u}' = A\tilde{u}$ with coefficient matrix

$$A = \begin{pmatrix} \left.\frac{df}{dx}\right|_{(x_e,y_e)} & \left.\frac{df}{dy}\right|_{(x_e,y_e)} \\ \left.\frac{dg}{dx}\right|_{(x_e,y_e)} & \left.\frac{dg}{dy}\right|_{(x_e,y_e)} \end{pmatrix}.$$

The matrix

$$J(x, y) = \begin{pmatrix} \frac{df(x,y)}{dx} & \frac{df(x,y)}{dy} \\ \frac{dg(x,y)}{dx} & \frac{dg(x,y)}{dy} \end{pmatrix}$$

is called the *Jacobian* of the nonlinear system (8.15). With this notation, the coefficient matrix of the linearization at an equilibrium (x_e, y_e) point is $J(x_e, y_e)$ and the linearization of the system at (x_e, y_e) is

$$\tilde{u}' = J(x_e, y_e)\tilde{u}.$$

A nonlinear planar autonomous system may have more than one equilibrium (e.g., see Example 8.1). We cannot refer to "the" linearization of a planar autonomous system, but instead must refer to the linearization *at an equilibrium*.

Example 8.4 *In Example 8.1 we found that the competition system*

$$x' = x - xy$$
$$y' = 2y - 2xy$$

has two equilibria: $(x_e, y_e) = (0, 0)$ *and* $(1, 1)$. *The Jacobian matrix for this system is*

$$J(x, y) = \begin{pmatrix} 1 - y & -x \\ -2y & 2 - 2x \end{pmatrix}.$$

At the equilibrium $(0, 0)$

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

which is the coefficient matrix of the linearization

$$u' = u$$
$$v' = 2v$$

at $(0,0)$. *The Jacobian at the equilibrium* $(1,1)$ *is*

$$J(1,1) = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$$

which is the coefficient matrix of the linearization

$$u' = -v$$
$$v' = -2u$$

at $(1,1)$.

We can also linearize second order equations at an equilibrium. The next example illustrates this.

Example 8.5 *The van der Pol equation*

$$x'' + \alpha\left(x^2 - 1\right)x' + x = 0$$

has equilibrium $x_e = 0$. *The equivalent system*

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha(x^2 - 1)y \end{aligned} \tag{8.18}$$

has equilibrium $(x_e, y_e) = (0,0)$. *The Jacobian of this system is*

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -1 - 2\alpha xy & -\alpha(x^2 - 1) \end{pmatrix}.$$

At the equilibrium

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$$

which is the coefficient matrix of the linearization

$$\begin{aligned} u' &= v \\ v' &= -u + \alpha v \end{aligned} \tag{8.19}$$

at $(0,0)$. *This first order system is equivalent to the linear second order equation*

$$u'' - \alpha u' + u = 0,$$

which is the linearization of the van der Pol equation at $x_e = 0$.

Now that we have learned how to linearize a system of differential equations at an equilibrium, we ask:

> What can an we learn anything about the system's orbits and its phase portrait from those of its linearization?

We studied phase portraits for linear homogeneous systems with constant coefficients in Chapter 6. Using the classification scheme developed in that chapter, we can identify the phase portrait type of the linearization (8.17). Since the linearization is an approximation to the nonlinear system for u and v small, *we anticipate that the phase portrait of the nonlinear system will resemble that of its linearization* – at least in a neighborhood of the equilibrium. Before examining some fundamental theorems that support this conclusion, we look at some examples.

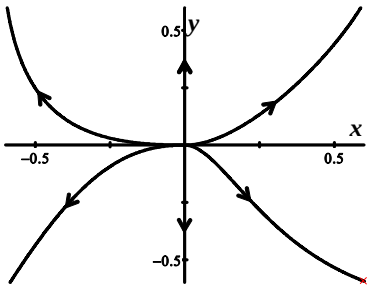


Figure 8.2
The phase plane portrait of (8.20) near the equilibrium $(x_e, y_e) = (0, 0)$.

Example 8.6 *Figure 8.2 shows plots of several orbits of the competition system*

$$\begin{aligned} x' &= x - xy \\ y' &= 2y - 2xy \end{aligned} \tag{8.20}$$

in a magnified neighborhood of the equilibrium $(0, 0)$*. These orbits show that near* $(0, 0)$ *the phase plane portrait appears very much like an unstable node for a linear system. In fact, from Example 8.4 we see that the phase portrait of the linearization at the origin is indeed an unstable node! This is because the eigenvalues* $\lambda = 1,\ 2$ *of* $J(0, 0)$*) are real and positive.*

Example 8.7 *As a second motivational example, consider the van der Pol equation*

$$x'' + \alpha \left(x^2 - 1\right) x' + x = 0.$$

which we looked at in Example 8.5. In Figure 8.3 appear graphs of orbits near the equilibrium $(0, 0)$ *for* $\alpha = -1$ *and for* 1*. The phase portrait resembles a stable spiral for* $\alpha = -1$ *and an unstable spiral for* $\alpha = 1$*.*

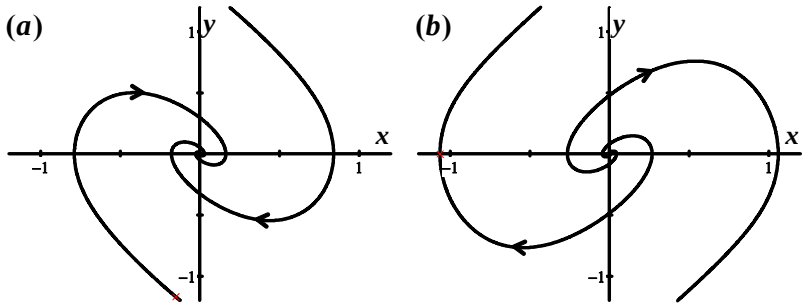


Figure 8.3. The phase plane portraits of the van der Pol equation near the equilibrium $(0,0)$ for (a) $\alpha = -1$ and (b) $\alpha = 1$.

What about the phase portrait of the linearization (8.19) at $(0,0)$? In Example 8.5 we calculated the Jacobian of the equivalent linear system and found that, when evaluated at the equilibrium $(0,0)$, the resulting matrix

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}.$$

This is the coefficient matrix of the linearization at $(0,0)$ and it has eigenvalues

$$\lambda = \frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4}.$$

Notice that λ is complex for values of α satisfying $|\alpha| < 2$. For such values of α the linearization at the origin does indeed have a spiral phase portrait. Moreover, it is a stable spiral if the $\mathrm{Re}\,\lambda = \alpha/2 < 0$ and unstable if $\mathrm{Re}\lambda = \alpha/2 > 1$. In particular $\alpha = -1$ the linearization has a stable spiral and for $\alpha = 1$ it has an unstable spiral. *Thus, the linearization and the computer generated (local) phase portraits do match!*

In the examples of Figures 7.2 and 7.3 the stability properties of the equilibria are the same as those of the linearization at the equilibrium. (So are the geometric properties of orbits, but first we focus just on stability properties). These are examples of the following general *Linearization Principle* for planar autonomous systems.

Theorem 8.1 ***(The Fundamental Theorem of Stability) Suppose*** (x_e, y_e) ***is an equilibrium of the autonomous system***

$$x' = f(x,y)$$
$$y' = g(x,y)$$

and let λ_1, λ_2 ***denote the eigenvalues of the Jacobian*** $J(x_e, y_e)$.

If λ_1 ***and*** λ_2 ***both have negative real parts, then the equilibrium*** (x_e, y_e) ***is (locally asymptotically) stable***[3].

If at least one root has a positive real part, then the equilibrium is unstable.

[3]Note: real roots are equal to their own real part.

A number λ with a negative real part is said to lie in the left half of the complex number plane. Thus, an equilibrium is stable if both roots of the characteristic polynomial at the equilibrium lie in the left half plane. It is unstable if at least one root lies in the right half plane.

Recall the trace-determinant criteria for stability and instability of a linear homogeneous system (Section 6.6 of Chapter 6). When these criteria are applied to the linearization of a nonlinear system, then from Theorem 8.1 we obtain the following result .

Theorem 8.2 ***An equilibrium*** (x_e, y_e) ***of the autonomous system***

$$x' = f(x, y)$$
$$y' = g(x, y)$$

is (locally asymptotically) stable if both inequalities

$$\frac{df}{dx} + \frac{dg}{dy} < 0 \quad \textbf{\textit{and}} \quad \frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} > 0$$

hold at the point $(x, y) = (x_e, y_e)$.

If

$$\frac{df}{dx} + \frac{dg}{dy} > 0 \quad or \quad \frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} < 0$$

at $(x, y) = (x_e, y_e)$, ***then the equilibrium is unstable.***

Example 8.8 *The trace and determinant of the Jacobian associated with the system*

$$x' = x - xy$$
$$y' = 2y - 2xy$$

are

$$\frac{df}{dx} + \frac{dg}{dy} = 3 - y - 2x$$
$$\frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} = 2 - 2x - 2y.$$

For the equilibrium $(x, y) = (0, 0)$ *we calculate that*

$$\frac{df}{dx} + \frac{dg}{dy} = 3 > 0.$$

and conclude by Theorem 8.2 that the equilibrium $(x_e, y_e) = (0, 0)$ *is unstable.*

For the equilibrium $(x, y) = (1, 1)$ *we calculate that*

$$\frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} = -2 < 0$$

and conclude by Theorem 8.2 that also the equilibrium $(x_e, y_e) = (1, 1)$ *is unstable.*

Example 8.9 *Consider the van der Pol system (8.6)*

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha(x^2 - 1)y. \end{aligned}$$

for which

$$\begin{aligned} \frac{df}{dx} + \frac{dg}{dy} &= -\alpha(x^2 - 1) \\ \frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} &= 1 + 2\alpha xy. \end{aligned}$$

At the equilibrium $(x_e, y_e) = (0, 0)$

$$\begin{aligned} \frac{df}{dx} + \frac{dg}{dy} &= \alpha \\ \frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} &= 1 > 0. \end{aligned}$$

By Theorem 8.2 the equilibrium $(x_e, y_e) = (0, 0)$ *is stable if* $\alpha < 0$ *and unstable if* $\alpha > 0$.

Figures 8.2 and 8.3 illustrate that the stability properties of the equilibrium and the linearization agree in those examples. These Figures also illustrate more, namely, that the geometry of the phase portraits are also similar. Specifically, the graph in Figure 8.2 looks like an unstable node for a linear system and the graphs in Figure 8.3 look like spirals for linear systems.

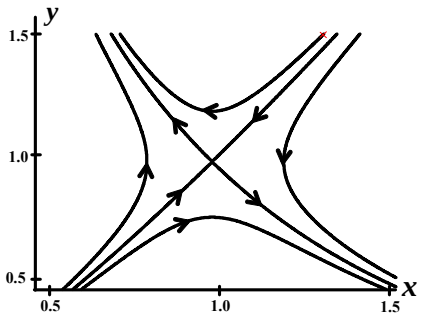


Figure 8.4. The phase plane portrait of (8.20) near the equilibrium $(x_e, y_e) = (1, 1)$.

Here is another example involving system (8.20) in Examples 8.6 and 8.8. In Figure 8.4 appear several orbits in a magnified neighborhood of the equilibrium $(x_e, y_e) = (1, 1)$. Near this equilibrium, the phase portrait is very much like a saddle for a linear system. Indeed, from Example 8.4 we find that the phase portrait of the linearization at $(1, 1)$ is a saddle. This is because the eigenvalues $\lambda = \pm\sqrt{2}$ of the Jacobian $J(1, 1)$ have opposite signs. Note that one root, $\lambda = \sqrt{2}$, is positive and Theorem 8.1 implies $(1, 1)$ is unstable. It appears that not only can one learn about stability and instability of hyperbolic equilibria from the linearization, but that one can also learn about the geometry of the phase portrait. This is the subject of the next section.

8.3 Local Phase Plane Portraits

We classified phase portraits of *linear* systems in Chapter 6, Section 6.4. The examples in Figures 8.2, 8.3 and 8.4 suggest that the phase portrait of a nonlinear system near an equilibrium is similar to that of the linearization (at that equilibrium).

The question is: will it always true for a nonlinear system of equations that the phase portrait near an equilibrium is a node if the phase portrait of the linearization at the equilibrium is a node? Will the phase portrait near an equilibrium be a spiral if that of its linearization at the equilibrium is a spiral? And so on.

To give an answer to these questions about comparing linear phase plane portraits to nonlinear phase plane portraits, we must first consider what we mean by a node and a spiral for a nonlinear system.

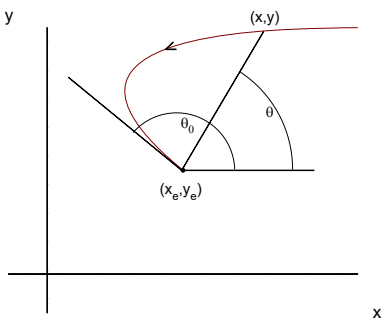


Figure 8.5

Let's begin with nodes. The characteristic feature of a node for linear systems is that all orbits approach the origin from a definite direction. That is to say, the polar angle $\theta(t)$ determined by a point $(x(t), y(t))$ lying on an orbit, using the origin as a reference point, approaches a limit θ_0 as $t \to +\infty$. We can adopt this same property into a definition of an equilibrium node for a nonlinear system by using the equilibrium (x_e, y_e) as the reference point. See Figure 8.5. For an unstable node the angle $\theta(t)$ approaches a limit as $t \to -\infty$. (The limit angle is not necessarily the same for all orbits.)

Moreover, for stable phase portraits (nodes or spirals) we require that the distance $r(t)$ from points $(x(t), y(t))$ lying on the orbit to the equilibrium (x_e, y_e) approaches 0 as $t \to +\infty$, whereas for unstable phase portraits the distance $r(t)$ approaches 0 as $t \to -\infty$.

We can use these characteristics of linear phase portraits to define nodes and spirals for nonlinear phase portraits.

Definition 8.1 *The (local) phase portrait near an equilibrium (x_e, y_e) of a nonlinear system is a **stable node** if $r(t) \to 0$ and the angle $\theta(t)$ approaches a limit as $t \to +\infty$. It is an **unstable node** if $r(t) \to 0$ and the angle $\theta(t)$ approaches a finite limit as $t \to -\infty$.*

*The (local) phase portrait near an equilibrium of a nonlinear system is a **stable spiral** if $r(t) \to 0$ and the angle θ increases without bound as $t \to +\infty$. It is an **unstable spiral** if $r \to 0$ and the angle θ increases without bound as $t \to -\infty$.*

In Chapter 6, Section 6.4, we studied one other fundamental type of phase portrait for linear systems, namely, the saddle point. The distinguishing characteristics of a saddle are the existence of two half line orbits that tend to the equilibrium $(0, 0)$ as $t \to +\infty$ (forming the stable manifold) and two half line orbits that tend to the equilibrium $(0, 0)$ as $t \to -\infty$ (forming the unstable manifold.). No other orbits approach $(0, 0)$ for $t \to +\infty$ or $t \to -\infty$. For nonlinear systems we define a saddle to have these same characteristics.

Definition 8.2 *An equilibrium is a **saddle** if*

(1) there are two orbits that tend to the equilibrium, each of whose angle $\theta(t)$ approaches a limit as $t \to +\infty$

(2) there are two orbits that tend to the equilibrium each of whose angle $\theta(t)$ approaches a limit as $t \to -\infty$.

(3) No other orbits approach the equilibrium as $t \to +\infty$ or $t \to -\infty$.

The two orbits that tend to the equilibrium as $t \to +\infty$ form the *stable manifold.* However, for a nonlinear system these orbits are not necessarily straight lines, as they are for a linear system.

Similarly, the two orbits that tend to the equilibrium as $t \to -\infty$ and that together form the *unstable manifold* are not necessarily straight lines.

For single nonlinear equations the Linearization Principle we learned that the equilibrium type (either attractor or repeller) is the same as that of the linearization at the equilibrium provided the equilibrium is hyperbolic, i.e. provided

$$\lambda = \left.\frac{df}{dx}\right|_{x_e} \neq 0.$$

Similarly, for a nonlinear system the type of an equilibrium is the same as that of its linearization provided the equilibrium is hyperbolic, as defined below.

Definition 8.3 *An equilibrium (x_e, y_e) is* ***hyperbolic*** *if the eigenvalues λ of the Jacobian $J(x_e, y_e)$ all have nonzero real parts.*

We are now in a position to compare local phase plane portraits to those of their linearizations.

Theorem 8.3 ***(Hartman-Grobman Theorem) A hyperbolic equilibrium of the planar autonomous system***

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{8.21}$$

is a

> ***stable (unstable) spiral if the linearization is a stable (unstable) spiral***
> ***stable (unstable) node if the linearization is a stable (unstable) node***
> ***saddle if the linearization has a saddle.***

In the case of a saddle, at the equilibrium the stable and unstable manifolds of the saddle are tangent to those of the linearization.

Since the phase portrait of the linearization system is determined by the eigenvalues of $J(x_e, y_e)$ we can, by means of the Hartman-Grobman Theorem, relate local phase portraits to the eigenvalues of the linearization.

Corollary 8.1 ***Suppose (x_e, y_e) is a hyperbolic equilibrium of a planar autonomous system (8.21).***

If the eigenvalues $\lambda = \alpha \pm i\beta$, $\beta \neq 0$, of the Jacobian $J(x_e, y_e)$ are complex, then (x_e, y_e) is a stable spiral if $\operatorname{Re}\lambda = \alpha < 0$. ***The point (x_e, y_e) is an unstable spiral if*** $\operatorname{Re}\lambda = \alpha > 0$.

Suppose the eigenvalues of the Jacobian $J(x_e, y_e)$ are real. If they are both negative, then (x_e, y_e) is a stable node. If they are both positive, then (x_e, y_e) is an unstable node. If they have different signs, then (x_e, y_e) is a saddle.

We can now give a complete local analysis of the equilibria for the system (8.20) we have been using for motivation in Examples 8.1, 8.4, and 8.8.

Example 8.10 *The nonlinear plane autonomous system*

$$\begin{aligned} x' &= x - xy \\ y' &= 2y - 2xy \end{aligned}$$

has two equilibria $(x_e, y_e) = (0,0)$ *and* $(1,1)$. *The Jacobians at these equilibria and their eigenvalues are*

$$\begin{aligned} J(0,0) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{and} \quad \lambda = 1, 2 \\ J(1,1) &= \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \quad \textit{and} \quad \lambda = -\sqrt{2}, \sqrt{2}. \end{aligned}$$

Both equilibria are hyperbolic (i.e., none has a 0 *real part). By Theorem 8.1 we find that* $(x_e, y_e) = (0,0)$ *is an unstable node and* $(x_e, y_e) = (1,1)$ *is a saddle.*

Example 8.11 *The nonlinear plane autonomous system*

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha(x^2 - 1)y \end{aligned} \tag{8.22}$$

which is equivalent to the second order van der Pol equation, has only one equilibrium, which is located at the origin $(x_e, y_e) = (0,0)$. *The Jacobian at the origin and its eigenvalues are*

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \quad \textit{and} \quad \lambda_\pm = \frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4}.$$

Whether these eigenvalues are real, positive or negative, complex, etc. depends on the value of α. *Therefore, we have to organize the analysis of the (local) phase portrait into cases based on values of* α. *Note first that the eigenvalues are complex if* $|\alpha| < 2$ *and real if* $|\alpha| \geq 2$. *A summary of all the options appear in the table below.*

	λ	*Local portrait*
$\alpha \leq -2$	*real, negative*	*hyperbolic stable node*
$-2 < \alpha < 0$	*complex, negative real part*	*hyperbolic stable spiral*
$\alpha = 0$	*complex,* 0 *real part*	*nonhyperbolic, linearization fails*
$0 < \alpha < 2$	*complex, positive real part*	*hyperbolic unstable spiral*
$2 \leq \alpha$	*real, positive*	*hyperbolic unstable node*

Example 8.12 *The nonlinear, plane autonomous system*

$$\begin{aligned} x' &= x_{in} - x - \frac{2x}{1+x}y \\ y' &= \frac{2x}{1+x}y - y \end{aligned}$$

is a particular case of a famous model called the "chemostat model". A chemostat is a container used to contain chemical and biological reactions. Chemostats are used in scientific studies and diverse applications ranging from gene splicing to brewing. The quantity x is the concentration of a substrate (nutrient) that is continuously pumped into the chemostat at a fixed rate (with concentration $x_{in} > 0$) and reacts with (or is consumed by) the quantity y. The mixture is well stirred and continuously pumped out at the same rate so as to maintain a fixed volume.

There are two equilibria (see Exercise 8.26):

$$(x_e, y_e) = (x_{in}, 0) \qquad \text{and} \qquad (1, x_{in} - 1)\,.$$

To perform a local analysis near each equilibrium, we calculate the Jacobian matrix

$$J(x,y) = \begin{pmatrix} -1 - \frac{2y}{(1+x)^2} & -\frac{2x}{1+x} \\ \frac{2}{(1+x)^2}y & \frac{x-1}{1+x} \end{pmatrix}.$$

For the first equilibrium $(x_{in}, 0)$

$$J(x_{in}, 0) = \begin{pmatrix} -1 & -\frac{2x}{1+x} \\ 0 & \frac{x_{in}-1}{1+x_{in}} \end{pmatrix}.$$

This is a triangular matrix and therefore the eigenvalues appear along the diagonal:

$$\lambda_1 = -1, \qquad \lambda_2 = \frac{x_{in} - 1}{1 + x_{in}}.$$

We get the results in the following table by applying Theorem 8.1.

	λ	*Local portrait near* $(x_{in}, 0)$
$x_{in} > 1$	*real, opposite signs*	*hyperbolic saddle*
$x_{in} = 1$	-1 *and* 0	*nonhyperbolic, linearization fails*
$x_{in} < 1$	*real, negative*	*hyperbolic, stable node*

For the second equilibrium $(1, x_{in} - 1)$, the eigenvalues of the Jacobian

$$J\,(1, x_{in} - 1) = \begin{pmatrix} -\frac{1}{2}\,(1 + x_{in}) & -1 \\ \frac{1}{2}(x_{in} - 1) & 0 \end{pmatrix}.$$

are

$$\lambda_1 = -1, \qquad \lambda_2 = \frac{1}{2}\,(1 - x_{in})\,.$$

We get the results in the following table by applying Theorem 8.1.

	λ	*Local portrait near* $(x_{in}, 0)$
$x_{in} > 1$	*real, negative*	*hyperbolic stable node*
$x_{in} = 1$	-1 *and* 0	*nonhyperbolic, linearization fails*
$x_{in} < 1$	*real, negative*	*hyperbolic, saddle*

Remark 3. For nonlinear systems we have not defined different types of nodes (improper, star, etc.) as we did for linear phase portraits in Chapter 6. Nor have we defined a center for nonlinear systems. For these cases the relationship between the phase portrait of a nonlinear system and that of its linearization is complicated and beyond the scope of this course. In the neighborhood of a *nonhyperbolic* equilibrium the phase portraits of a planar autonomous system and that of its linearization need not be, and often are not, of the same type.

We emphasize that Theorem 8.3 and its Theorem 8.1 describe phase portraits *only in a neighborhood of an equilibrium*. The *global* phase portrait of a planar autonomous system may look considerably different from the phase portrait of its linearization, as we will see in the next section.

8.4 Global Phase Plane Portraits

The *global* phase portrait of a planar autonomous system may look considerably different from the phase portrait of its linearization. For example, consider the phase portrait of the competition system

$$\begin{aligned} x' &= x - xy \\ y' &= 2y - 2xy \end{aligned} \tag{8.23}$$

shown in Figure 8.6. The phase portraits shown in Figures 8.2 and 8.4 are magnifications of this phase portrait near the two equilibria $(x_e, y_e) = (0, 0)$ and $(1, 1)$ respectively.

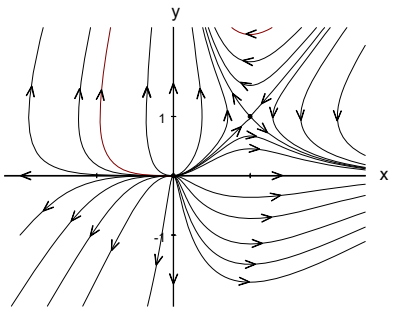


Figure 8.6

Figure 8.7a shows the phase portrait of the system

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha(x^2 - 1)y \end{aligned} \tag{8.24}$$

associated with the van der Pol equation

$$x'' + \alpha\left(x^2 - 1\right)x' + x = 0$$

for $\alpha = 1$.

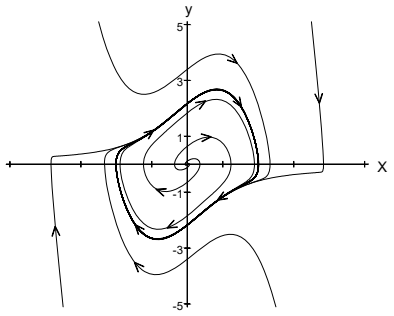


Figure 8.7a

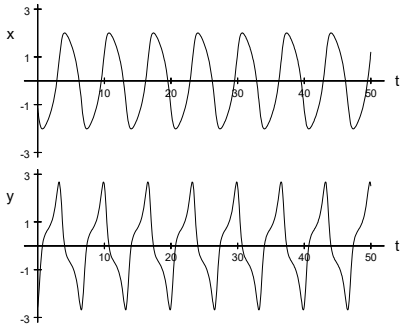


Figure 8.7b

The unstable spiral shown in Figure 8.3b is a magnification of this phase portrait near the equilibrium $(x_e y_e) = (0, 0)$. It shows orbits near this unstable spiral equilibrium spiraling outward. In Figure 8.7a, however, we see that orbits far enough away from the equilibrium do not spiral outward, but instead spiral *inward*. A notable feature of the phase portrait shown in Figure 8.7a is what appears to be a closed loop that separates the outward spiraling orbits from the inward spiraling orbits. This closed loop is (apparently) itself an orbit of the van der Pol system, as can be seen by choosing a point on the loop (e.g., $(x_0, y_0) = (-1.14, -2.57)$) and calculating the orbit starting at that initial point.

Graphs of both components $x(t)$ and $y(t)$ of the solution pair corresponding to this loop orbit appear in Figure 8.7b. Notice both $x(t)$ and $y(t)$ appear to be periodic functions of t (repeating with a period of between 6 and 7 time units).

A periodic function $x = x(t)$ of period p satisfies $x(t + p) = x(t)$ for all t. The graph of a periodic function on the interval $0 \leq t \leq p$ repeats itself identically on intervals of length p, i.e., on the intervals $p \leq t \leq 2p$, $2p \leq t \leq 3p$ and so on (and also on the intervals $-2p \leq t \leq -p$, $-3p \leq t \leq -2p, ...$). That nonequilibrium solutions can be periodic is an important feature of systems of two (or more) equations. Recall that nonequilibrium solutions of a *single* autonomous equation are either monotonically increasing or decreasing and therefore cannot be periodic.

A (nonequilibrium) periodic solution pair $x(t)$ and $y(t)$ satisfies $x(t + p) = x(t)$ and $y(t + p) = y(t)$ for all t. As a result *the orbit of periodic solution pair is a closed loop in the phase plane*. Moreover, although not so obvious, the converse is true: *a closed loop orbit is associated with periodic solution pairs.* See Exercise 8.75.

That is to say, closed loop orbits arise from and only from periodic solutions.

The closed loop orbit of a periodic solution is called a *cycle*. If other orbits approach a cycle as $t \to +\infty$ (or $-\infty$), then it is called a *limit cycle*. Figure 8.7a shows that the van der Pol equation (8.24) with $\alpha = 1$ has a limit cycle.

In Chapter 3 we learned that the only points in the phase line portrait that can be approached by orbits of single autonomous equations (as either $t \to +\infty$ or $-\infty$) are equilibria. From the van der Pol equation example we see that for systems of equations, on the other hand, orbits do not necessarily approach equilibria. A new type of "limit set" or "attractor" is possible for systems, namely, a limit cycle.

In constructing the phase plane portrait of a system, an important problem is to determine the sets of points approached by orbits as $t \to +\infty$ and $-\infty$. Given the added dimension in plane autonomous systems, we have to make more precise about what is meant by attracting points or sets.

We call the set of points approached by the orbit as $t \to +\infty$ the *forward (or omega) limit set* of the orbit and denote the set by S^+.

Definition 8.4 *Suppose $\tilde{x}(t)$ is a solution of a plane autonomous system. A point $\tilde{x}^*$ is a* ***forward (or omega) limit point*** *of the solution's orbit if there exists a sequence $t_n \to +\infty$ such that*

$$\lim_{t \to +\infty} \tilde{x}(t) = \tilde{x}^*.$$

A point $\tilde{x}^*$is a ***backward (or alpha) limit point*** *of the solution's orbit if there exists a sequence $t_n \to -\infty$ such that*

$$\lim_{t \to -\infty} \tilde{x}(t) = \tilde{x}^*.$$

The collection of all forward limit points is called the ***forward limit set*** *of the orbit, which we denote by S^+. The collection of all backward limit points is the backward limit set of the orbit, which we denote by S^1.*

A fundamental fact about the limit sets $S^{\pm}$ is that they are *invariant sets*. By this is meant that if a point $\tilde{x}^*$ lies in a limit set, then the entire orbit passing throughout that point lies in the limit set (for *all* t, positive and negative). *Thus, forward and backward limit sets $S^{\pm}$ are a collection of orbits.* The simplest example is a limit set that consists solely of an equilibrium $\tilde{x}_e$.

To illustrate these ideas consider the van der Pol system (8.24) with $\alpha = 1$. Figure 8.7a suggests that the limit is the forward limit set of all orbits (except for the equilibrium at $(0,0)$). The equilibrium $(0,0)$ is the backward limit set of all orbits inside the the limit cycle. The orbits outside the limit cycle are unbounded as $t \to -\infty$ and have no backward limit set. In this example, the forward and backward limit sets consist of single orbits (either the equilibrium or the limit cycle). However, as we will see in Example 8.14 below, *it is possible for limit sets to consist of more than one orbit.*

The following famous theorem provides information about the limit sets of forward (and backward) bounded orbits.[4] As always, in the planar autonomous system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{8.25}$$

we assume $f(x,y)$, $g(x,y)$ and their partial derivatives with respect to x and y are continuous so that the Fundamental Existence and Uniqueness Theorem is in effect.

Theorem 8.4 ***(Poincaré-Bendixson Theorem, Version 1) Let S^+ be the forward limit set of an orbit of the plane autonomous system (8.25) that is bounded as $t \to +\infty$. One of the following alternatives is true:***

(a) *S^+* ***contains an equilibrium***
(b) *S^+* ***is a limit cycle.***

These two alternatives also hold for the backward limit set S^- of an orbit bounded as $t \to -\infty$.

[4] An orbit is *bounded* as $t \to +\infty$ if it does not get arbitrarily far from the origin, or in other words, remains inside a circle of sufficiently large radius for all $t \geq 0$. Similarly an orbit is bounded as $t \to -\infty$ if it remains inside a circle of sufficiently large raduis for all $t \leq 0$.

Theorem 8.4 provides two alternatives for the limit set of a bounded orbit. If, for a particular system, we can rule out one alternative, then the remaining alternative must hold. A typical application of the Poincaré-Bendixson Theorem proceeds by eliminating one of the alternatives.

For example, if by some means or other we can be show that an orbit cannot have an equilibrium in its limit set (which rules out alternative (a)), then it follows that the orbit approaches a limit cycle (alternative (b) must hold). This is the case in the van der Pol equation (8.24) with $\alpha = 1$, as seen in Figure 8.7b. Here's how we argue. The only equilibrium of the system, namely the origin $(0, 0)$, is an unstable spiral (as was shown by an application of the Linearization Principle). Therefore, the origin cannot be in the forward limit set of an orbit. By so ruling out the alternative (a) in Theorem 8.4, we conclude that the second alternative (b) must be true. That is to say, any bounded orbit of the van der Pol system approaches a limit cycle.

One issue left unresolved in this argument is the (forward) boundedness of orbits. If there are no bounded orbits, then of course our conclusion that bounded orbits approach a limit cycle tells us nothing at all. This is a general issue with regard to the Poincaré-Bendixson Theorem 8.4. The theorem is only about bounded orbits, so in order to use the theorem one must be able to prove that orbits (or at least some orbits) of the system under consideration are in fact bounded.

There are methods for determining when orbits of plane autonomous systems are bounded. We will not pursue any such methods in this course. We will just rely on direction field evidence. For example, suppose we examine the direction field of a system and notice that the arrows point inward everywhere on all edges of the observed window. (One could, in fact, try to prove this by investigating the differential equations.) Then all orbits in the window are (forward) bounded, since they cannot leave the window without violating the direction field.

Remark 4. It is in general a difficult mathematical problem to establish the existence of a limit cycle (i.e., periodic solutions) of a planar autonomous system. The second alternative in Theorem 8.4 is a powerful tool that can often provide the existence of limit cycles in applications.

In addition to the Linearization Principle and the Poincaré-Bendixson Theorem 8.4 many other facts and techniques are known that help to sketch the phase plane portrait of autonomous systems. The list below contains a few. (Also see Exercise 8.39.) These facts are particularly useful when attempting to eliminate one of the two alternatives in the Poincaré-Bendixson Theorem 8.4.

Useful Facts

#1 A cycle must surround at least one equilibrium.

#2 If a forward limit set S^+ contains a stable node or a stable spiral, then it contains only that equilibrium point (that is to say, the orbit approaches the equilibrium as $t \to +\infty$).

#3 A forward limit set S^+ cannot contain an unstable node or unstable spiral point (It can, however, contain saddles.)

A forward limit set S^+ can, of course, contain (and hence contain only) a stable equilibrium. A forward limit set S^+ can also contain saddles, although by Useful Fact #3, S^+ can contain no other kinds of unstable equilibria. We will see an example in which S^+ contains saddles in Example 8.14 below.

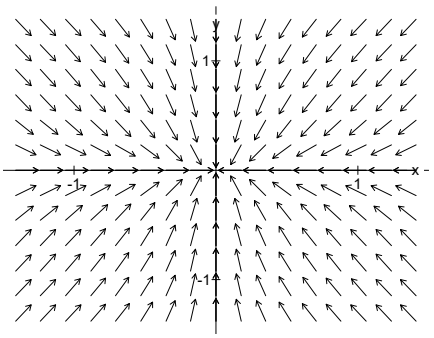


Figure 8.8

Example 8.13 *Consider the planar autonomous system*

$$\begin{aligned} x' &= -x - xy^2 \\ y' &= -2y - x^2 y. \end{aligned} \tag{8.26}$$

The equilibrium equations

$$\begin{aligned} -x\left(1 + y^2\right) &= 0 \\ -y\left(2 + x^2\right) &= 0 \end{aligned}$$

have only one solution

$$(x_e, y_e) = (0, 0).$$

The Jacobian

$$J(x, y) = \begin{pmatrix} -1 - y^2 & -2xy \\ -2xy & -2 - x^2 \end{pmatrix}$$

evaluated at the equilibrium

$$J(x, y) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

has eigenvalues $\lambda = -1, -2$. Therefore, origin is a stable node.

In this example we will rule out alternative (b) in the Poincaré-Bendixson Theorem 8.4; that is, we will show that there can be no cycle orbit.

We begin by noting that a cycle, if it existed, would have to encircle the origin (Useful Fact #1). Therefore, a cycle orbit would of necessity have to pass sequentially through the four quadrants in the plane. But an investigation of the direction field shows that it is impossible for an orbit to do this. For example, the SW direction in the first quadrant and the SE direction in the second quadrant rule out an orbit passing between these two quadrants.

We conclude that alternative (b) of the Poincaré-Bendixson Theorem 8.4 fails to hold in this example and that alternative (a) therefore must hold, that is to say, S^+ must contain an equilibrium point. But the origin is the only equilibrium point in this example and therefore $(0, 0) \in S^+$. Since the origin is a stable node, Useful Fact #2 tells us that S^+ is in fact the origin.

We've now shown that all forward bounded orbits approach the origin as $t \to +\infty$.

The direction field shown in Figure 8.8 indicates that orbits (at least those in the window shown) are forward bounded. Simply note that the direction field arrows point inward everywhere on the boundary of the window. (For a more rigorous validation of the bounded of orbits see Exercises 8.39 and 8.41.)

The limit set of an orbit does not necessarily consists of a single equilibrium or a limit cycle. Limit sets can consist of more than one orbit. In particular, a limit set might contain several equilibria. The following example illustrates this possibility.

Example 8.14 *Consider the planar autonomous system*

$$\begin{aligned} x' &= \left(x^2 - 1\right) y \\ y' &= \left(1 - y^2\right) \left(x + \frac{3}{10} y\right). \end{aligned} \tag{8.27}$$

In this example we will determine the limit set of those orbits starting near the origin $(0, 0)$*, which is an equilibrium. The Jacobian*

$$J(x, y) = \begin{pmatrix} 2xy & x^2 - 1 \\ 1 - y^2 & \frac{3}{10}\left(1 - y^2\right) - 2y\left(x + \frac{3}{10} y\right) \end{pmatrix}$$

at the origin

$$J(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & \frac{3}{10} \end{pmatrix}$$

has complex eigenvalues

$$\lambda = \frac{3}{20} \pm \frac{\sqrt{391}}{20} i$$

with positive real part. The origin is therefore an unstable spiral and, as a result, it cannot belong to a forward limit set S^+*.*

There are six other equilibria are:

$$(1, 1),\ (-1, 1),\ (-1, -1),\ (1, -1) \tag{8.28}$$

and

$$(1, -10/3),\ (-1, 10/3). \tag{8.29}$$

The eigenvalues of the Jacobian $J(x, y)$ *evaluated at each one of the four equilibria (8.28) are real and of opposite signs, which shows that these equilibria are saddles.*

The remaining two equilibria (8.29) are stable nodes. Of course, orbits starting near either one of these equilibria will approach it. However, not all orbits in the plane will approach one of these stable nodes.

Figure 8.9 shows an orbit spirally counterclockwise outward from the origin (which is consistent with the origin being an unstable spiral) and seemingly approaching the square whose corners are the four saddle equilibria (8.28).

The sides of the square are each themselves orbits. This can be seen by investigating the differential equations in the system (8.27) when an initial condition is chosen on a side of the square. For example, take $(x_0, y_0) = (1, 0)$*. By inspection we see that the solution components are*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 \\ y(t) \end{pmatrix} \tag{8.30}$$

where $y(t)$ solves the initial value problem

$$y' = \left(1 - y^2\right)\left(1 + \frac{3}{10}y\right), \quad y(0) = 0.$$

The phase line portrait of this nonlinear autonomous equation for y is

$$\longrightarrow -\frac{10}{3} \longleftarrow -1 \longrightarrow 1 \longleftarrow .$$

From this phase line portrait, we see that the initial condition $y_0 = 0$ yields a solution $y(t)$ that connects -1 to $+1$ as t runs from $-\infty$ to $+\infty$. The final result is that the orbit of the solution (8.30) is the right side of the square in the phase plane portrait in Figure 8.9 that connects the two saddle $(1, -1)$ and $(1, 1)$. Similar considerations show that the other three sides are orbits.

It follows that orbits starting inside the square formed by the four saddles remain inside this square, and therefore are bounded. To do an application of the Poincaré-Bendixson Theorem 8.4 we would have to rule out one of the two alternatives (a) or (b). It turns out that alternative (b) can be ruled out. That is, it can be shown that there are no cycles. We will not attempt to do that here, as it is not so easy. Instead, we just note that the computer simulation shown in Figure 8.9 shows an orbit that seemingly "spirals out" to the square formed by the four saddles. It can in fact be proved that the forward limit set S^+ of any orbit inside the square is the square.

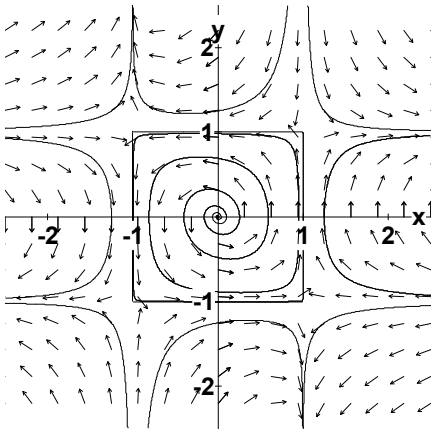


Figure 8.9

The forward limit set S^+ described in Example 8.14 is the square formed by the four saddle equilibria (8.28). Note that in that example the limit set S^+ consists of eight different orbits, four equilibria and four orbits that "connect" the equilibria. Orbits that connect two different equilibria (as $t \to +\infty$ and $t \to -\infty$) are called *heteroclinic orbits* . The limit S^+ in Example 8.14) is an example of a *cycle chain*. This is a set of saddles connected by heteroclinic orbits to form a loop. With these concept at hand we can state a second version of the Poincaré-Bendixson Theorem.

Theorem 8.5 ***(Poincaré-Bendixson Theorem, Version 2) Suppose the plane autonomous system (8.25) has a finite number of equilibria. Let S^+ be the forward limit set of an orbit bounded as $t \to +\infty$. Then one of the following alternatives is true:***

(a) *S^+ is an equilibrium*
(b) *S^+ is a limit cycle*
(c) *S^+ is a cycle chain.*

These two alternatives also hold for the backward limit set S^- of an orbit bounded as $t \to -\infty$.

In this section we have seen, for planar autonomous systems, that limit sets of orbits are not necessarily equilibria, as they are for single autonomous equations. By going up in dimension, from dimension 1 to 2, we create the possibility of new kinds of limit sets and attractors, namely, limit cycles or cycle chains. In Section 8.6 we will see, perhaps not surprisingly, that further increases in dimension (i.e., in the number of equations in the system) can increase the complexity of limit sets even further.

8.5 Bifurcations & The Hopf Bifurcation Theorem

The phase plane portrait of a system might change in significant ways if the numerical values of parameters appearing in the equations are changed. For example, the phase portrait of the linear system

$$\begin{aligned} x' &= px + y \\ y' &= -x + py \end{aligned}$$

changes from a stable spiral to an unstable spiral as the value assigned to the parameter p changes from negative to positive. The reason for this is that the eigenvalues of the coefficients matrix (Jacobian)

$$\begin{pmatrix} p & 1 \\ -1 & p \end{pmatrix}$$

are the complex numbers $\lambda = p \pm i$ with real part p. Such a fundamental change in the phase portrait of a system is called a bifurcation.

We studied bifurcations in the phase line portraits of single autonomous equations in Chapter 3 (Section 3.1.4). We classified fundamental types of bifurcations, which for single equations necessarily involve only equilibrium configurations. In this section we will see that each of those types of bifurcations – blue-sky, pitchfork, and transcritical bifurcations – can also occur in plane autonomous. For systems, however, we will also see that bifurcations can involve entries other than equilibria, namely, cycles.

8.5.1 Local Bifurcations of Equilibria

We will not attempt a general theory of local equilibrium bifurcations in this course. In this section we will look at an example of each of the three typical bifurcations that we studied for a single autonomous equation in Chapter 3 – namely, blue-sky, pitchfork and transcritical – as they occur in a plane autonomous system.

One way to construct a plane autonomous system example of any given bifurcation type for a single autonomous equation, is as follows. Consider a planar autonomous system of the form

$$\begin{aligned} x' &= f(x, p) \\ y' &= -y. \end{aligned} \tag{8.31}$$

In this kind of a system, the equations are uncoupled from one another and can be treated separately.

The second equation implies every solution satisfies $y(t) \to 0$ as $t \to +\infty$. As a result, all orbits of the plane autonomous system (8.31) approach the x-axis.

The equilibria, clearly, have the form $(x_e, 0)$ (i.e., they lie on the x-axis) where x_e is an equilibrium of the first equation in (8.31).

The x-axis is invariant. That is to say, an orbit starting on the x-axis at time t_0 remains on the x-axis (for $t > t_0$ and $t < t_0$). This is because for an initial condition $y_0 = 0$ we have, from the second equation in (8.31) that $y(t) = 0$ for all t What do orbits on the x-axis do? They are determined by the first equation in (8.31), which we note is a single autonomous equation for x containing a bifurcation parameter p of the type we studied this type of equation in Chapter 3. Any bifurcation in the phase line portrait of the first equation will correspond to a bifurcation for the planar autonomous system (8.31).

For example, consider the system

$$\begin{aligned} x' &= x^2 - p \\ y' &= -y. \end{aligned} \tag{8.32}$$

In Example 3.16 of Chapter 3 we saw that the equation $x' = x^2 - p$ undergoes a blue-sky bifurcation at the critical value $p_0 = 0$. For $p < 0$ this equation has no equilibria and for $p > 0$ it has two equilibria $x_e = \pm\sqrt{p}$. The phase line portrait of this equation appears on the x-axis in the phase plane portrait of (8.32).

Recall that all other orbits in the phase plane approach the x-axis. An orbit can approach the x-axis as either a horizontal asymptote (as $t \to +\infty$ or $t \to -\infty$) or by approaching an equilibrium on the x-axis. The latter case can only occur is $p \geq 0$ when the plane autonomous has equilibria

$$(x_e, y_e) = (\sqrt{p}, 0) \quad \text{and} \quad (-\sqrt{p}, 0)$$

The Jacobian

$$J(x, y) = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$$

evaluated at the equilibrium $\left(\sqrt{p}, 0\right)$ is

$$J\left(\sqrt{p}, 0\right) = \begin{pmatrix} 2\sqrt{p} & 0 \\ 0 & -1 \end{pmatrix}.$$

The roots of the characteristic equation are $\lambda_1 = 2\sqrt{p} > 0$ and $\lambda_2 = -1 < 0$ and therefore this equilibrium is a saddle.

The Jacobian evaluated at the equilibrium $\left(-\sqrt{p}, 0\right)$ is

$$J\left(-\sqrt{p}, 0\right) = \begin{pmatrix} -2\sqrt{p} & 0 \\ 0 & -1 \end{pmatrix}.$$

The roots of the characteristic equation are $\lambda_1 = -2\sqrt{p} < 0$ and $\lambda_2 = -1 < 0$ and therefore this equilibrium is a stable node.

In summary, for p less than the critical (bifurcation) point $p_0 = 0$ there are no equilibria and for p greater than $p_0 = 0$ there is a saddle and a node. The blue-sky bifurcation in the phase line portrait of the first equation in (8.32) has given rise to this bifurcation involving a saddle and a node in the phase plane of the system (8.32).

This bifurcation scenario can be seen in Figure 8.10. It has the characteristics of the blue-sky bifurcation of a single equation in that the equilibrium count goes from 0 to 2 as the bifurcation occurs and that one of the two equilibria is stable and one is unstable. However, for a plane autonomous system, since the equilibria involved are a saddle and a node, the bifurcation is more commonly called a *saddle-node bifurcation* (rather than a blue-sky bifurcation).

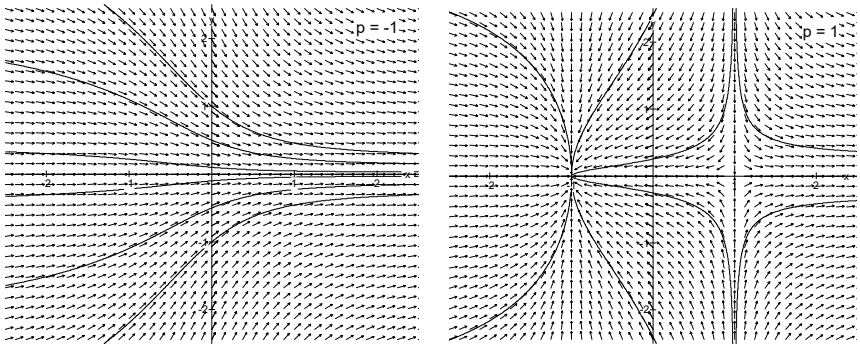


Figure 8.10. Phase portraits (8.32) with $p = -1$ and $p = 1$.

In Chapter 3 we drew bifurcation diagrams as a convenient graphical way to summarize the bifurcations that occur for single equations. These diagrams show the equilibria plotted against the parameter p. We can also draw bifurcation diagrams for planar systems of equations.

If, however, we plot equilibria against the parameter p, we will have to draw in three dimensions, since equilibria are now ordered pairs. While this certainly can be done, three dimensional pictures are often difficult to draw and understand. Therefore, the usual practice to plot a single representative of the equilibria against the parameter p. For example, one might plot just the x-coordinate or just the y-coordinate of the equilibria against p. A bifurcation diagram for system (8.32) appears in Figure 8.11.

In a similar fashion, using a system of the form (8.31), we can provide illustrative examples of pitchfork and transcritical bifurcations in planar autonomous systems by choosing appropriate expressions for $f(x, p)$ in (8.31).

Referring to Examples 3.17 and 3.18 in Chapter 3 the choices $f(x, p) = px - x^3$ and $f(x, p) = px - x^2$ in (8.32) produce systems with a pitchfork and a transcritical bifurcation at $p_0 = 0$ respectively. We leave it to the reader for the details. See Exercises 8.43 and 8.44. We instead give an application that involves transcritical bifurcations.

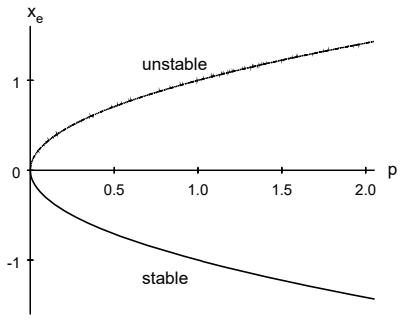


Figure 8.11. A plot of the x component of the equilibria against the parameter p for system (8.32).

Example 8.15 *The system*

$$\begin{aligned} x' &= (p - x)\, x - xy \\ y' &= (1 - 2y)\, y - xy \end{aligned} \tag{8.33}$$

arises in competition theory. This system describes the dynamics of two populations each of which, when isolated from one another, grow according to the logistic equations

$$x' = (p - x)\, x$$
$$y' = (1 - 2y)\, y,$$

but when interacting each has negative effect on the other's per capita growth rates.

For example, when the two populations are interacting, population y negatively effects the per capita growth rate x'/x of population x by an amount proportional to its population size y. This gives rise to the term $-xy$ in the first equation of system (8.33). A similar effect of population x on the per capita growth rate of population y gives rise to the same term in the second equation of the system.

The parameter $p > 0$ is positive. It is the carrying capacity (attractor equilibrium) of the x population when the y population is absent. Note that the carrying capacity of the y population in the absence of the x population is fixed, in this example, at $y = 1/2$.

The equilibrium equations

$$(p - x)\, x - xy = 0$$
$$(1 - 2y)\, y - xy = 0$$

yield four equilibria

$$(x_e, y_e) = (0, 0)\,, \quad (p, 0)\,, \quad \left(0, \frac{1}{2}\right) \text{ and } \left(2\left(p - \frac{1}{2}\right), 1 - p\right). \tag{8.34}$$

The first equilibrium represents the absence of both species. The second and third equilibria represent the absence of one population and the presence of the other. The fourth equilibrium allows for the coexistence of the two competing populations (provided its components x_e and y_e are both positive) and it is to this equilibrium that we turn our attention.

In this application only equilibria and orbits with non-negative values of x and y are relevant (since they denote population numbers or densities). The first three equilibria in (8.34) are therefore relevant in applications for all $p > 0$ values, while the fourth equilibrium is relevant if and only if $1/2 \le p \le 1$.

Note when $p = 1/2$ that the third and fourth equilibria in (8.34) coincide; otherwise they are distinct equilibria. This intersection of the two equilibria is a transcritical bifurcation (see Chapter 3). Another transcritical bifurcation occurs at $p = 1$ where the second and fourth equilibrium in (8.34) coincide. See Figure 8.12.

In Example 8.15 we dealt only with the existence and bifurcations of equilibria for the competition system (8.33). Recall (Chapter 3) that it is typical for transcritical bifurcations to exhibit an exchange of stability between the two crossing branches of equilibria. We can investigate the stability properties of the bifurcating equilibria in Example 8.15 by use of the Linearization Principle to see whether or not the Exchange of Stability Principle holds true.

Two transcritical bifurcations occur for system (8.34), one at $p = 1/2$ and the other at $p = 1$. We will show here that the Exchange of Stability Principle holds at $p = 1/2$ and leave it for the reader to investigate the bifurcation at $p = 1$ (Exercise 8.54).

The transcritical bifurcation at $p = 1/2$ involve the the third and fourth equilibria in (8.34). The Jacobian of the system is

$$J(x,y) = \begin{pmatrix} p-2x-y & -x \\ -y & 1-4y-x \end{pmatrix}.$$

When the Jacobian is evaluated at the equilibrium $(x_e, y_e) = (0, 1/2)$ we obtain the matrix

$$J\left(0,\frac{1}{2}\right) = \begin{pmatrix} p-\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 \end{pmatrix}$$

whose eigenvalues are $\lambda = -1$ and $p - 1/2$. If follows that

$$(x_e, y_e) = \left(0, \tfrac{1}{2}\right) \text{ is a } \begin{cases} \text{stable node if } p < \frac{1}{2} \\ \text{saddle if } \frac{1}{2} < p. \end{cases}$$

An analysis of the equilibrium $(x_e, y_e) = \left(2\left(p-\frac{1}{2}\right), 1-p\right)$ is a little more complicated, but still algebraically manageable. The Jacobian at this equilibrium

$$J\left(2\left(p-\frac{1}{2}\right), 1-p\right) = \begin{pmatrix} 1-2p & 1-2p \\ p-1 & -2+2p \end{pmatrix}$$

whose eigenvalues of

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left(-1+\sqrt{5-12p+8p^2}\right) \\ \lambda_2 &= \frac{1}{2}\left(-1-\sqrt{5-12p+8p^2}\right). \end{aligned}$$

Are these eigenvalues real or complex? A little bit of algebra shows this about the expression under the radical:

$$0 < 5-12p+8p^2 < 1 \quad \text{if } \tfrac{1}{2} < p < 1$$

$$1 < 5-12p+8p^2 \quad \text{otherwise.}$$

Thus, eigenvalues are real. Moreover, the $\lambda_2 < 0$ for all $p > 0$ and

$$\lambda_1 < 0 \quad \text{if } \tfrac{1}{2} < p < 1$$

$$\lambda_1 > 0 \quad \text{otherwise.}$$

It follows that

$$(x_e, y_e) = \left(2\left(p-\tfrac{1}{2}\right), 1-p\right) \text{ is a } \begin{cases} \text{saddle if } p < \frac{1}{2} \\ \text{stable node if } \frac{1}{2} < p < 1 \\ \text{saddle if } 1 < p. \end{cases}$$

One result of this analysis is that as p increases through the critical value $p_0 = 1/2$ (and the third and fourth equilibria intersect), an exchange of stability occurs between these two equilibria. See Figure 8.12.

In terms of the application to competition theory, when $p < p_0$ the extinction equilibrium $(0, 1/2)$ is stable, a fact indicating that population x goes extinct as $t \to +\infty$. Only when p is increased greater than p_0 can population x survive, as indicated by the stability of the fourth equilibrium in which the x component is positive. (To strengthen these conclusions obtained from *local* stability results supplied by the Linearization Principle, one would need to investigate the global phase portraits.)

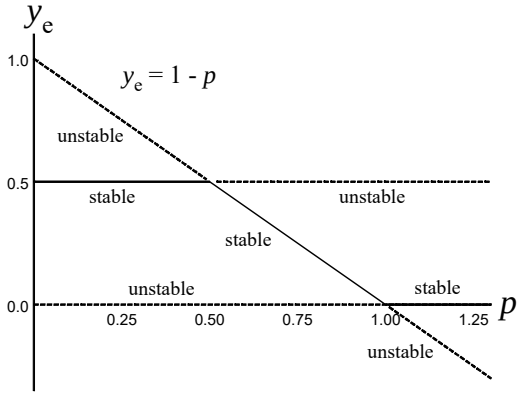


Figure 8.12. A plot of the y_e equilibrium component against the parameter p for system (8.33).

In this section we looked at bifurcations involving equilibria. In the next section we consider a new kind of bifurcation that involves cycles.

8.5.2 Hopf Bifurcation of Limit Cycles

An investigation of the bifurcation examples in the previous section reveals that at the bifurcation value $p = p_0$ the equilibria are nonhyperbolic and, because of this, the Linearization Principle fails to hold. Moreover, in these examples the equilibria at bifurcation are nonhyperbolic because one of eigenvalues of the Jacobian (evaluated at the equilibrium) equals 0. This is characteristic of equilibrium bifurcations.

Another important type of bifurcation occurs when an equilibrium is nonhyperbolic because an eigenvalue of the Jacobian has zero real part, but is *not* equal to 0, that is to say, an eigenvalue has the form $\lambda = \beta i$, $\beta \neq 0$. Actually, since complex roots appear in conjugate pairs, at such a nonhyperbolic equilibrium the eigenvalues are $\lambda = \pm\beta i$, $\beta \neq 0$. In such a case, at the bifurcation point the linearization of the plane autonomous system at the equilibrium is a center and, as a result, the Linearization Principle fails to hold.

Here is an example.

Example 8.16 *The origin* $(x_e, y_e) = (0, 0)$ *is an equilibrium of the planar autonomous system*

$$\begin{aligned} x' &= px + y - x^3 \\ y' &= -x + py - y^3. \end{aligned} \tag{8.35}$$

The Jacobian evaluated at $(0, 0)$ *is*

$$J(0,0) = \begin{pmatrix} p & 1 \\ -1 & p \end{pmatrix}.$$

The roots of the characteristic polynomial are

$$\lambda = p \pm i.$$

By the Linearization Principle $(0,0)$ *is a stable spiral for* $p < 0$ *and an unstable spiral for* $p > 0$*. The value* $p_0 = 0$ *is therefore a bifurcation point. Furthermore, for* $p = 0$ *the origin is nonhyperbolic (because* $\operatorname{Re}\lambda = 0$*).*

For $p < 0$*, orbits near the origin spiral into the origin as* $t \to +\infty$*. For* $p > 0$*, orbits near the origin spiral away. Where do these orbits go? Are the unbounded as* $t \to +\infty$*? Figure 8.13 shows an example that indicates that these orbits are bounded as* $t \to +\infty$ *and that the limit set of these orbits is a cycle.*

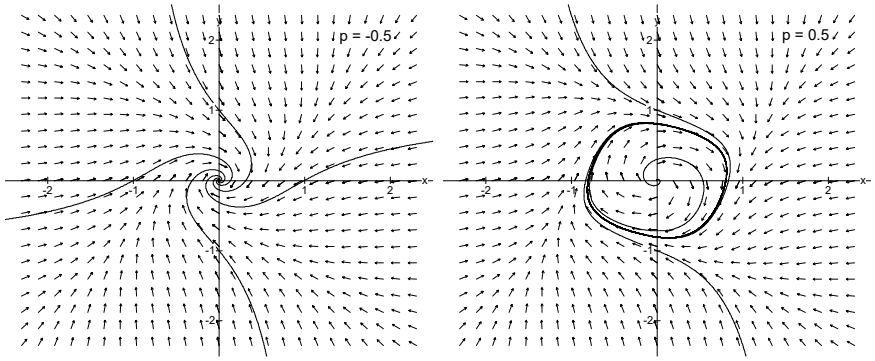


Figure 8.13. Sample direction fields and orbits of (8.35) for a negative and a positive value of p.

In Example 8.16, the equilibrium $(0,0)$ of the plane autonomous system (8.35) loses stability and a limit cycle is created as p increases through $p_0 = 0$. This is an example of a famous theorem – called the Hopf Bifurcation Theorem – concerning bifurcations that create limit cycles. A full detailed statement of this theorem is too technical for the level of this book. Roughly speaking, however, the theorem says the following.

Theorem 8.6 ***(Hopf Bifurcation Criteria). Suppose an equilibrium*** $\tilde{x}(p)$ ***of the system***

$$x' = f(x, y, p)$$
$$y' = g(x, y, p)$$

loses stability as the parameter p ***passes through a critical value*** p_0 ***(increasing or decreasing) and does so because a pair of complex conjugate eigenvalues***

$$\lambda(p) = \alpha(p) \pm \beta(p)i$$

of the Jacobian evaluated at the equilibrium crosses transversely from the left to the right half complex plane (or vice versa). By this is meant

$$\alpha(p_0) = 0, \quad \beta(p_0) \neq 0, \quad \left.\frac{d\alpha}{dp}\right|_{p_0} \neq 0. \tag{8.36}$$

Then[5] ***limit cycles bifurcate from the equilibrium at*** $p = p_0$. ***These limit cycles exist for*** p ***(near and) either greater than or less than*** p_0. ***The cycles encircle the equilibrium, have amplitudes that shrinks to zero and periods that approach*** $2\pi/\beta(p_0)$ ***as*** $p \to p_0$.

Remark 5. It is also possible to determine, by means of formulas too complicated for this book (that involve the technical condition mentioned in the footnote), whether the bifurcating limit cycles are stable (i.e., attract all nearby orbits as $t \to +\infty$) or unstable (do not attract all nearby orbits) and whether they exist for $p < p_0$ or for $p > p_0$. In practice, one often uses the Hopf Bifurcation Criteria (8.36) to locate possible Hopf bifurcation points and then relies on computer examples to determine whether limit cycles really exist and are stable or unstable.

For the system (8.35) we have $\alpha(p) = p$ and $\beta(p) = 1$ and the Hopf criteria (8.36) are satisfied with $p_0 = 0$. The computer generated examples in Figure 8.13 corroborates that a bifurcation to a stable limit cycle occurs at this value of p.

Example 8.17 *The system*

$$\begin{aligned} x' &= 1 - (p+1)\,x + x^2 y \\ y' &= px - x^2 y \end{aligned} \tag{8.37}$$

is an example of a model for a idealized chemical reaction called the Brusselator reaction. In this system x and y are chemical concentrations and $p > 0$ is a positive constant.

We can algebraically solve the equilibrium equations

$$\begin{aligned} 1 - (p+1)\,x + x^2 y &= 0 \\ px - x^2 y &= 0 \end{aligned}$$

to find the equilibrium $(x_e, y_e) = (1, p)$. (Hint: add the two equations.) The Jacobian

$$J(x,y) = \begin{pmatrix} -p - 1 + 2xy & x^2 \\ p - 2xy & -x^2 \end{pmatrix}$$

evaluated at the equilibrium is

$$J(1,p) = \begin{pmatrix} p-1 & 1 \\ -p & -1 \end{pmatrix}.$$

and has eigenvalues

$$\lambda(p) = \frac{1}{2}\,(p-2) \pm \frac{1}{2}\sqrt{p\,(p-4)}$$

which are complex if $p < 4$, in which case we write

$$\lambda(p) = \frac{1}{2}\,(p-2) \pm \frac{1}{2} i \sqrt{p\,(4-p)}.$$

[5] There is another technical condition needed which we ignore.

The real part $\alpha(p) = (p-2)/2$ *and imaginary part* $\beta(p) = \sqrt{p(4-p)}/2$ *satisfy the criteria (8.36) with* $p_0 = 2$*. The graphs in Figure 8.14 indicate that a Hopf bifurcation of stable limit cycles occurs at* $p_0 = 2$*.*

In terms of the application, we find that the chemical concentrations will equilibrate if $p < 2$ *but will settle into sustained periodic oscillations if* $p > 2$*.*

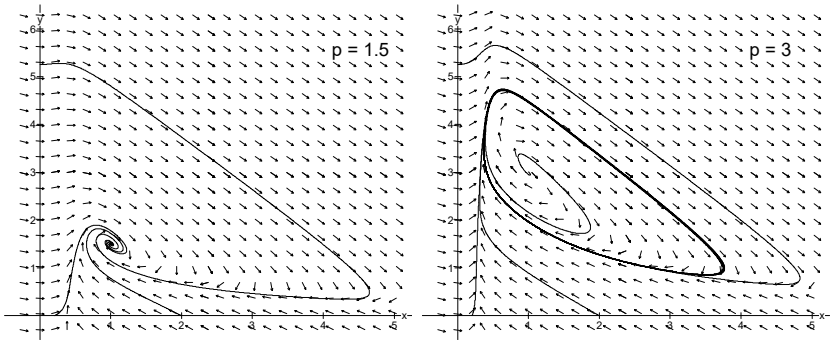


Figure 8.14. These two sample direction fields and orbits of the Brusselator system (8.37) show the a stable spiral equilibrium for $p < 2$ and an unstable spiral equilibrium encircled by a stable limit cycle for $p > 2$.

We have seen that bifurcation diagrams provide a useful way to summarize equilibrium bifurcations that occur in a system. To include Hopf bifurcations in such a graph we need to devise a way to "plot" a cycle in the diagram. One way to do this, is to plot both the maximum and the minimum of one component of the cycles (say, the x component) against the parameter value p. Figure 8.15a shows an example of how a Hopf bifurcation would appear in such a plot. One must indicate on such a graph, or in its caption, that the plot represents a cycle, so that the plot is not mistaken for a pitchfork bifurcation of equilibria.

Another way to plot a Hopf bifurcation is shown in Figure 8.15b. In this plot, all values (i.e. the range) of one component of the cycle are plotted above the corresponding value of p. Since these values will cover an entire interval of numbers ranging from the maximum to the minimum of the component, the plot appears as a vertical line segment lying above the corresponding value of p.

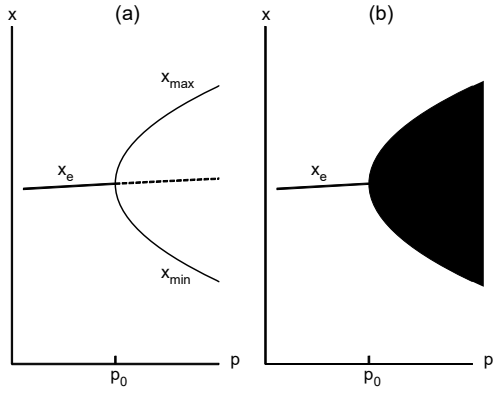


Figure 8.15. (a) The maximum and the minimum of the x component of the cycle are plotted. (b) All values of the x component of the cycle are plotted.

8.6 Higher Dimensional Systems

As the number of equations in a system of differential equations increases, the properties of solutions and orbits can become more complicated. As we have seen, the solutions of a single autonomous equation are monotonic and bounded orbits approach an equilibrium as $t \to +\infty$. We have also seen that bounded orbits of systems of two autonomous equations do not, however, always approach equilibria; solutions can oscillate and orbits approach limit cycles or cycle chains. In moving from systems of one to systems of two equations, i.e., by moving from one to two dimension phase space, we found that solution and orbit properties become more complex. In particular, the list of possible attractors increases from one type – equilibria – to three types – equilibria, cycles, and cycle chains.

We should, then, expect more complexity, and a longer list of possible attractors, to arise if we move up to three and higher dimensional systems. And this indeed occurs. It is possible for orbits in three or more dimensions to approach attractors that are none of the three we encountered in two dimensional systems. Moreover, attractors in even three dimensional systems can be so irregular and complex that they are called "chaotic". The move from two to three dimensions is significant in that we can no long provide a short list of possible attractors with easily tractable characteristics.

So of the analytic tools we've studied can be extended to higher dimensional systems. The analysis of equilibria, by means of the Linearization Principle, is the most basic tool. the Hopf bifurcation Theorem for limit cycles also holds in higher dimensions. On the other hand, the analytic study of more complex solutions and exotic attractors is difficult, and for their investigation one relies a great deal on computer explorations.

8.6.1 The Linearization Principle

In 1963 E. N. Lorenz, a meteorologist studying the dynamics of a layer of fluid heated from below (as part of his study of atmospheric weather patterns), investigated the nonlinear system of three first order equations

$$\begin{aligned} x' &= \sigma (y - x) \\ y' &= \rho x - y - xz \\ z' &= -\beta z + xy. \end{aligned} \tag{8.38}$$

The system contains three positive parameters: σ (the Prandtl number), ρ (the Rayleigh number), and β (the aspect ratio).

Lorenz investigated the dynamics of solutions of (8.38) for values of ρ considerably larger than 1 and found that some very complicated orbits can result. Although it was later shown that this system is an accurate approximation to the original fluid dynamic problem only for Rayleigh numbers ρ near 1, the Lorenz equations (8.38) have become a prototypical mathematical example of a system with a so-called "strange" or "chaotic" attractor. In this section, however, we will only look at the equilibrium solutions of the Lorenz system. We'll have a look at a strange attractor for this system in Section 8.6.2.

A solution of the system (8.38) is a triple of functions $\tilde{x}(t) = \text{col}(x(t), y(t), z(t))$ for which the points $(x(t), y(t), z(t))$ define an orbit in three dimensional Euclidean space (called

phase space). For this reason, we say (8.40) is a three dimensional system. More generally, a system

$$\begin{aligned} x_1' &= f_1(x_1, \ldots, x_n) \\ x_2' &= f_2(x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, \ldots, x_n) \end{aligned} \tag{8.39}$$

of n first order (autonomous) equations for n unknown functions

$$\tilde{x}(t) = \operatorname{col}(x_1(t), x_2(t), \cdots, x_n(t))$$

is an *n-dimensional system*. An equilibrium solution is a constant solution

$$\tilde{x} = \operatorname{col}(e_1, e_2, \cdots, e_n)$$

whose components must satisfy the *equilibrium equations*

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ f_2(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \ldots, x_n) &= 0. \end{aligned}$$

Example 8.18 *A commonly studied special case of the Lorenz system (8.38) is*

$$\begin{aligned} x' &= 10(y - x) \\ y' &= \rho x - y - xz \\ z' &= -\frac{8}{3}z + xy. \end{aligned} \tag{8.40}$$

This system arises from (8.38) by setting $\sigma = 10$ and $\beta = 8/3$ and leaving only the one parameter ρ unspecified. The equilibrium equations for this system are

$$\begin{aligned} 10(y - x) &= 0 \\ \rho x - y - xz &= 0 \\ -\frac{8}{3}z + xy &= 0. \end{aligned}$$

The first equation implies $y = x$, which when substituted into the remaining equations yields two equations

$$\begin{aligned} (\rho - 1 - z)\, y &= 0 \\ -\frac{8}{3}z + y^2 &= 0 \end{aligned}$$

for two unknowns y and z. The first of these equations implies

$$y = 0 \quad or \quad z = \rho - 1.$$

The first choice $y = 0$ and the second equation imply $z = 0$ and we have the equilibrium

$$(x, y, z) = (0, 0, 0).$$

The second choice $z = \rho - 1$ and the second equation imply $y^2 = 8(\rho - 1)/3$. If $0 < \rho \leq 1$ the only equilibrium of the system (8.40) is the origin,

$$(x, y, z) = (0, 0, 0).$$

On the other hand, if $\rho > 1$, there are three equilibria:

$$(x, y, z) = \begin{cases} (0, 0, 0) \\ \left(\sqrt{\frac{8}{3}(\rho - 1)}, \sqrt{\frac{8}{3}(\rho - 1)}, \rho - 1\right) \\ \left(-\sqrt{\frac{8}{3}(\rho - 1)}, -\sqrt{\frac{8}{3}(\rho - 1)}, \rho - 1\right) \end{cases}$$

Thus, there is a pitchfork bifurcation of equilibria at the bifurcation value $\rho_0 = 1$.

Suppose $\tilde{x}_e = \text{col}(e_1, e_2, ..., e_n)$ is an equilibrium of the system (8.39). We obtained the linearization of the system at the equilibrium from the Taylor approximation of

$$f_i(x_1, \ldots, x_n) \approx \sum_{j=1}^{n} \left.\frac{df_i}{dx_j}\right|_{(e_1,\ldots,e_n)} (x_j - e_j).$$

Namely, letting $y_i = x_i - e_i$ we obtain the system of n linear equations

$$y_i' = \sum_{j=1}^{n} \left.\frac{df_i}{dx_j}\right|_{(e_1,\ldots,e_n)} y_j$$

which is the *linearization of (8.39) at the equilibrium* $(e_1, e_2, ..., e_n)$. The coefficient matrix of this linear system is the Jacobian

$$J(x_1, x_2, ..., x_n) = \begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_n} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_n} \\ \vdots & \vdots & & \vdots \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_n} \end{pmatrix}$$

of the nonlinear system (8.39) in which all derivatives in this matrix are evaluated at the equilibrium $\tilde{x}_e$. That is to say, the coefficient matrix of the linearization is $J(e_1, e_2, ..., e_n)$.

The $n \times n$ matrix $J(e_1, e_2, ..., e_n)$ has n eigenvalues (if you count multiplicities) which are the roots of the *characteristic polynomial*

$$\det(J(e_1, e_2, ..., e_n) - \lambda I)$$

of the coefficient matrix has degree n. Here

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is the $n \times n$ identity matrix. By the Fundamental Theorem of Algebra the characteristic polynomial has n roots (allowing for complex roots and counting multiplicities); these roots are the *characteristic roots* (or the *eigenvalues*) of the coefficient matrix.

The following theorem is the n-dimensional generalization of Theorem 8.1 and embodies the *Linearization Principle* for systems of arbitrary dimension. The stability definitions given in Section 8.2 for planar autonomous systems extend straightforwardly to systems of any dimension. The following theorem is sometimes referred to as the *Fundamental Stability Theorem.*

Theorem 8.7 ***(The Linearization Principle) Suppose the point*** $\tilde{x}_e = (e_1, e_2, ..., e_n)$ ***is an equilibrium of the n-dimensional system (8.39).***

If all eigenvalues of the Jacobian $J(e_1, e_2, ..., e_n)$ ***have negative real parts, then the equilibrium is (locally asymptotically) stable.***

If at least one eigenvalue has positive real part, then the equilibrium is unstable.

Example 8.19 *The Jacobian of the Lorenz system (8.40) is*

$$J(x, y, z) = \begin{pmatrix} -10 & 10 & 0 \\ \rho - z & -1 & -x \\ y & x & -\frac{8}{3} \end{pmatrix}.$$

For the equilibrium $\tilde{x}_e = \text{col}(0, 0, 0)$ *the Jacobian*

$$J(0, 0, 0) = \begin{pmatrix} -10 & 10 & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}.$$

has the cubic characteristic polynomial

$$\det\left(\lambda I - J(0, 0, 0)\right) = \left(\lambda + \frac{8}{3}\right)\left(\lambda^2 + 11\lambda - 10\rho + 10\right)$$

whose three roots

$$\lambda = -\frac{8}{3}, \quad \frac{1}{2}\left(-11 \pm \sqrt{121 - 40\,(1 - \rho)}\right)$$

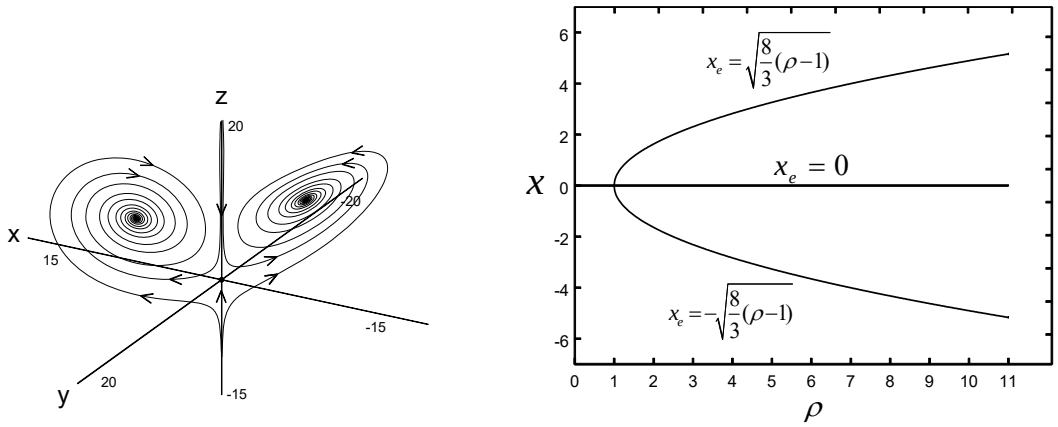
are the eigenvalues of $J\,(0, 0, 0)$. *If* $\rho < 1$ *all three eigenvalues are real and negative. By Theorem 8.7 the origin* $\tilde{x}_e = \text{col}(0, 0, 0)$ *is stable if* $\rho < 1$.

If $\rho > 1$ *then one root, namely*

$$\lambda = \frac{1}{2}\left(-11 + \sqrt{121 - 40\,(1 - \rho)}\right),$$

is positive and the origin is unstable.

Thus, at the pitchfork bifurcation point $\rho_0 = 1$ *the origin loses stability. See Figure 8.1.*



(a) (b)

Figure 8.16. (a) Orbits of the Lorenz system (8.40) with $\rho = 10$. The equilibrium $(x, y, z) = (0, 0, 0)$ is unstable. The two equilibria $(x, y, z) = \left(2\sqrt{6}, 2\sqrt{6}, 9\right)$ and $\left(-2\sqrt{6}, -2\sqrt{6}, 9\right)$ are stable spirals. (b) A bifurcation diagram showing the pitchfork bifurcation that occurs, using ρ as a bifurcation parameter, in the Lorenz system (8.40) at $\rho_0 = 1$.

When $\rho = 1$ in the preceding example, $\lambda = 0$ is an eigenvalue and the origin is nonhyperbolic. As a result, the Linearization Principle (Theorem 8.7) does not apply. (This does not mean the equilibrium is unstable. It simply means we cannot draw any conclusion from the Theorem 8.7.)

In Example 8.19 the equilibrium $\tilde{x}_e = \text{col}(0, 0, 0)$ of the Lorenz system (8.40) loses stability as ρ increases through 1. The origin is the only equilibrium for $\rho < 1$ (Example 8.18), but for $\rho > 1$ there are two additional equilibria, i.e., a pitchfork bifurcation occurs at $\rho = 1$. It turns out that these two additional equilibria are stable for $\rho > 1$ but close to 1. See Exercise 8.64. For ρ sufficiently large, however, each of these stable equilibria loses stability and all three equilibria are unstable! We will see an illustration in the next Example below.

Recall that the Hopf bifurcation criteria in two dimensions (Theorem 8.6) involve the destabilization of an equilibrium as a parameter passes through a critical value. The destabilization is caused by a pair of complex conjugate eigenvalues whose real part changes sign. At the critical parameter value the complex eigenvalues have the form $\lambda = \pm\beta i$, $\beta \neq 0$. This indicator of a possible Hopf bifurcation. to a limit cycle (and the Hopf Bifurcation Theorem) remains valid in higher dimensions. These Hopf criteria involve a pair of complex eigenvalues, even though in higher dimensions there will be more eigenvalues.

Example 8.20 *In Exercise 8.65 the reader is asked to show, by means of the Linearization Principle, that the equilibrium* $\tilde{x}_e = \text{col}(0, 0, 0)$ *of the system*

$$\begin{aligned} x' &= 2x + 2y - 2xy - \frac{1}{10}x^2 \\ y' &= pz - y - xy \\ z' &= \frac{1}{5}(x - z) \end{aligned} \tag{8.41}$$

is unstable for all p. In this example we are interested in other equilibria of this system. We consider two cases: $p = 1/2$ *and* 1.

For $p = 1/2$ *the system has two other equilibria (found by solving the equilibrium equations with the help of a computer or calculator)*

$$E_1 : (x_e, y_e, z_e) = (11.59, \ -0.2897, \ 11.59)$$
$$E_2 : (x_e, y_e, z_e) = \left(-2.589, \ 6.472 \times 10^{-2}, \ -2.589\right).$$

The eigenvalues of the Jacobian

$$J(x, y, z) = \begin{pmatrix} 2 - 2y - \frac{1}{5}x & 2 - 2x & 0 \\ -y & -1 - x & p \\ \frac{1}{5} & 0 & -\frac{1}{5} \end{pmatrix}$$

evaluated at each these equilibria in turn are (found with the aid of a computer or calculator)

$$E_1 : \lambda \approx -12.11 \text{ and } -0.2100 \pm 0.4216i$$
$$E_2 : \lambda \approx 1.729, \ 2.085 \text{ and } -3.717 \times 10^{-2}.$$

Therefore, by the Linearization Principle, we see that

$$\text{for } p = \frac{1}{2} \ E_1 \text{ is stable and } E_2 \text{ is unstable.}$$

For $p = 1$ *the system has two equilibria*

$$E_1 : (x_e, y_e, z_e) = (5.844, \ 0.8540, \ 5.844)$$
$$E_2 : (x_e, y_e, z_e) = (-6.8443, \ 1.171, \ -6.844).$$

The eigenvalues of the Jacobian evaluated at these equilibria are

$$E_1 : \lambda \approx -8.035 \text{ and } 5.684 \times 10^{-2} \pm 0.4259i$$
$$E_2 : \lambda \approx, -7.392 \times 10^{-2} \text{ and } 3.372\,4 \pm 3.482i.$$

Therefore, by the Linearization Principle we see that

$$\text{for } p = 1 \ E_1 \text{ is unstable and } E_2 \text{ is unstable.}$$

As p *is increased from* $1/2$ *to* 1, *the equilibrium* E_2 *remains unstable. The equilibrium* E_1, *however, loses stability because the real part of a complex pair of characteristic roots changes from negative to positive. This suggests a Hopf bifurcation might have occurred at some critical value of* p *somewhere between* $1/2$ *and* 1. *The existence of a limit cycle when* $p = 1$ *is corroborated by the computer simulations shown in Figure 8.17.*

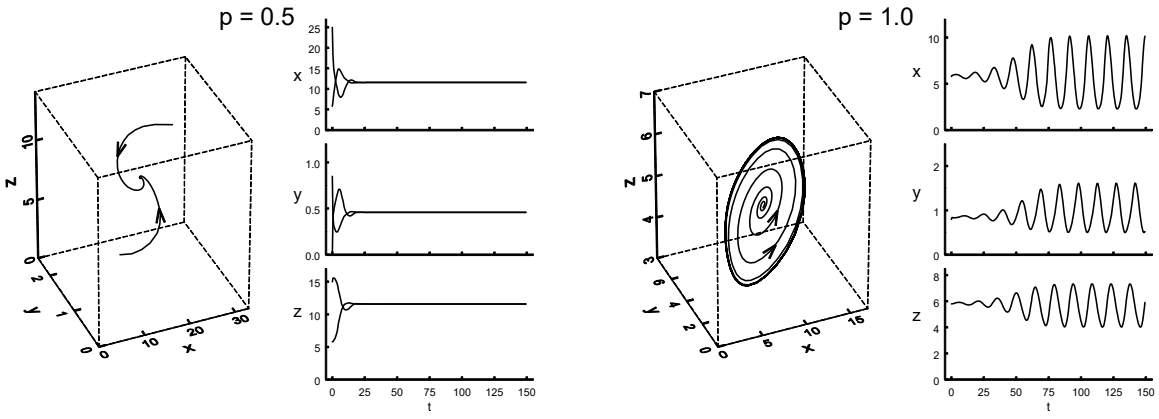


Figure 8.17. (a) Two orbits of the three dimensional system (8.41) with $p = 1/2$ are shown approaching the stable equilibrium E_1. (b) For $p = 1$, there exists a limit cycle, which is shown approached by an orbit starting near the unstable equilibrium E_1.

8.6.2 Strange Attractors and Chaos

Orbits of systems consisting of three or more equations can be very complicated. Bounded orbits can approach unusual sets in phase space that are neither equilibria or cycles. Solutions can exhibit such irregular oscillations that they are often called "chaotic" and attractors can be so exotic that they are called "strange attractors". These are generic terms and many different formal definitions of these concepts are available. A mathematical study of these kinds of solutions is usually very difficult. In this section we will look briefly at only one example of an exotic chaotic solution, using the computer as an aide. The example comes from the Lorenz system (8.40).

We saw in Example 8.19 that when $\rho < 1$ the point $\tilde{x}_e = \text{col}(0, 0, 0)$ is the only equilibrium of the Lorenz system (8.40) and that this equilibrium is (locally asymptotically) stable. For $\rho > 1$ the origin is unstable and there are two additional equilibria (see Example 8.18), both of which are (locally asymptotically) stable for ρ near 1 (see Exercise 8.64). As ρ increases, these two equilibria become unstable through a Hopf bifurcation to limit cycles which, in turn, themselves destabilize as orbits and their attractors become more complicated.

For example, Figure 8.18 shows part of the attractor when $\rho = 28$. This attractor is a double winged looking "surface" whose two wings are centered on the two (unstable) of the equilibria. An orbit on this surface moves in a complicated manner. It circles around one of the wings several times before embarking on an excursion to the other wing, around which it then circles before returning to the first branch, and so on indefinitely. These episodic excursions occur irregularly and the number consecutive circuits flown around each wing varies unpredictably.

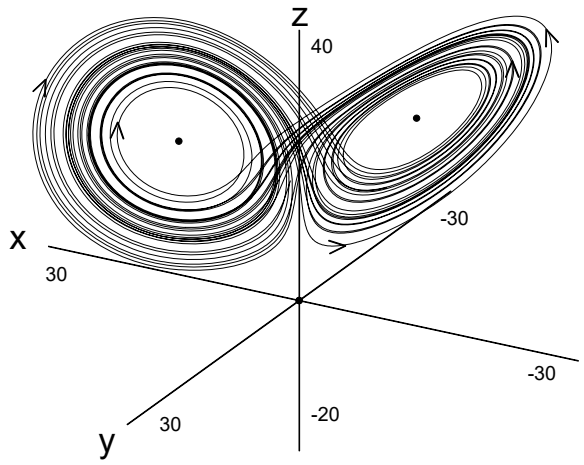


Figure 8.18

The irregular nature of the oscillations in the components $x(t)$, $y(t)$ and $z(t)$ of the orbit in Figure 8.18 is seen in Figure 8.19. This kind of solution is called *chaotic*. An important feature of chaos is that solutions whose initial conditions are very close together do not remain close together as t increases. This divergence of solutions (or orbits) that start arbitrarily close together is called *sensitivity to initial conditions*. See Figure 8.20. This property is a hallmark of chaos and has important consequences in applications. It means that small errors or perturbations in initial conditions result in drastically different long term predictions. Given that errors in measuring initial conditions and/or external disturbances are inevitable in applications, this property raises serious questions concerning the ability to make long term predictions in such systems.

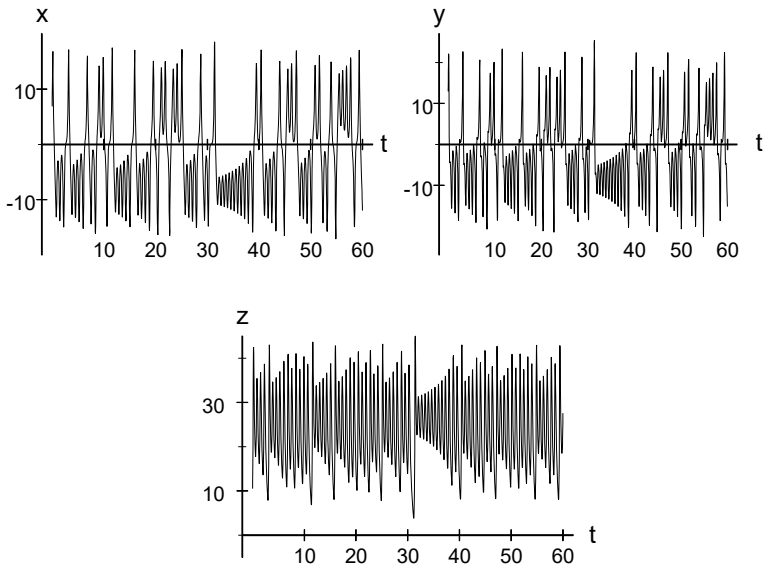


Figure 8.19. The x, y and z components of the orbits in Figure 8.18 show irregular oscillations.

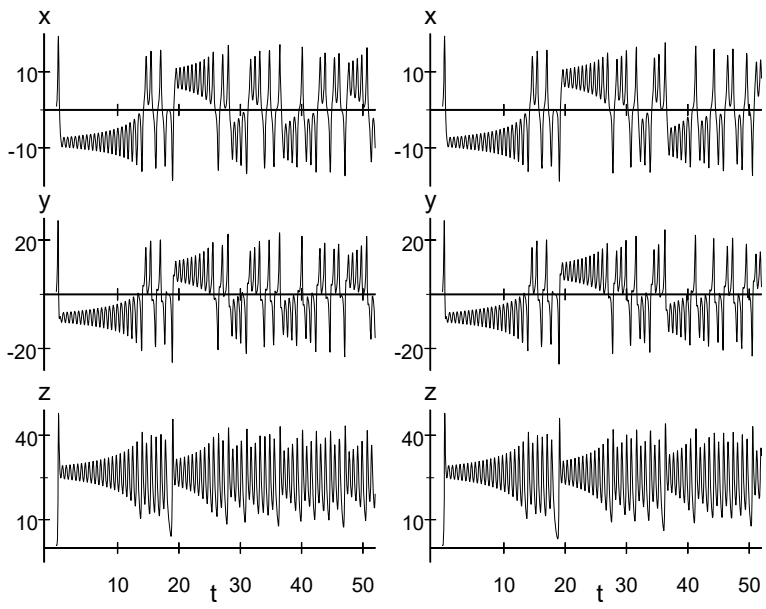


Figure 8.20. Two solution triples of the Lorenz system (8.40) with $\rho = 28$ with slightly different initial conditions. For $t > 30$ the solutions have little in common. (a) Initial condition $(x, y, z) = (1, 1, 1)$. (b) Initial condition $(x, y, z) = (1.0001, 1, 1)$.

8.7 Chapter Summary

In this chapter we studied nonlinear systems of autonomous equations. We studied techniques used to determine the stability properties of equilibria and learned methods that aid in the construction of phase portraits. The Linearization Principle (Theorems 8.1 and 8.7 and Theorem 8.1) relates the phase portrait near an equilibrium to that of the linearization at the equilibrium. The eigenvalues associated with the coefficient matrix of the linearization (i.e., the eigenvalues of the Jacobian of the system evaluated at the equilibria) determine the nature of the phase portrait in a neighborhood of the equilibrium (provided it is hyperbolic). We saw how the bifurcation scenarios for equilibria that we classified for single autonomous equations in Chapter 3 (blue-sky or saddle-node, pitchfork, and transcritical) occur in planar autonomous systems as well. Unlike the phase line portrait of a single autonomous equation, however, the global phase portrait of a system is not easily determined from the local phase portraits near equilibria. Poincaré-Bendixson theory provides techniques for constructing the global phase plane portrait. Moreover, this theory classifies possible attractors into only three types: equilibria, cycles, and cycle chains. Unfortunately, Poincaré-Bendixson theory does not extend to systems of dimension three and higher. Another technique for analyzing cycles, the Hopf Bifurcation Theorem, does extend to higher dimensions. We saw, moreover, that in three or higher dimensions attractors are not necessarily of the three types allowed in Poincaré-Bendixson theory, but can be considerable more complicated.

8.8 Exercises

Exercise 8.1 *Use a computer or calculator to help find all equilibria of the following systems and higher order equations.*

(a) $\begin{cases} x' = x - e^{-y} \\ y' = x - y \end{cases}$

(b) $\begin{cases} x' = y - \frac{1}{2-x} \\ y' = 1 - 3ye^{-x^2} \end{cases}$

(c) $\begin{cases} x' = \ln\left(\frac{1}{1+2x^2}\right) - y \\ y' = -3x - 4y \end{cases}$

(d) $\begin{cases} x' = \frac{3x}{1+y^2} - 1 \\ y' = y - x^2 \end{cases}$

(e) $x'' + 2x' + xe^{-x} = \frac{1}{4}$

(f) $x'' + x - \cos x = 0$

Exercise 8.2 *In the systems and equations below, $r > 0$ is a positive constant. Use geometric methods to study the equilibria. Without solving the equilibrium equations algebraically, determine those values of r for which there are no equilibria and those for which there are equilibria. In the latter case, determine how many equilibria there are.*

(a) $\begin{cases} x' = x^2 + y^2 - r^2 \\ y' = (x-3)^2 + y^2 - 4 \end{cases}$

(b) $\begin{cases} x' = x^2 + y^2 - r^2 \\ y' = x + y - 1 \end{cases}$

(c) $\begin{cases} x' = rx - y \\ y' = 6x + y - 8x^2 + 2x^3 \end{cases}$

(d) $\begin{cases} x' = y + x^2 y - x^2 \\ y' = -x + \frac{1}{r}y \end{cases}$

(e) $x'' + 2x' + xe^{-x} - r = 0$

(f) $x'' + x + \frac{r}{x-1} = 0$

Exercise 8.3 *Find all equilibria for the chemostat equations*

$$x' = (x_{in} - x)\, d - \frac{1}{\gamma}\frac{mx}{a+x}y$$
$$y' = \frac{mx}{a+x}y - dy.$$

In these equations all coefficients are positive constants.

For each of the systems and higher order equations below:

(a) Find all equilibria.

(b) Calculate Jacobian at each equilibrium.

(c) Apply Theorem 8.1, if possible, to determine the stability or instability of each equilibrium.

(d) Apply Theorem 8.2, if possible, to determine the stability of each equilibrium.

Exercise 8.4 $\begin{cases} x' = x - y^2 \\ y' = x - y \end{cases}$

Exercise 8.5 $\begin{cases} x' = x\,(2x - y) \\ y' = x^2 + y - 8 \end{cases}$

Exercise 8.6 $\begin{cases} x' = x\,(1 - x - y) \\ y' = y\,(2 - x - 4y) \end{cases}$

Exercise 8.7 $\begin{cases} x' = y - xy \\ y' = -x + xy \end{cases}$

Exercise 8.8 $\begin{cases} x' = 1 - x^2 - y^2 \\ y' = x - y \end{cases}$

Exercise 8.9 $\begin{cases} x' = x - x^2 y \\ y' = 1 - x^2 + xy \end{cases}$

Exercise 8.10 $x'' + x' + \sin x = 0$

Exercise 8.11 $x'' + xx' + x - x^3 = 0$

Exercise 8.12 $x'' + px' + qx^3 = 0$ *where* p *and* $q \neq 0$ *are constants (Duffing's equation)*

Exercise 8.13 $mx'' + k \sin x = 0$, *where* m *and* k *are positive constants (the frictionless pendulum equation)*

Exercise 8.14 *Consider the nonlinear system*

$$x' = \left(\frac{3}{2} - x - 2y\right) x$$
$$y' = \left(-\frac{1}{4} + x\right) y.$$

This is an example of a predator-prey system in which x *is the density of prey and* y *is the density of predator.*

(a) Find all equilibria.
(b) Calculate the Jacobian and evaluate it at each equilibrium.
(c) Determine the stability of each equilibrium.

Exercise 8.15 *Consider the nonlinear system*

$$x' = x - e^{-y}$$
$$y' = x - y.$$

(a) Find numerical approximations to all equilibria.
(b) Calculate the Jacobian and evaluate it at each equilibrium.
(c) Determine the stability of each equilibrium.

Exercise 8.16 *Show both roots of the quadratic $\lambda^2 + \beta\lambda + \gamma$ have negative real parts if and only if $\beta > 0$ and $\gamma > 0$.*

Exercise 8.17 *Show if either $\beta < 0$ or $\gamma < 0$ then at least one root of the quadratic $\lambda^2 + \beta\lambda + \gamma$ has a positive real part.*

Find the equilibria of the following systems and determine whether they are hyperbolic or nonhyperbolic. Determine the phase portrait in the neighborhood of all hyperbolic equilibria.

Exercise 8.18 $\begin{cases} x' = x - y^2 \\ y' = x - y \end{cases}$

Exercise 8.19 $\begin{cases} x' = x\,(2x - y) \\ y' = x^2 + y - 8 \end{cases}$

Exercise 8.20 $\begin{cases} x' = x\,(1 - x - y) \\ y' = y\,(2 - x - 4y) \end{cases}$

Exercise 8.21 $\begin{cases} x' = y - xy \\ y' = -x + xy \end{cases}$

Exercise 8.22 $\begin{cases} x' = 1 - x^2 - y^2 \\ y' = x - y \end{cases}$

Exercise 8.23 $\begin{cases} x' = x - x^2 y \\ y' = 1 - x^2 + xy \end{cases}$

Exercise 8.24 *Consider the predator-prey system*

$$\begin{aligned} x' &= \left(\frac{3}{2} - x - 2y\right) x \\ y' &= \left(-\frac{1}{4} + x\right) y. \end{aligned}$$

(a) Find all equilibria.

(b) Calculate the Jacobian matrix $J(x, y)$.

(c) Which equilibria are hyperbolic?

(d) Determine the phase portrait in the neighborhood of each equilibrium.

Exercise 8.25 *The system*

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha\left(x^2 - 1\right) y. \end{aligned}$$

is equivalent to the second order van der Pol equation. The only equilibrium is $\tilde{x}_e = \mathrm{col}(0, 0)$.

(a) Calculate the Jacobian matrix at $(0, 0)$.

(b) Determine the phase portrait in the neighborhood of $(0, 0)$. Identify any non-hyperbolic cases. Your answer will depend on the coefficient α.

Exercise 8.26 *Find the equilibria of the nonlinear system in Example 8.12.*

Exercise 8.27 *The general chemostat model equations are*

$$x' = (x_{in} - x)\, d - \frac{1}{\gamma}\frac{mx}{a+x}y$$
$$y' = \frac{mx}{a+x}y - dy.$$

All coefficients are positive constants. When $m \neq d$ there are two equilibria

$$\tilde{x}_e = (x_e, y_e) = \begin{cases} (x_{in}, 0) \\ \left(\frac{ad}{m-d}, \left(x_{in} - \frac{ad}{m-d}\right)\gamma\right) \end{cases} .$$

In applications only non-negative solutions are of interest. Therefore, assume

$$m > d \quad \text{and} \quad x_{in} > \frac{ad}{m-d}.$$

(If $x_{in} = ad/(m-d)$ then the second equilibrium coincides with the first.)
(a) Calculate the Jacobian matrix $J(x_e, y_e)$ for each equilibrium $\tilde{x}_e = \mathrm{col}\,(x_e, y_e)$.
(b) Determine the phase portrait near each equilibrium. Identify any nonhyperbolic cases.
(Hint: use the trace/determinant criteria.)

Use the Poincaré-Bendixson Theorems, the analytic tools in this section, and a computer sketch of the direction field, to determine the limit set S^+ of orbits of the following plane autonomous systems.

Exercise 8.28 $\begin{cases} x' = -x + y - xy^2 \\ y' = x - 2y \end{cases}$

Exercise 8.29 $\begin{cases} x' = -x - xy^2 \\ y' = -2y - x^2 y \end{cases}$

Exercise 8.30 $\begin{cases} x' = y - x^3 \\ y' = -x + y - y^3 \end{cases}$

Exercise 8.31 $\begin{cases} x' = x - y - x\,(x^2 + y^2) \\ y' = 2x - y\,(x^2 + y^2) \end{cases}$

Exercise 8.32 $\begin{cases} x' = 1 - x - xy^2 \\ y' = -2y - y^3 \end{cases}$

Exercise 8.33 $\begin{cases} x' = -2x \\ y' = 2 - y - x^2 y \end{cases}$

Exercise 8.34 *Consider the system*

$$x' = x + y - x\left(x^2 + y^2\right)$$
$$y' = -x + y - y\left(x^2 + y^2\right).$$

(a) Show $\tilde{x}_e = \mathrm{col}(0,0)$ is the only equilibrium.

(b) Use a computer to explore the phase portrait of the system. Formulate a conjecture about nonequilibrium orbits. Does there appear to be a limit cycle?

(c) Calculate the Jacobian and determine the phase portrait in the neighborhood of the equilibrium $(0,0)$. Does this agree with your observations in (b)?

(d) The distance from the equilibrium $\tilde{x}_e = \mathrm{col}(0,0)$ to a point $\tilde{x}(t) = \mathrm{col}(x(t), y(t))$ on an orbit is $r(t) = \sqrt{x^2(t) + y^2(t)}$. Show $r(t)$ satisfies the first order differential equation $r' = (1 - r^2)\, r$. Draw the phase line portrait of this equation for $r \geq 0$.

(e) Show the polar angle $\theta(t) = \tan^{-1}(y(t)/x(t))$ satisfies $\theta' = -1$.

(f) Use (d) and (e) to draw the phase plane portrait of the system. Compare the result with your observations in (b).

Exercise 8.35 *Consider the system*

$$x' = px + y - x\left(x^2 + y^2\right)$$
$$y' = -x + py - y\left(x^2 + y^2\right).$$

where p is a constant.

(a) Show $\tilde{x}_e = \mathrm{col}(0,0)$ is the only equilibrium of the system.

(b) Use a computer to explore the phase portrait of the system for selected values of p, both positive and negative. Formulate a conjecture about nonequilibrium orbits and limit cycles.

(c) Calculate the Jacobian and determine the phase portrait in the neighborhood of the equilibrium $\tilde{x}_e = \mathrm{col}(0,0)$. Does this agree with your observations in (b)?

(d) The distance from the equilibrium $\tilde{x}_e = \mathrm{col}(0,0)$ to a point $\tilde{x}(t) = \mathrm{col}(x(t), y(t))$ on an orbit is $r(t) = \sqrt{x^2(t) + y^2(t)}$. Show $r(t)$ satisfies the first order differential equation $r' = (p - r^2)\, r$. Draw the phase line portrait of this equation for $r \geq 0$.

(e) Show the polar angle $\theta(t) = \tan^{-1}(y(t)/x(t))$ satisfies $\theta' = -1$.

(f) Use (d) and (e) to draw the phase plane portrait of the system. How does the portrait depend on p? Compare your result with your observations in (b).

Exercise 8.36 *Consider the planar autonomous system*

$$x' = (1 - x - y)\, x$$
$$y' = (2 - x - y)\, y.$$

(a) Show the x-axis and y-axis consist of orbits.

(b) Show orbits starting in the first quadrant remain in the first quadrant for all t.

(c) Show there are no cycles in the first quadrant.

(d) What happens to a bounded orbit as $t \to +\infty$? Justify your answer.

Exercise 8.37 *Consider the planar autonomous system*

$$S' = -SI$$
$$I' = SI - I$$

(a) Show the S-axis and the I-axis consist of orbits.
(b) Show orbits starting in the first quadrant remain in the first quadrant for all t.
(c) Show there are no cycles in the first quadrant.
(d) What happens to a bounded orbit as $t \to +\infty$? Justify your answer.

Exercise 8.38 *Apply the Linearization Principle to each of the six equilibria of the system (8.27). Classify the hyperbolic equilibria and identify any nonhyperbolic equilibria.*

Exercise 8.39 *The distance from the origin $(0,0)$ to a point $\tilde{x}(t) = (x(t), y(t))$ on an orbit is $r(t) = \sqrt{x^2(t) + y^2(t)}$. If $r(t)$ is decreasing at t, then the orbit is moving toward the origin at the point $(x(t), y(t))$. A calculation shows*

$$\frac{dr(t)}{dt} = \frac{xx' + yy'}{r} = \frac{xf(x,y) + yg(x,y)}{r}.$$

One way to show orbits are (forward) bounded as $t \to +\infty$ is to show they cannot move away from the origin, at least at all points far away from the origin. Thus, if $xf(x,y)+yg(x,y) \leq 0$ for r sufficiently large (say for r greater than some positive number r_0), then orbits are (forward) bounded. Use this test to show that the orbits of the following systems are bounded as $t \to +\infty$.

(a) $\begin{cases} x' = -x - xy^2 \\ y' = -2y - x^2 y \end{cases}$

(b) $\begin{cases} x' = -x + y - x\left(x^2 + y^2\right) \\ y' = -x - 2y - y\left(x^2 + y^2\right) \end{cases}$

(c) $\begin{cases} x' = x + y - x\left(x^2 + y^2\right) \\ y' = -x + y - y\left(x^2 + y^2\right) \end{cases}$

(d) $\begin{cases} x' = px + y - x\left(x^2 + y^2\right) \\ y' = -x + py - y\left(x^2 + y^2\right) \\ \text{where } p \text{ is a real number} \end{cases}$

Exercise 8.40 *The following is called the **Dulac Criterion**. Suppose there exists a function $\mu = \mu(x,y)$ such that*

$$\frac{d}{dx}(\mu f) + \frac{d}{dy}(\mu g) \neq 0$$

for all (x,y) in a simply connected region D of the plane.[6] *Then the planar autonomous system*

$$x' = f(x,y), \quad y' = g(x,y)$$

has no cycle in D.

[6] A simply connected region in the plane is one with no holes in it. In a simply connected domain a closed loop can encircle no point lying outside of the domain.

(a) Use the Dulac Criterion with $\mu = 1/xy$ to prove part (c) in Exercise 8.36.
(b) Use the Dulac Criterion with $\mu = 1/I$ to prove part (c) in Exercise 8.37.
(c) Use the Dulac Criterion with $\mu = 1$ to show the system

$$x' = x - xy^2 + y^3$$
$$y' = 3y - yx^2 + x^3$$

has no cycle in the circle of radius 2 centered at the origin.
(d) Use the Dulac Criterion with $\mu = 1$ to show the system

$$x' = x + y - xy^2 - x^2y + y^2$$
$$y' = y - 2yx^2 + x^2$$

has no cycle inside the ellipse $2x^2 + 2xy + y^2 = 2$.

Exercise 8.41 *An **Extended Dulac Criterion** states that if*

$$\frac{d}{dx}(\mu f) + \frac{d}{dy}(\mu g) \neq 0$$

for some function $\mu = \mu(x, y)$ and all x and y, then an orbit of $x' = f(x, y)$, $y' = g(x, y)$ is either unbounded or approaches an equilibrium as $t \to +\infty$.[7] Apply this criterion to the systems below. HINT: First try $\mu = 1$ then, if that fails to work, try $\mu = x^p y^q$ for some appropriate numbers p and q.

(a) $\begin{cases} x' = -x - xy^2 \\ y' = -2y - x^2y \end{cases}$
(b) $\begin{cases} x' = -x + y - xy \\ y' = 2x - y + \frac{1}{2}y^2 \end{cases}$
(c) $\begin{cases} x' = x + y + 2x^3y \\ y' = 1 - \frac{1}{2}y + x^2 \end{cases}$
(d) $\begin{cases} x' = 1 + x^2 - xy^2 \\ y' = -3xy + \frac{2}{3}y^3 \end{cases}$
(e) $\begin{cases} x' = x + y^2 + x^3 \\ y' = -x + y + yx^2 \end{cases}$
(f) $\begin{cases} x' = -x - y^2 - x^3 \\ y' = x - y - yx^2 \end{cases}$

Exercise 8.42 *(For readers who have studied multi-variable calculus). Use Green's Theorem to prove (by a contradiction argument) the Dulac Criterion in Exercise 8.40.*

Exercise 8.43 *Find all equilibria of the system*

$$x' = px - x^3$$
$$y' = -y.$$

Apply the Linearization Principle to classify each equilibrium. Describe the phase plane portrait, and how it depends on the parameter p. Show how a pitchfork bifurcation occurs in this system.

[7] C.C. McCluskey and J.S. Muldowney, SIAM Review 40, No. 4 (1998), 931-934

Exercise 8.44 *Find all equilibria of the system*

$$\begin{aligned} x' &= px - x^2 \\ y' &= -y. \end{aligned}$$

Apply the Linearization Principle to classify each equilibrium. Describe the phase plane portrait, and how it depends on the parameter p. Show how a transcritical bifurcation occurs in this system.

Classify the bifurcations that occur in the following systems with parameter p.

Exercise 8.45 $\begin{cases} x' = p - x^2 - y^2 \\ y' = 1 - x - y \end{cases}$

Exercise 8.46 $\begin{cases} x' = \left[(x-1)^2 + y^2 - p\right](x^2 + y^2) \\ y' = x\,(x^2 + y^2) \end{cases}$

Exercise 8.47 $\begin{cases} x' = (p - 2x^2 - y^2)\left[(x-1)^2 + y^2\right] \\ y' = (x-1)\left[(x-1)^2 + y^2\right] \end{cases}$

Exercise 8.48 $\begin{cases} x' = y + (1-x)(2-x) \\ y' = y - px^2 \end{cases}$

Exercise 8.49 $\begin{cases} x' = px - xy \\ y' = py - xy \end{cases}$

Exercise 8.50 $\begin{cases} x' = (x - 2p)(p - y) \\ y' = (y + p)(p - x) \end{cases}$

Exercise 8.51 $\begin{cases} x' = y - \ln x \\ y' = px - y \end{cases}$

Exercise 8.52 $\begin{cases} x' = y - e^{-x} \\ y' = y + x^2 - p \end{cases}$

Exercise 8.53 *The nonlinear, second order equation $x'' + (p - \cos x)\sin x = 0$, $p > 0$, is called the rotating pendulum equation. It models the motion of a swinging pendulum whose pivot rotates in a circle; x is the angle made by the pendulum with the vertical. Note $x_e = 0$ is an equilibrium for all p. Does this equilibrium undergo a bifurcation as a function of $p > 0$?*

Exercise 8.54 *Show a transcritical bifurcation of equilibria and an exchange of stability occur in the competition system (8.33) at the critical value $p_0 = 1$.*

Exercise 8.55 *Apply the Linearization Principle to the equilibrium $\tilde{x}_e = \mathrm{col}(0,0)$ of the competition system (8.33).*

For each of the systems below show the equilibrium $\tilde{x}_e = \text{col}(0,0)$ undergoes a Hopf bifurcation by supplementing the criteria (8.36) with computer sketches of the phase plane portrait. Identify the bifurcation value p_0.

Exercise 8.56 $\begin{cases} x' = 1 + (p-1)x + (p-2)y - e^{xy} \\ y' = x + y - y^3 \end{cases}$

Exercise 8.57 $\begin{cases} x' = 1 - e^{px+y} \\ y' = 1 - x^2 y - e^{-x+py} \end{cases}$

Exercise 8.58 $\begin{cases} x' = px + y - x\left(x^2+y^2-1\right)\left(x^2+y^2-4\right) \\ y' = -3x + py - y\left(x^2+y^2-1\right)\left(x^2+y^2-4\right) \end{cases}$

Exercise 8.59 $\begin{cases} x' = px + 2y + x\left(x^2+y^2-1\right) \\ y' = -x + py + 3y\left(x^2+y^2-1\right) \end{cases}$

Exercise 8.60 $\begin{cases} x' = y - x^3 \\ y' = -2x - py - x^3 - xy^2 \end{cases}$

Exercise 8.61 $\begin{cases} x' = 1 + x - e^x + y \\ y' = -x - y\ln\left(p + x^2 + y^2\right) \end{cases}$

Exercise 8.62 *The origin $(x,y,z) = (0,0,0)$ is an equilibrium of each of the systems below. Find the Jacobian of the system and apply the Linearization Principle to determine the stability of the origin (if possible).*

(a) $\begin{cases} x' = x(1-x) - xy - xz \\ y' = -y + xy \\ z' = -z + xz \end{cases}$

(b) $\begin{cases} x' = x - ye^{-z} \\ y' = y - ze^{-x} \\ z' = z - xe^{-y} \end{cases}$

(c) $\begin{cases} x' = -x^2 - xy^2 - xz \\ y' = y - xy - y^2 - yz^2 \\ z' = -z - zx^2 - yz - z^2 \end{cases}$

(d) $\begin{cases} x' = \sin(x+y+z) \\ y' = \ln(1+2x+y-z) \\ z' = -z + x^2 + y^2 \end{cases}$

(e) $\begin{cases} x' = -1 + e^{y-2x} \\ y' = -z \\ z' = -1 + e^{y-2z} \end{cases}$

(f) $\begin{cases} x' = x - \sin(5x - y) \\ y' = -2y + z - x^2 \\ z' = x - 5ze^{-y} \end{cases}$

Exercise 8.63 *Find the equilibria of the systems below and determine their stability properties by applying the Linearization Principle.*

$$(a)\begin{cases} x' = x + y - xy - x^2 \\ y' = 4z - y - xy \\ z' = x - z \end{cases}$$

$$(b)\begin{cases} x' = x + y - xy - x^2 \\ y' = -4z - y - xy \\ z' = x - z \end{cases}$$

Exercise 8.64 *Show the two equilibria*

$$\tilde{x}_e = \text{col}\left(\sqrt{8\left(\rho-1\right)/3},\ \sqrt{8\left(\rho-1\right)/3}, \rho - 1\right)$$
$$\tilde{x}_e = \text{col}\left(-\sqrt{8\left(\rho-1\right)/3},\ -\sqrt{8\left(\rho-1\right)/3}, \rho - 1\right)$$

of the Lorenz equations (8.40) are stable for $\rho > 1$ close to 1. (Hint Let $\lambda = \lambda(\rho)$ be the characteristic root (eigenvalue) that equals 0 at $\rho = 1$. Show $d\lambda(1)/d\rho < 0$.)

Exercise 8.65 . *Use the Linearization Principle to show the equilibrium $\tilde{x}_e = \text{col}(0,0,0)$ of the system (8.41) is unstable for all $p > 0$. (HINT: find the characteristic polynomial and argue that it always has a positive root. You need not calculate this root, nor the other two roots.)*

Exercise 8.66 *This exercise is a computer investigation of some bifurcations and chaos that occurs in the three dimensional (Roessler) system*

$$x' = -y - z$$
$$y' = x + by$$
$$z' = c + z(x - a)$$

with $a = 5.7$ and $b = 0.2$.

(a) Use a computer program to determine what orbits do as $t \to +\infty$ for values of c ranging from 10.0 down to 0.2. Describe the bifurcations that occur. (A "bifurcation" here means a significant change in what orbits approach as $t \to +\infty$.)

(b) How would you describe the orbits when $b = 0.2$?

Exercise 8.67 *(a) If $\tilde{x}(t) = \text{col}\left(x\left(t\right), y\left(t\right)\right)$ is a solution of the planar autonomous system (8.1) and if τ is any real number, show the translation $\tilde{x}(t+\tau) = \text{col}(x(t+\tau), y(t+\tau))$ is also a solution.*

(b) Show a solution and any of its translates all give the same orbit in the phase plane.

(c) Prove the following: if two solutions $\tilde{x}_1(t) = \text{col}(x_1(t), y_1(t))$ and $\tilde{x}_2(t) = \text{col}(x_2(t), y_2(t))$ of a planar autonomous system have the same orbit, then each is a translate of the other. (HINT. Pick any point $\tilde{x}_0 = \text{col}(x_0, y_0)$ on the common orbit. Then for some t_1 and t_2 we have $\tilde{x}_1(t_1) = \tilde{x}_0$ and $\tilde{x}_2(t_2) = \tilde{x}_0$. Show $\tilde{x}_2(t)$ is a translate of $\tilde{x}_1(t)$ with $\tau = t_1 - t_2$.)

Exercise 8.68 *Suppose two orbits have a point $\tilde{x}_0 = \text{col}(x_0, y_0)$ in common. Prove the orbits must therefore be identical. This fact shows different orbits cannot have a point in common in the phase plane. (HINT. Let $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ be solutions that give the two orbits. Show they are translates of each other. For some t_1 and t_2 we have $\tilde{x}_1(t_1) = \tilde{x}_0$ and $\tilde{x}_2(t_2) = \tilde{x}_0$. Take $\tau = t_1 - t_2$.)*

Exercise 8.69 *Find the equilibria of the planar autonomous systems below. If it applies, use the Linearization Principle to determine their stability properties.*

(a) $\begin{cases} x' = x + y + x^2 + y^2 \\ y' = y + xy \end{cases}$

(b) $\begin{cases} x' = y^2 - (x-1)^2 \\ y' = 1 + x - 2y \end{cases}$

(c) $\begin{cases} x' = -y - x^2 \\ y' = -x + xy^2 \end{cases}$

(d) $\begin{cases} x' = y - x^2 \\ y' = -x + xy^2 \end{cases}$

(e) $\begin{cases} x' = -x - xy^2 \\ y' = x - 2y^2 \end{cases}$

(f) $\begin{cases} x' = x - xy^2 \\ y' = x - 2y^2 \end{cases}$

Exercise 8.70 *Find the equilibria of the planar autonomous systems below. Using p as a parameter identify all equilibrium bifurcations.*

(a) $\begin{cases} x' = x + py + x^2 + y^2 \\ y' = y + xy \end{cases}$

(b) $\begin{cases} x' = y^2 - (x-1)^2 \\ y' = p + x - 2y \end{cases}$

(c) $\begin{cases} x' = x - xy^2 \\ y' = p + x - 2y^2 \end{cases}$

(d) $\begin{cases} x' = py - x^2 \\ y' = -x + xy^2 \end{cases}$

(e) $\begin{cases} x' = x + y - y^2 \\ y' = x - y + p^2y^2 \end{cases}$

(f) $\begin{cases} x' = px - xy^2 \\ y' = x - 2y^2 \end{cases}$

(g) $\begin{cases} x' = x - xy \\ y' = p + x - 2y \end{cases}$

(h) $\begin{cases} x' = x - x^3y - p \\ y' = -y + xy^2 \end{cases}$

Exercise 8.71 *The origin $\tilde{x}_e = \text{col}(0,0)$ is an equilibrium of the systems below. For which values of p do the Hopf bifurcation criteria (8.36) hold at $\tilde{x}_e = \text{col}\,(0,0)$? Use a computer to determine if stable limit cycles bifurcate or not. The coefficient p satisfies $-1 < p < 1$.*

(a) $\begin{cases} x' = px - 2y - x\sin\,(x^2 + y^2) \\ y' = x - y\sin\,(x^2 + y^2) \end{cases}$

(b) $\begin{cases} x' = (p-2)\,x + (p+5)\,y - 10xy^2 \\ y' = -x + (p+2)\,y - 10yx^2 \end{cases}$

Exercise 8.72 *(a) Find all equilibria of the system*

$$\begin{aligned} x' &= (1 - x - 2y)\,x \\ y' &= (1 - 2x - y)\,y. \end{aligned}$$

(b) Find the Jacobian of this system.

(c) Determine the local phase portrait in a neighborhood of each equilibrium (if possible).

Exercise 8.73 *(a) Find the equilibria of the system*

$$\begin{aligned} x' &= y + \alpha\left(x - \frac{1}{3}x^3\right), \quad \alpha > 0 \\ y' &= -x. \end{aligned}$$

(b) Find the Jacobian of this system and evaluate it at the equilibrium. Show the equilibrium is hyperbolic for all $\alpha > 0$.

(c) Determine the local phase plane portrait near the equilibrium.

(d) Use a computer to study the phase plane portrait for selected values of α *between* 0 *and* 2. *What do you conclude about the local phase portrait and stability of the equilibrium? What do orbits do as* $t \to +\infty$*?*

(e) Use a computer to study the phase plane portrait in the case $\alpha = 0$. *What do orbits do as* $t \to +\infty$. *(HINT: be sure to use a sufficiently small step size.)*

(f) If $x = x(t)$, $y = y(t)$ *is a solution pair of, show* $x = x(t)$ *solves the van der Pol equation (8.7).*

Exercise 8.74 *(a) Show* $\tilde{x}_e = \mathrm{col}\,(b/d, 0)$ *is an equilibrium of the AIDS equations (8.4). Find all other equilibria.*

(b) Find the Jacobian of the system.

(c) Determine the phase portrait in the neighborhood of $\tilde{x}_e = \mathrm{col}\,(b/d, 0)$. *(All coefficients are positive.)*

Exercise 8.75 *The orbit of a (nonequilibrium) periodic solution pair of a planar autonomous system (8.1) is a closed loop. The object of this exercise is to prove the converse. That is, if the orbit of a solution is a closed loop, then the solution pair is periodic.*

(a) Let $\tilde{x}\,(t) = \mathrm{col}(x(t), y(t))$ *be a solution pair of (8.1). Let* τ *be any real number. Show* $\tilde{x}\,(t + \tau)$ *is also a solution of (8.1).*

(b) Suppose the orbit associated with a non-equilibrium solution $\tilde{x}\,(t) = \mathrm{col}(x(t), y(t))$ *is self intersecting. That is, suppose there are two different values of* t, *say* $t_1 > t_2$, *for which* $\mathrm{col}(x(t_1), y(t_1)) = \mathrm{col}(x(t_2), y(t_2))$. *Show* $\tilde{x}\,(t) = \mathrm{col}(x(t), y(t))$ *is periodic. (Hint. Let* $\tau = t_2 - t_1$ *and use (a) and the Fundamental Existence and Uniqueness Theorem 4.1 to prove* $\tilde{x}(t + \tau) = \tilde{x}(t)$ *for all* t.*)*

Since a non-equilibrium loop orbit is self intersecting it follows that it is associated with a periodic orbit.

Exercise 8.76 *Use us a computer program to investigate the nonlinear system*

$$x' = (1 - y)x$$
$$y' = (-1 + x)y.$$

Describe the orbits lying in the positive quadrant (i.e. the first quadrant $x > 0$, $y > 0$). Are there any cycles? Any limit cycles? What does the Linearization Principle tell you about the equilibrium $(x_e, y_e) = (1, 1)$?

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Appendix A

Answers to Exercises

A.1 Preliminaries

EXERCISE 0.1. The equation is first order. This is because the first order derivative x' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.3. The equation is first order. This is because the first order derivative x' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.5. The equation is second order. This is because the second order derivative x'' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.7. The equation is first order. This is because the first order derivative x' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.9. The equation is first order. This is because the first order derivative x' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.11. The equation is first order. This is because the first order derivative x' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.13. The equation is third order. This is because the third order derivative x''' appears in the equation and no higher order derivative appears in the equation.
EXERCISE 0.15. $x(t) = e^{-3t}$ is a solution because

$$x' + 3x = -3e^{-3t} + 3e^{-3t} = 0$$

EXERCISE 0.17. $x(t) = -e^{-3t}$ is a solution because

$$x' + 3x = -3\left(-e^{-3t}\right) + 3\left(-e^{-3t}\right) = 0$$

EXERCISE 0.19. $x(t) = e^{2t}$ is not a solution because

$$x' - 2tx = 2e^{2t} - 2te^{2t} = 2(1-t)e^{2t} \neq 0$$

EXERCISE 0.21. $x(t) = -7e^{t^2}$ is a solution because

$$x' - 2tx = -7e^{t^2}(2t) - 2t\left(-7e^{t^2}\right) = 0$$

EXERCISE 0.23. $x(t) = t^{-3/2}$ is a solution because

$$2x' + 3x^{5/3} = 2\left(-\frac{3}{2}t^{-5/2}\right) + 3\left(t^{-3/2}\right)^{5/3} = -3t^{-5/2} + 3t^{-5/2} = 0$$

EXERCISE 0.25. $x(t) = (t-1)^{-3/2}$ is a solution because

$$2x' + 3x^{5/3} = 2\left(-\frac{3}{2}\right)(t-1)^{-5/2} + 3\left((t-1)^{-3/2}\right)^{5/3} = 0$$

EXERCISE 0.27. $x(t) = t^{3/2}$ is not a solution because

$$2x' + 3x^{5/3} = 2\left(\frac{3}{2}\right)t^{1/2} + 3\left(t^{3/2}\right)^{5/3} \neq 0$$

EXERCISE 0.29. $x(t) = (t-2)^{-2/3}$ is not a solution because

$$\begin{aligned} 2x' + 3x^{5/3} &= 2\left(-\frac{2}{3}\right)(t-2)^{-5/3} + 3\left((t-2)^{-2/3}\right)^{5/3} \\ &= -4(t-2)^{-5/3} + 3(t-2)^{-10/9} \neq 0 \end{aligned}$$

EXERCISE 0.31. $x(t) = e^{-2t}$ is not a solution because

$$x'' - 5x' + 6x = 4e^{-2t} - 5(-2)e^{-2t} + 6e^{-2t} = 20e^{-2t} \neq 0$$

EXERCISE 0.33. $x(t) = e^{3t}$ is a solution because

$$x'' - 5x' + 6x = 9e^{3t} - 5(3)e^{3t} + 6e^{3t} = 0$$

EXERCISE 0.35. $x(t) = 5e^{2t}$ is a solution because

$$x'' - 5x' + 6x = 20e^{2t} - 5\left(10e^{2t}\right) + 6\left(5e^{2t}\right) = 0$$

EXERCISE 0.37. $x(t) = e^{2t} + e^{3t}$ is a solution because

$$x'' - 5x' + 6x = \left(4e^{2t} + 9e^{3t}\right) - 5\left(2e^{2t} + 3e^{3t}\right) + 6\left(e^{2t} + e^{3t}\right) = 0$$

EXERCISE 0.39.

(a) $x(t) = e^{-5t}$ is a solution because

$$x' + 5x = -5e^{-5t} + 5e^{-5t} = 0$$

(b) $x(t) = 3e^{-5t}$ is a solution because

$$x' + 5x = -15e^{-5t} + 5\left(3e^{-5t}\right) = 0$$

(c) $x(t) = 5e^{-3t}$ is not a solution because

$$x' + 5x = -15e^{-3t} + 5\left(5e^{-3t}\right) = 10e^{-3t} \neq 0$$

EXERCISE 0.41.

(a) $x(t) = t^{-1}$ is a solution because

$$x' + x^2 = -\frac{1}{t^2} + \left(\frac{1}{t}\right)^2 = 0$$

(b) $x(t) = 2t^{-1}$ is not a solution because

$$x' + x^2 = -\frac{2}{t^2} + \left(\frac{2}{t}\right)^2 = \frac{2}{t^2} \neq 0$$

(c) $x(t) = (t-2)^{-1}$ is a solution because

$$x' + x^2 = -\frac{1}{(t-2)^2} + \left(\frac{1}{t-2}\right)^2 = 0$$

EXERCISE 0.43.

(a) $x(t) = \ln t$ is a solution because

$$tx'' + x' = t\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t}\right) = 0$$

(b) $x(t) = 1$ is a solution because

$$tx'' + x' = t(0) + (0) = 0$$

(c) $x(t) = t$ is not a solution because

$$tx'' + x' = t(0) + (1) = 1 \neq 0$$

EXERCISE 0.44. $x(t) = e^{4t}$ is a solution because

$$x''' - 4x'' - 4x' + 16x = 64e^{4t} - 4\left(16e^{4t}\right) - 4\left(4e^{4t}\right) + 16e^{4t} = 0$$

EXERCISE 0.46. $x(t) = ce^{4t}$ is a solution for any constant c because

$$x''' - 4x'' - 4x' + 16x = 64ce^{4t} - 4\left(16ce^{4t}\right) - 4\left(4ce^{4t}\right) + 16\left(ce^{4t}\right) = 0$$

EXERCISE 0.48. $x(t) = \frac{1}{2}e^{-2t}$ is a solution because

$$x''' - 4x'' - 4x' + 16x = -4e^{-2t} - 4\left(2e^{-2t}\right) - 4\left(-e^{-2t}\right) + 16\left(\frac{1}{2}e^{-2t}\right) = 0$$

EXERCISE 0.50. $x(t) = e^{6t}$ is not a solution because

$$\begin{aligned} x''' - 4x'' - 4x' + 16x &= 216e^{6t} - 4\left(36e^{6t}\right) - 4\left(6e^{6t}\right) + 16\left(e^{6t}\right) \\ &= 64e^{6t} \neq 0 \end{aligned}$$

EXERCISE 0.51. $x(t) = e^t$ is a solution because

$$x'' + x' - 2x = e^t + e^t - 2e^t = 0$$

EXERCISE 0.53. $x(t) = e^t e^{-2t} = e^{-t}$ is not a solution because

$$x'' + x' - 2x = e^{-t} + \left(-e^{-t}\right) - 2e^{-t} = -2e^{-t} \neq 0$$

EXERCISE 0.55. Yes, because

$$\begin{aligned} x' &= 4e^{4t} = 2\left(e^{4t}\right) - \left(-2e^{-4t}\right) = 2x - y \\ y' &= -8e^{4t} = -6e^{4t} + \left(-2e^{4t}\right) = -6x + y \end{aligned}$$

EXERCISE 0.57. Is a solution pair because

$$\begin{aligned} x' &= e^{t} = 4\left(e^{t}\right) + 3\left(-e^{t}\right) = 4x + 3y \\ y' &= -e^{t} = -2\left(e^{t}\right) - \left(-e^{t}\right) = -2x - y \end{aligned}$$

EXERCISE 0.59. Is not a solution pair because

$$\begin{aligned} x' &= e^{t} \neq 4\left(e^{t}\right) + 3\left(e^{t}\right) = 4x + 3y \\ y' &= e^{t} \neq -2\left(e^{t}\right) - \left(e^{t}\right) = -2x - y \end{aligned}$$

EXERCISE 0.61. Is a solution pair because

$$\begin{aligned} x' &= 6e^{2t} = 4\left(3e^{2t}\right) + 3\left(-2e^{2t}\right) = 4x + 3y \\ y' &= -4e^{2t} = -2\left(3e^{2t}\right) - \left(-2e^{2t}\right) = -2x - y \end{aligned}$$

EXERCISE 0.63. Is a solution pair because

$$\begin{aligned} x' &= e^{t} + 6e^{2t} \\ &= 4\left(e^{t} + 3e^{2t}\right) + 3\left(-e^{t} - 2e^{2t}\right) = 4x + 3y \\ y' &= -e^{t} - 4e^{2t} \\ &= -2\left(e^{t} + 3e^{2t}\right) - \left(-e^{t} - 2e^{2t}\right) = -2x - y \end{aligned}$$

EXERCISE 0.65. Is a solution pair for all c_1 and c_2 because

$$\begin{aligned} x' &= c_1e^{t} + 6c_2e^{2t} \\ &= 4\left(c_1e^{t} + 3c_2e^{2t}\right) + 3\left(-c_1e^{t} - 2c_2e^{2t}\right) = 4x + 3y \\ y' &= -c_1e^{t} - 4c_2e^{2t} \\ &= -2\left(c_1e^{t} + 3c_2e^{2t}\right) - \left(-c_1e^{t} - 2c_2e^{2t}\right) = -2x - y \end{aligned}$$

EXERCISE 0.66.

Exercise 15. The exponential function $x(t) = e^{-3t}$ is solution for all t.

EXERCISE 0.67.

Exercise 21. The function $x(t) = -7e^{t^2}$ is a solution for all t.

Exercise 23. The function $x(t) = t^{-3/2} = 1/\left(\sqrt{t}\right)^3$ a solution for all $t > 0$.

Exercise 25. The function $x(t) = (t-1)^{-3/2} = 1/\left(\sqrt{t-1}\right)^3$ a solution for all $t > 1$.

EXERCISE 0.68. Let $y = x'$. Then

$$y' = x'' = -x' + 3x$$

and

$$\begin{aligned} x' &= y \\ y' &= 3x - y \end{aligned}$$

EXERCISE 0.70. Let $y = x'$. Then

$$y' = x'' = \frac{1}{3}\left(1 + 6xx' - 12x^2\right)$$

and

$$\begin{aligned} x' &= y \\ y' &= -4x^2 + 2xy + \frac{1}{3} \end{aligned}$$

EXERCISE 0.72. Let $y = x'$ and $z = y'$. Then

$$z' = x''' = \frac{1}{2}\left(-3 + 6x'' - 4x' - x\right)$$

and

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -\frac{1}{2}x - 2y + 3z - \frac{3}{2} \end{aligned}$$

EXERCISE 0.74. Let $y = x'$. Then

$$\begin{aligned} y' &= x'' \\ &= \cos t - 2x' - 4x \end{aligned}$$

and

$$\begin{aligned} x' &= y \\ y' &= -4x - 2y + \cos t \end{aligned}$$

EXERCISE 0.76. Let $y = x'$. Then

$$y' = x'' = t^{-2}\left(-\left(x'\right)^2 - \cos x\right)$$

and

$$\begin{aligned} x' &= y \\ y' &= -t^{-2}y^2 - t^{-2}\cos x \end{aligned}$$

EXERCISE 0.78. Let $y = x'$, $w = z'$. Then

$$\begin{aligned} y' &= x'' = -2y - x + z \\ w' &= z'' = -w + 2x - z \end{aligned}$$

and

$$\begin{aligned} x' &= y \\ y' &= -2y - x + z \\ z' &= w \\ w' &= -w + 2x - z \end{aligned}$$

EXERCISE 0.80. Let $y = x'$ and $w = z'$. Then

$$\begin{aligned} y' = x'' &= \frac{1}{2}\left(x' - 2z' - 4x + 8z\right) \\ w' = z'' &= \sin t - 2x' + z' + x - 3z \end{aligned}$$

and

$$\begin{aligned} x' &= y \\ y' &= -2x + \frac{1}{2}y + 4z - w \\ z' &= w \\ w' &= x - 2y - 3z + w + \sin t \end{aligned}$$

EXERCISE 0.82. The equation is linear because it has the form $x' = p(t)x + q(t)$ with $p(t) = 2$ and $q(t) = 1$.
EXERCISE 0.84. The equation is nonlinear (because of the x^2 term).
EXERCISE 0.86. The equation is linear because it has the form $x' = p(t)x + q(t)$ with $p(t) = t^{-2}$ and $q(t) = 0$.
EXERCISE 0.88. The equation is nonlinear (because of the $\sin x$ term).
EXERCISE 0.90. The equation is nonlinear (because of the xx' term).
EXERCISE 0.92. The equation is linear because it is linear in x'', x' and x with coefficients t^2, t and 1.
EXERCISE 0.94. The equation is nonlinear (because of the $(1 - x)\,x$ term).
EXERCISE 0.96. The equation is nonlinear (because of the e^{-x} term).
EXERCISE 0.98. The system is linear because both equations are linear. The first equation is linear in x', x and y and the second equation is linear in y', x and y.
EXERCISE 0.100. The system is nonlinear (because of the xy term)
EXERCISE 0.102. The equation is linear because it has the form $x' = p(t)x + q(t)$ with $p(t) = -a$ and $q(t) = ar$.
EXERCISE 0.104. The equation is nonlinear (because for the equation to be linear $df(x)/dx$ would have to be a constant and $d^2f(x)/dx^2$ would equal 0).
EXERCISE 0.106. The equation is nonlinear (because of the term $\sin x$).
EXERCISE 0.108. The equation is nonlinear (because of the term $\ln(ty)$).
EXERCISE 0.110. The system is linear because both equations are linear. The first equation is linear in x', x and y and the second equation is linear in y', x and y.
EXERCISE 0.111. We can rewrite the equation as the linear equation $x' = (\ln 2)\,x$.

A.2 Chapter 1: First Order Equations

EXERCISE 1.1.

$$x(t) = \int \left(1 + t^2\right) dt = t + \frac{1}{3}t^3 + c$$

EXERCISE 1.3.

$$x(t) = \int e^{2t} dt = \frac{1}{2}e^{2t} + c$$

EXERCISE 1.5.

$$x(t) = \int t^2 dt = \frac{1}{3}t^3 + c$$
$$x(1) = \frac{1}{3} + c = 2 \Longrightarrow c = \frac{5}{3} \Longrightarrow x(t) = \frac{1}{3}t^3 + \frac{5}{3}$$

EXERCISE 1.7.

$$x(t) = \int te^{-t} dt = -te^{-t} - e^{-t} + c$$
$$x(0) = -1 + c = 1 \Longrightarrow c = 2 \Longrightarrow x(t) = -te^{-t} - e^{-t} + 2$$

EXERCISE 1.10. Theorem 1.1does not apply since $f(t,x) = t^2/x^2$ is not defined (and therefore cannot be continuous) at $x = x_0 = 0$. No conclusion can be drawn from the theorem.

EXERCISE 1.12. Theorem 1.1applies since

$$f(t,x) = \tan x \quad \text{and} \quad \frac{df(t,x)}{dx} = \tan^2 x + 1$$

are continuous for x near $x_0 = 0$ (and t near $t_0 = 0$). There exists a unique solution on an interval containing $t_0 = 0$.

EXERCISE 1.13. Theorem 1.1does not apply since $f(t,x) = \tan x$ is not defined (and therefore cannot be continuous) at $x = x_0 = \pi/2$. No conclusion can be drawn from the theorem..

EXERCISE 1.15. Theorem 1.1applies since

$$f(t,x) = \frac{1}{\sin x} \quad \text{and} \quad \frac{df(t,x)}{dx} = -\frac{\cos x}{\sin^2 x}$$

are continuous for t near $t_0 = 0$ and x near $x_0 = \pi/2$ (where $\sin x \neq 0$). There exists a unique solution on an interval containing $t_0 = 0$.

EXERCISE 1.17. From

$$f(t,x) = |x| \quad \text{and} \quad \frac{df(t,x)}{dx} = \frac{x}{|x|}$$

we see that both of these functions are continuous for x near $x_0 = 10$ and for all t. Therefore Theorem 1.1 applies and we conclude that there exists a unique solution on an interval containing $t_0 = 0$.

EXERCISE 1.19. If $a > 0$ then

$$f(t,x) = \ln(a-x) \quad \text{and} \quad \frac{df(t,x)}{dx} = \frac{1}{x-a}$$

are continuous for t near $t_0 = 0$ and x near $x_0 = 0$ (where $x \neq a$). Thus, when $a > 0$ Theorem 1.1 implies there is a unique solution on an interval containing $t_0 = 0$. For $a \leq 0$, f is not defined for all x near $x_0 = 0$ (since $a - x < 0$ for $x > 0$) and no conclusion can be drawn from the theorem.

EXERCISE 1.21. If $|a| > 2$ then

$$f(t,x) = \left(a^2 - x^2\right)^{1/2} \quad \text{and} \quad \frac{df(t,x)}{dx} = -x\left(a^2 - x^2\right)^{-1/2}$$

are continuous for t near $t_0 = 1$ and x near $x_0 = 2$. Thus, Theorem 1.1applies when $|a| > 2$ and there is a unique solution on an interval containing $t_0 = 1$. For $|a| \leq 2$ no conclusion can be drawn from the theorem.

EXERCISE 1.23. The function $f(t,x) = \ln(t^2 + x^2)$ is continuous and differentiable if $t \neq 0$, $x \neq 0$. Therefore, if $t_{_0} \neq 0$, $x_0 \neq 0$ then the initial value problem has a unique solution on an interval containing t_0. For $t_0 = 0$ and $x_0 = 0$ nothing can be concluded from Theorem 1.1.

EXERCISE 1.25. The function $f(t,x) = \tan bx$ is continuous and differentiable for all x such that $\cos bx \neq 0$. Therefore, if

$$x_0 \neq \frac{1}{2b}(2n+1)\pi$$

for all $n = 0, \pm 1, \pm 2, \pm 3, \cdots$, then the initial problem has a unique solution on an interval containing t_0. For any other x_0 nothing can be concluded from Theorem 1.1.

EXERCISE 1.27 Since the cube root function is defined and continuous for all values of its argument, $f(t,x) = t^{1/3} + x^{2/3}$ is defined and continuous for all values of t and x. For this same reason, $df(t,x)/dx = 2x^{-1/3}/3$ is defined and continuous except for $x = 0$. Therefore, Theorem 1.1 applies to all initial value problems $x(t_0) = x_0$ with $x_0 \neq 0$ and we conclude, for such initial problems, that there exists a unique solution on some interval containing t_0. If $x_0 = 0$, we can conclude nothing about the initial value problem from Theorem 1.1.

EXERCISE 1.29 Since the cube root function is defined and continuous for all values of its argument, $f(t,x) = (1 - e^{x-1})^{4/3}$ is defined and continuous for all values of t and x. For this same reason, $df(t,x)/dx = (4/3)(1 - e^{2x-1})^{1/3}(-2e^{2x-1})$ is defined for all values of t and x. Therefore, Theorem 1.1 applies to all initial value problems $x(t_0) = x_0$ and we conclude any initial value problem has a unique solution on some interval containing t_0.**EXERCISE 1.31.** For any point such that $t_0 \neq x_0$ because the function $f(t,x) = \ln|x - t|$ is continuous and continuously differentiable in x for all $t \neq x$. From Theorem 1.1we conclude that this initial value problem has a unique solution defined on an interval containing t_0. Nothing can be concluded from the theorem when $x_0 = t_0$.

EXERCISE 1.33. $f(t,x) = |x|$ and $df/dx = x/|x|$ are continuous for $x \neq 0$. If $x_0 \neq 0$ then Theorem 1.1applies and we conclude that the initial value problem has a unique solution on an interval containing t_0.

EXERCISE 1.35. $f(t,x) = \sqrt{1-x}$ is a composite function made from the two functions $1-x$ (which is continuously differentiable for all t and x) and the square root function $\sqrt{x}$ (which is continuous differentiable at any positive $x > 0$). Therefore, $f(t,x)$ is continuously differentiable for all t and all $x < 1$. Since $x_0 = 0 < 1$ Theorem 1.1applies and we conclude that this initial value problem has a unique solution on an interval containing $t_0 = 1$.

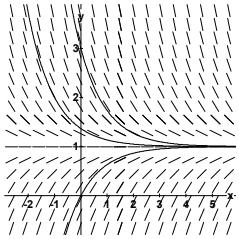
EXERCISE 1.37. Although $f(t,x) = (x+t)^{1/3}$ is continuous at $t = 0$ and $x = 0$ (it is the composite of two continuous functions), the derivative

$$\frac{df(t,x)}{dx} = \frac{1}{3}(x+t)^{-2/3}$$

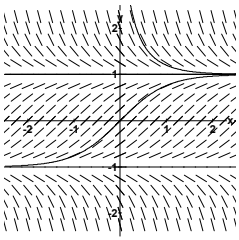
is not continuous (not even defined) at $t = 0$ and $x = 0$. Therefore, the Fundamental Exixtence and Uniqueness Theorem does not apply and no conclusion can be drawn from that theorem.

EXERCISE 1.40. Polynomials in x and t are continuous and have continuous derivatives (of all orders) for all x and t. Therefore, Theorem 1.1applies to any initial value problem.

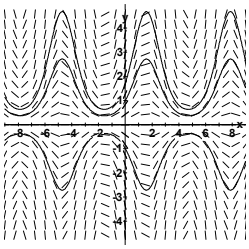
EXERCISE 1.44.



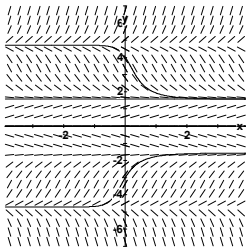
EXERCISE 1.46.



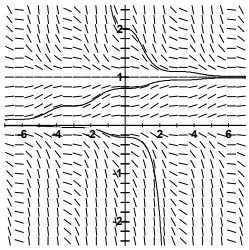
EXERCISE 1.48.



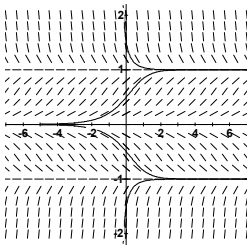
EXERCISE 1.50.



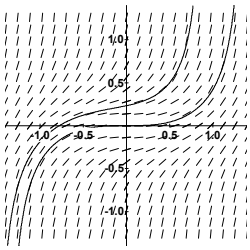
EXERCISE 1.52.



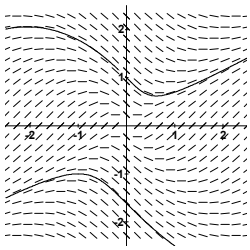
EXERCISE 1.54.



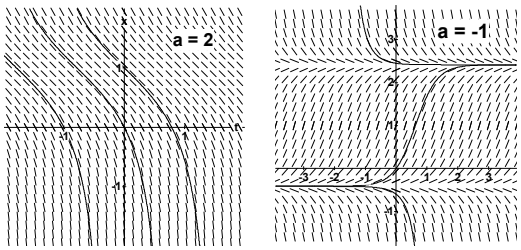
EXERCISE 1.55.



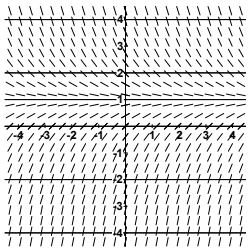
EXERCISE 1.57.



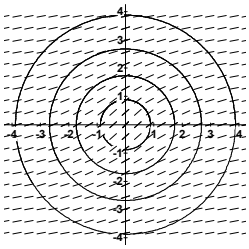
EXERCISE 1.60. For $a > 1$ all solutions appear to decrease. For $a < 1$ there is a horizontal region (lying between two parallel, horizontal straight lines) in the t, x plane in which solutions increase and outside of which solutions decrease.



EXERCISE 1.62. The isoclines are horizontal straight lines of the form $x = 1 - m$ where m, the associated slope, is any constant.



EXERCISE 1.64. The isoclines are circles, centered at the origin, of the form $t^2 + x^2 = \frac{1}{m^2} - 1$ where m, the associated slope, is any constant satisfying $0 < m < 1$.



EXERCISE 1.67. Solve $x(t) = t + m$ for $m = x - t$ and choose $f(t, x) = x - t$:

$$x' = x - t$$

EXERCISE 1.69. Solve $x(t) = t + 1/m$ for $m = 1/(x - t)$ and choose $f(t, x) = 1/(x - t)$:

$$x' = \frac{1}{x - t}$$

EXERCISE 1.71. Solve $2x^2 + 3t^2 = m^{1/3}$ for $m = (2x^2 + 3t^2)^3$ and choose $f(t, x) = (2x^2 + 3t^2)^3$:

$$x' = \left(2x^2 + 3t^2\right)^3$$

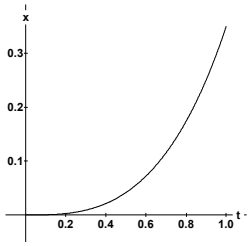
EXERCISE 1.80.

(a) $x(0.8) \approx 1.6094571436$, which has five correct significant digits

(b) $s = 0.1$ and $s = 0.1$
(c)

Step size s	$x(1) \approx$ Runge-Kutta	Absolute Error
0.10000	1.6094571436	-1.9232×10^{-5}
0.05000	1.6094402828	-2.371×10^{-6}
0.02500	1.6094381088	-1.97×10^{-7}
0.01250	1.6094379264	-1.4×10^{-8}
0.00625	1.6094379134	1.0×10^{-9}

The error goes down by approximately a fraction of $1/16 = (1/2)^4$ at each step, as is to be expected from the fourth order Runge-Kutta Algorithm.
EXERCISE 1.83. $s = 0.003125$ because there is virtually no change in the graph from $s = 0.00625$.



EXERCISE 1.84. $s = 0.0125$ because there is visibly no change in the graph from $s = 0.025$.
EXERCISE 1.85. $s = 0.025$ because there is visibly no change in the graph from $s = 0.05$.

A.3 Chapter 2: Linear First Order Equations

EXERCISE 2.1. The equation is linear with $p(t) = t^2$ and $q(t) = t$.
EXERCISE 2.3. The equation is nonlinear because of the x^2 term.
EXERCISE 2.5. The equation is nonlinear because of the $\sin x$ term.
EXERCISE 2.7. The equation is nonlinear because of the xx' term.
EXERCISE 2.9. The equation is linear with

$$p(t) = \frac{1}{5}\left(t^2 + \sin t\right) \quad \text{and} \quad q(t) = -\frac{1}{5}\left(\cos 3t + \frac{1}{t^2+1}\right).$$

EXERCISE 2.11. The equation is linear nonhomogeneous with $p(t) = t^2$ and $q(t) = -1$.
EXERCISE 2.13. The equation is linear homogeneous with $p(t) = t^2$ and $q(t) = 0$.
EXERCISE 2.15. The equation is linear nonhomogeneous with $p(t) = 3e^{-t}$ and $q(t) = -te^{-t}$.

EXERCISE 2.18.

$$p(t) = -3 \Longrightarrow P(t) = \int (-3)dt = -3t$$
$$x\left(t\right) = ce^{P(t)} = ce^{-3t}$$

EXERCISE 2.20.

$$p(t) = \frac{1}{t} \Longrightarrow P(t) = \int \frac{1}{t}dt = \ln|t|$$
$$x\left(t\right) = ce^{P(t)} = ce^{\ln|t|} = c|t| \text{ for } t \neq 0$$

EXERCISE 2.22.

$$p(t) = e^{-3t} \Longrightarrow P(t) = \int e^{-3t}dt = -\frac{1}{3}e^{-3t}$$
$$x\left(t\right) = ce^{P(t)} = c\exp\left(-\frac{1}{3}e^{-3t}\right)$$

EXERCISE 2.24.

$$p(t) = \frac{t}{1+t^2} \Longrightarrow P(t) = \int \frac{t}{1+t^2}dt = \frac{1}{2}\ln\left(1+t^2\right)$$
$$\Longrightarrow x\left(t\right) = ce^{P(t)} = ce^{\ln\left(1+t^2\right)/2} = c\sqrt{1+t^2}$$

EXERCISE 2.26.

$$p(t) = t\sin t \Longrightarrow P(t) = \int \left(t\sin t\right)dt = \sin t - t\cos t$$
$$x\left(t\right) = ce^{P(t)} = c\exp\left(\sin t - t\cos t\right)$$

EXERCISE 2.28.

$$p(t) = \frac{1}{a} \Longrightarrow P(t) = \int \frac{1}{a}dt = \frac{1}{a}t$$
$$x\left(t\right) = ce^{P(t)} = ce^{t/a}$$

EXERCISE 2.30.

$$p(t) = e^{at} \Longrightarrow P(t) = \int e^{at}dt = \begin{cases} \frac{1}{a}e^{at} & \text{if } a \neq 0 \\ t & \text{if } a = 0 \end{cases}$$

$$x(t) = ce^{P(t)} = \begin{cases} c\exp\left(\frac{1}{a}e^{at}\right) \text{ if } a \neq 0 \\ ce^{t} \text{ if } a = 0 \end{cases}$$

EXERCISE 2.32. Using $p(t) = -2$ (hence $P(t) = -2t$) and $q(t) = 12$ in the Variation of Constants formula we obtain

$$\begin{aligned} x(t) &= ce^{P(t)} + e^{P(t)} \int^t e^{-P(u)} q(u)\, du \\ &= ce^{-2t} + e^{-2t} \int^t e^{2u} 12 du \\ &= ce^{-2t} + 6e^{-2t}e^{2t} \\ &= ce^{-2t} + 6. \end{aligned}$$

EXERCISE 2.34. Using $p(t) = t$ (hence $P(t) = t^2/2$) and $q(t) = t$ in the Variation of Constants formula we obtain

$$\begin{aligned} x(t) &= ce^{P(t)} + e^{P(t)} \int^t e^{-P(u)} q(u)\, du \\ &= ce^{t^2/2} + e^{t^2/2} \int^t e^{-u^2/2} u du \\ &= ce^{t^2/2} - e^{t^2/2}e^{-t^2/2} \\ &= ce^{t^2/2} - 1. \end{aligned}$$

EXERCISE 2.36. Using $p(t) = a$ (hence $P(t) = at$) and $q(t) = \cos bt$ in the Variation of Constants formula we obtain

$$\begin{aligned} x(t) &= ce^{P(t)} + e^{P(t)} \int^t e^{-P(u)} q(u)\, du \\ &= ce^{at} + e^{at} \int^t e^{-au} \cos bu du. \end{aligned}$$

Now

$$\int^t e^{-au} \cos bu du = \begin{cases} \frac{1}{a^2+b^2} e^{-at} (b \sin bt - a \cos bt) & \text{if } a^2 + b^2 \neq 0 \\ t & \text{if } a = b = 0 \end{cases}$$

and therefore

$$x(t) = \begin{cases} ce^{at} + \frac{1}{a^2+b^2} (b \sin bt - a \cos bt) & \text{if } a^2 + b^2 \neq 0 \\ c + t & \text{if } a = b = 0 \end{cases}$$

EXERCISE 2.38. Using $p(t) = -1/t$ (hence $P(t) = -\ln|t|$) and $q(t) = t^{-2/3}$ in the Variation of Constants formula we obtain

$$\begin{aligned} x(t) &= ce^{P(t)} + e^{P(t)} \int^t e^{-P(u)} q(u)\, du \\ &= ce^{-\ln|t|} + e^{-\ln|t|} \int^t e^{\ln|u|} u^{-2/3} du \\ &= c|t|^{-1} + |t|^{-1} \int^t |u|\, u^{-2/3} du. \end{aligned}$$

For $u > 0$ we have $|u| = u$ and

$$\int^t |u|\, u^{-2/3} du = \int^t u^{1/3} du = \frac{3}{4} t^{4/3}$$

and

$$x(t) = c\frac{1}{t} + \frac{1}{t}\frac{3}{4} t^{4/3} = c\frac{1}{t} + \frac{3}{4} t^{1/3}.$$

For $u < 0$ we have $|u| = -u$ and

$$\int^t |u|\, u^{-2/3} du = -\int^t u^{1/3} du = -\frac{3}{4} t^{4/3}$$

and

EXERCISE 2.40. Using $p(t) = t$ (hence $P(t) = t^2/2$) and $q(t) = -1$ in the Variation of Constants formula we obtain

$$\begin{aligned} x(t) &= ce^{P(t)} + e^{P(t)} \int^t e^{-P(u)} q(u)\, du \\ &= ce^{t^2/2} + e^{t^2/2} \int^t e^{-u^2/2} (-1)\, du \\ &= ce^{t^2/2} - e^{t^2/2} \int^t e^{-u^2/2} du. \end{aligned}$$

EXERCISE 2.43.

$$\begin{aligned} &x(t) = ce^{\pi t} \Longrightarrow x(1) = ce^{\pi} = -2 \Longrightarrow c = -2e^{-\pi} \\ &x(t) = -2e^{-\pi} e^{\pi t} \end{aligned}$$

EXERCISE 2.45.

$$\begin{aligned} &x(t) = ce^{\tan^{-1} t} \Longrightarrow x(1) = ce^{\pi/4} = e^{\pi} \Longrightarrow c = e^{3\pi/4} \\ &x(t) = e^{3\pi/4} e^{\tan^{-1} t} \end{aligned}$$

EXERCISE 2.47.

$$\begin{aligned} &x(t) = ce^{-\frac{1}{a}\cos at} \Longrightarrow x(0) = ce^{-\frac{1}{a}} = 1 \Longrightarrow c = e^{\frac{1}{a}} \\ &x(t) = \exp\left(\frac{1 - \cos at}{a}\right) \end{aligned}$$

EXERCISE 2.49. (a) From the Variation of Constants Formula we obtain the general solution

$$x(t) = ce^{3t} + \frac{2}{3}.$$

Then

$$x(0) = c + \frac{2}{3} = 5 \Longrightarrow c = \frac{13}{3}$$

and the solution of the initial value problem is

$$x(t) = \frac{13}{3}e^{3t} + \frac{2}{3}.$$

(b) Using $p(t) = 3$ (therefore $P(t) = \int_{t_0}^{t} p(u)\,du = 3t$) and $q(t) = -2$ in the Variation of Constants Formula for initial value problems, we calculate

EXERCISE 2.51. (a) From the Variation of Constants Formula we obtain the general solution (we take $t > 0$ because the initial condition $t_0 = 2$ is positive)

$$x(t) = ct + \frac{1}{3}t^4.$$

Then

$$x(2) = 2c + \frac{16}{3} = 0 \Longrightarrow c = -\frac{8}{3}$$

and the solution of the initial value problem is

$$x(t) = -\frac{8}{3}t + \frac{1}{3}t^4.$$

(b) For $t > 0$ we have, using $p(t) = 1/t$ and $t_0 = 2$ (therefore $P(t) = \int_{t_0}^{t} p(u)\,du = \ln t - \ln 2$) and $q(t) = t^3$ in the Variation of Constants Formula for initial value problems, we calculate

$$\begin{aligned}
x(t) &= x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(u)} q(u)\,du \\
&= (0)\, e^{\ln t - \ln 2} + e^{\ln t - \ln 2} \int_{2}^{t} e^{-\ln u + \ln 2} u^3 du \\
&= \frac{1}{2}t \int_{2}^{t} \frac{2}{u} u^3 du = t\left(\frac{1}{3}u^3\right)\Big|_{u=2}^{u=t} \\
&= t\left(\frac{1}{3}t^3 - \frac{1}{3}2^3\right) \\
&= -\frac{8}{3}t + \frac{1}{3}t^4.
\end{aligned}$$

EXERCISE 2.53. (a) From the Variation of Constants Formula we obtain the general solution

$$x(t) = ce^{\frac{1}{a}\sin at} - b.$$

Then

$$\begin{aligned}
x(0) &= c - b = 0 \Longrightarrow c = b \\
x(t) &= be^{\frac{1}{a}\sin at} - b
\end{aligned}$$

(b) Using $p(t) = \cos at$ and $t_0 = 0$ (therefore $P(t) = \int_{t_0}^{t} p(u)\,du = \frac{1}{a}\sin at$) and $q(t) = b\cos at$ in the Variation of Constants Formula for initial value problems, we calculate

$$\begin{aligned} x(t) &= x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(u)} q(u)\,du \\ &= 0 \cdot e^{\frac{1}{a}\sin at} + e^{\frac{1}{a}\sin at} \int_0^t e^{-\frac{1}{a}\sin au} b\cos au du \\ &= e^{\frac{1}{a}\sin at}\left(-be^{-\frac{1}{a}\sin au}\right)\Big|_{u=0}^{u=t} \\ &= e^{\frac{1}{a}\sin at}\left(-be^{-\frac{1}{a}\sin at} + b\right) \\ &= be^{\frac{1}{a}\sin at} - b. \end{aligned}$$

EXERCISE 2.55. (a) From the Variation of Constants Formula we obtain the general solution

$$c\frac{1}{t} + \frac{1}{t}\ln\left(1+t^2\right)$$

(we take $t > 0$ because the initial condition $t_0 = 1$ is positive). Then

$$x(1) = c + \ln 2 = \ln 8 \Longrightarrow c = \ln 4$$

and the solution of the initial value problem is

$$x(t) = \frac{1}{t}\ln 4 + \frac{1}{t}\ln\left(1+t^2\right).$$

(b) For $t > 0$ we have, using $p(t) = -1/t$ and $t_0 = 1$ (therefore $P(t) = \int_{t_0}^{t} p(u)\,du = -\ln t$) and $q(t) = \frac{2}{1+t^2}$ in the Variation of Constants Formula for initial value problems, we calculate

$$\begin{aligned} x(t) &= x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(u)} q(u)\,du \\ &= (\ln 8)\, e^{-\ln t} + e^{-\ln t} \int_1^t e^{\ln u} \frac{2}{1+u^2} du \\ &= \frac{1}{t}\ln 8 + \frac{1}{t}\int_1^t \frac{2u}{1+u^2} du = \frac{1}{t}\ln 8 + \frac{1}{t}\left(\ln\left(1+u^2\right)\right)\Big|_{u=1}^{u=t} \\ &= \frac{1}{t}\ln 8 + \frac{1}{t}\left(\ln\left(1+t^2\right) - \ln 2\right) \\ &= \frac{1}{t}\ln 4 + \frac{1}{t}\ln\left(1+t^2\right). \end{aligned}$$

EXERCISE 2.57.

$$x(t) = x_0 e^{r(t-t_0)}$$

equals $2x_0$ when

$$t - t_0 = \frac{1}{r}\ln 2,$$

which is independent of x_0.

EXERCISE 2.62.

(a) The graphs of all solutions have a horizontal asymptote, i.e., the solutions all tend to a finite limit as $t \to +\infty$. Each solution tends to a different limit, however.

(b)

$$x(t) = x_0 \exp\left(2 - 2e^{-0.5t}\right)$$

and therefore the limit

$$\lim_{t \to +\infty} x(t) = x_0 e^2$$

exists and depends on x_0.

EXERCISE 2.65.

(a) The Runge-Kutta algorithm was used to obtain the following table of approximations.

$s =$ step size	$x(0.99) \approx$
0.010000	−0.510402
0.005000	−0.503585
0.002500	−0.503295
0.001250	−0.503281
0.000625	−0.503280

(b) The graph is highly oscillatory (has a very short period, i.e. a high frequency) and appears exactly periodic.

(c)

$$x(t) = ce^{\frac{1}{60\pi}\sin 60\pi t} - 100 \Longrightarrow x(0) = c - 100 = 0 \Longrightarrow c = 100$$

so

$$x(t) = 100e^{\frac{1}{60\pi}\sin 60\pi t} - 100$$

and $x(0.99) \approx -0.503280$.

(d) The solution in (c) is periodic with period

$$\frac{2\pi}{60\pi} = \frac{1}{30} \approx 0.03333$$

and frequency 30.

EXERCISE 2.67. $x_h(t) = ce^t$ and $x(t) = x_h(t) + x_p(t)$ so $x(t) = ce^t + t^{100}e^t$.

EXERCISE 2.69. $x_h(t) = ce^{-t}$ and by the superposition principle $x_p(t) = x(t) + y(t) = 2e^{-3t} + e^t$. Therefore $x(t) = ce^{-t} + 2e^{-3t} + e^t$.

EXERCISE 2.71. $x_h(t) = ce^t$ and $x(t) = x_h(t) + x_p(t) = ce^t + 10$. The initial condition implies $c = -5$ so $x(t) = 5e^t + 10$.

EXERCISE 2.73. $x_h(t) = ce^{rt}$ and $x(t) = x_h(t) + x_p(t) = ce^{rt} + h/r$. The initial condition implies $c = -h/r$ so $x(t) = -he^{rt}/r + h/r$.

EXERCISE 2.79. The method is applicable ($q(t)$ has the form $t^m e^{at}$ with $m = 2, a = -1$).

EXERCISE 2.81. The method is not applicable (the coefficient $p(t) = t$ of x' is not a constant).

EXERCISE 2.83. The method is applicable ($q(t)$ is a constant multiple of e^{at} with $a = 2$).

EXERCISE 2.85. The method is applicable ($q(t)$ is a constant multiple of $t^m e^{at}$ with $m = 1$, $a = -2$).

EXERCISE 2.87. The method is applicable ($q(t)$ is a constant multiple of $t^m e^{at}$ with $m = 1$, $a = b$).

EXERCISE 2.89. **(a)** The general solution of the associated homogeneous equation is $x_h(t) = ce^{0.5t}$. $q(t)$ is a multiple of $e^{0.2t}$,which generates no new, independent functions upon repeated differentiation. Since this function is not a solution of the associated homogeneous equation we construct the "guess" $x_p(t) = ke^{0.2t}$.

(b) A substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned} 0.2ke^{0.2t} &= 0.5ke^{0.2t} - 0.3e^{-.2t} \\ (-0.3k + 0.3)\, e^{0.2t} &= 0 \\ -0.3k + 0.3 &= 0 \\ k &= 1 \end{aligned}$$

and $x_p(t) = e^{0.2t}$.

EXERCISE 2.91. **(a)** The general solution of the associated homogeneous equation is $x_h(t) = ce^{3t}$. $q(t)$ is a multiple of e^{3t} ,which generates no new, independent functions upon repeated differentiation. Since this function is a solution of the associated homogeneous equation we construct the "guess" $x_p(t) = kte^{3t}$.

(b) A substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned} 3kte^{3t} + ke^{3t} &= 3kte^{3t} - 15e^{3t} \\ (k + 15)\, e^{3t} &= 0 \\ k &= -15 \end{aligned}$$

and $x_p(t) = -15te^{3t}$.

EXERCISE 2.93. **(a)** The general solution of the associated homogeneous equation is $x_h(t) = ce^{-2t/3}$. $q(t)$ is a multiple of $e^{-t}\sin t$, which generates one new, independent function upon repeated differentiation: $e^{-t}\cos t$. Since neither of these functions is a solution of the associated homogeneous equation, we construct the "guess" $x_p(t) = k_1 e^{-t}\cos t + k_2 e^{-t}\sin t$.

(b) A substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned} &k_1 e^{-t}\sin t - k_1 e^{-t}\cos t + k_2 e^{-t}\cos t - k_2 e^{-t}\sin t \\ &= -\frac{2}{3}\left(k_1 e^{-t}\cos t + k_2 e^{-t}\sin t\right) - \frac{15}{16}e^{-t}\sin t \end{aligned}$$

$$\left(-\frac{1}{3}k_1 - k_2 + \frac{15}{16}\right)e^{-t}\cos t + \left(k_1 - \frac{1}{3}k_2\right)e^{-t}\sin t = 0$$

and

$$-\frac{1}{3}k_1 - k_2 + \frac{15}{16} = 0, \qquad k_1 - \frac{1}{3}k_2 = 0$$

whose solution is $k_1 = \frac{9}{32}$, $k_2 = \frac{27}{32}$. Thus, $x_p(t) = \frac{9}{32}e^{-t}\cos t + \frac{27}{32}e^{-t}\sin t$.

EXERCISE 2.95. **(a)** The general solution of the associated homogeneous equation is $x_h(t) = ce^{-t}$. $q(t)$ is a multiple of $t^4\cos 2t$, which upon repeated differentiation generates

the new, independent functions: $t^3\cos 2t$, $t^2\cos 2t$, $t\cos 2t$, $\cos 2t$, $t^4\sin 2t$, $t^3\sin 2t$, $t^2\sin 2t$, $t\sin 2t$ and $\sin 2t$. Since none of these functions is a solution of the associated homogeneous equation, we construct the "guess"

$$\begin{aligned} x_p(t) = {} & k_1 t^4\cos 2t + k_2 t^3\cos 2t + k_3 t^2\cos 2t \\ & + k_4 t\cos 2t + k_5\cos 2t + k_6 t^4\sin 2t + k_7 t^3\sin 2t \\ & + k_8 t^2\sin 2t + k_9 t\sin 2t + k_{10}\sin 2t. \end{aligned}$$

(b) A substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned} & 2k_6 t^4\cos 2t - 2k_1 t^4\sin 2t \\ & + (4k_1 + 2k_7)\, t^3\cos 2t + (-2k_2 + 4k_6)\, t^3\sin 2t \\ & + (3k_2 + 2k_8)\, t^2\cos 2t + (-2k_3 + 3k_7)\, t^2\sin 2t \\ & + (2k_3 + 2k_9)\, t\cos 2t + (-2k_4 + 2k_8)\, t\sin 2t \\ & + (k_4 + 2k_{10})\cos 2t + (-2k_5 + k_9)\sin 2t \\ & = -\big(k_1 t^4\cos 2t + k_2 t^3\cos 2t + k_3 t^2\cos 2t + k_4 t\cos 2t + k_5\cos 2t \\ & + k_6 t^4\sin 2t + k_7 t^3\sin 2t + k_8 t^2\sin 2t + k_9 t\sin 2t + k_{10}\sin 2t\big) \\ & + 5t^4\cos 2t \end{aligned}$$

or

$$\begin{aligned} & (2k_6 + k_1 - 5)\, t^4\cos 2t + (-2k_1 + k_6)\, t^4\sin 2t \\ & + (4k_1 + 2k_7 + k_2)\, t^3\cos 2t + (-2k_2 + 4k_6 + k_7)\, t^3\sin 2t \\ & + (3k_2 + 2k_8 + k_3)\, t^2\cos 2t + (-2k_3 + 3k_7 + k_8)\, t^2\sin 2t \\ & + (2k_3 + 2k_9 + k_4)\, t\cos 2t + (-2k_4 + 2k_8 + k_9)\, t\sin 2t \\ & + (k_4 + 2k_{10} + k_5)\cos 2t + (-2k_5 + k_9 + k_{10})\sin 2t \\ & = 0 \end{aligned}$$

and

$$\begin{aligned} 2k_6 + k_1 - 5 &= 0 \\ -2k_1 + k_6 &= 0 \\ 4k_1 + 2k_7 + k_2 &= 0 \\ -2k_2 + 4k_6 + k_7 &= 0 \\ 3k_2 + 2k_8 + k_3 &= 0 \\ -2k_3 + 3k_7 + k_8 &= 0 \\ 2k_3 + 2k_9 + k_4 &= 0 \\ -2k_4 + 2k_8 + k_9 &= 0 \\ k_4 + 2k_{10} + k_5 &= 0 \\ -2k_5 + k_9 + k_{10} &= 0 \end{aligned}$$

whose solution is:

$$\begin{aligned} & k_1 = 1, \quad k_2 = \frac{12}{5}, \quad k_3 = -\frac{132}{25}, \quad k_4 = \frac{168}{125}, \quad k_5 = \frac{984}{625} \\ & k_6 = 2, \quad k_7 = -\frac{16}{5}, \quad k_8 = -\frac{24}{25}, \quad k_9 = \frac{576}{125}, \quad k_{10} = -\frac{912}{625}. \end{aligned}$$

Thus,

$$\begin{aligned}x_p(t) &= t^4\cos 2t + \frac{12}{5}t^3\cos 2t - \frac{132}{25}t^2\cos 2t \\ &\quad + \frac{168}{125}t\cos 2t + \frac{984}{625}\cos 2t + 2t^4\sin 2t - \frac{16}{5}t^3\sin 2t \\ &\quad - \frac{24}{25}t^2\sin 2t + \frac{576}{125}t\sin 2t - \frac{912}{625}\sin 2t \\ &\approx t^4\cos 2t + 2.4t^3\cos 2t - 5.28t^2\cos 2t \\ &\quad + 1.574\cos 2t + 1.344t\cos 2t - 3.2t^3\sin 2t + 2t^4\sin 2t \\ &\quad + 4.608t\sin 2t - 1.459\sin 2t - 0.96t^2\sin 2t\end{aligned}$$

EXERCISE 2.97. **(a)** The general solution of the associated homogeneous equation is $x_h(t) = ce^{pt}$. $q(t)$ is a multiple of t^3e^{at}, which upon repeated differentiation generates the new, independent functions: t^2e^{at}, te^{at} and e^{at}. If $a \neq p$ then none of these functions is a solution of the associated homogeneous equation and we construct the "guess"

$$x_p(t) = k_1t^3e^{at} + k_2t^2e^{at} + k_3te^{at} + k_4e^{at}.$$

If $a = p$ then the list contains a solution of the associated homogeneous equation and we construct the "guess"

$$x_p(t) = k_1t^4e^{at} + k_2t^3e^{at} + k_3t^2e^{at} + k_4te^{at}.$$

(b) If $a \neq p$ a substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned}&ak_1t^3e^{at} + 3k_1t^2e^{at} + ak_2t^2e^{at} + 2k_2te^{at} + ak_3te^{at} + k_3e^{at} + ak_4e^{at} \\ &= p\left(k_1t^3e^{at} + k_2t^2e^{at} + k_3te^{at} + k_4e^{at}\right) + \frac{1}{3}t^3e^{at}\end{aligned}$$

or

$$\begin{aligned}&\left((a-p)k_1 - \frac{1}{3}\right)t^3e^{at} + (3k_1 + (a-p)k_2)\,t^2e^{at} \\ &+ (2k_2 + (a - pk_3))\,te^{at} + (k_3 + (a-p)k_4)\,e^{at} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(a-p)k_1 - \tfrac{1}{3} &= 0 \\ 3k_1 + (a-p)k_2 &= 0 \\ 2k_2 + (a-p)k_3 &= 0 \\ k_3 + (a-p)k_4 &= 0\end{aligned}$$

whose solution is

$$\begin{aligned}k_1 &= -\frac{1}{3}\frac{1}{p-a}, \quad k_2 = -\frac{1}{(p-a)^2} \\ k_3 &= -\frac{2}{(p-a)^3}, \quad k_4 = -\frac{2}{(p-a)^4}.\end{aligned}$$

Thus, if $a \neq p$ the solution is

$$x_p(t) = -\frac{1}{3}\frac{1}{p-a}t^3e^{at} - \frac{1}{(p-a)^2}t^2e^{at} - \frac{2}{(p-a)^3}te^{at} - \frac{2}{(p-a)^4}e^{at}.$$

If $a = p$ a substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned} &ak_1t^4e^{at} + (4k_1 + ak_2)t^3e^{at} + (3k_2 + ak_3)t^2e^{at} + (2k_3 + ak_4)te^{at} + k_4e^{at} \\ &= a\left(k_1t^4e^{at} + k_2t^3e^{at} + k_3t^2e^{at} + k_4te^{at}\right) + \frac{1}{3}t^3e^{at} \end{aligned}$$

or

$$\left(4k_1 - \frac{1}{3}\right)t^3e^{at} + (3k_2)t^2e^{at} + (2k_3)te^{at} + k_4e^{at} = 0$$

and

$$\begin{aligned} 4k_1 - \tfrac{1}{3} &= 0 \\ 3k_2 &= 0 \\ 2k_3 &= 0 \\ k_4 &= 0 \end{aligned}$$

whose solution is $k_1 = 1/12, \quad k_2 = k_3 = k_4 = 0$. Thus, the solution is $x_p(t) = t^4e^{at}/12$.

EXERCISE 2.99. **(a)** The general solution of the associated homogeneous equation is $x_h(t) = ce^t$. $q(t)$ is a multiple of $\cos 2t$, which upon repeated differentiation generates only one new, independent function: $\sin 2t$. Since none of these functions is a solution of the associated homogeneous equation, we construct the "guess"

$$x_p(t) = k_1\cos 2t + k_2\sin 2t.$$

(b) A substitution of $x_p(t)$ in (a) into the differential equation yields

$$\begin{aligned} -2k_1\sin 2t + 2k_2\cos t &= k_1\cos 2t + k_2\sin 2t + 2\cos 2t \\ (-k_1 + 2k_2 - 2)\cos 2t + (-2k_1 - k_2)\sin 2t &= 0 \end{aligned}$$

and

$$\begin{aligned} -k_1 + 2k_2 - 2 &= 0 \\ -2k_1 - k_2 &= 0 \end{aligned}$$

whose solution is $k_1 = -\frac{2}{5}, \quad k_2 = \frac{4}{5}$. Thus, the solution is $x_p(t) = -\frac{2}{5}\cos 2t + \frac{4}{5}\sin 2t$.

EXERCISE 2.102. Solve $x' = -x + e^t$ for $x(t) = e^t/2$ and $x' = -x + \sin t$ for $x(t) = (\sin t - \cos t)/2$. Form the linear combination

$$x_p(t) = 2\left(\frac{1}{2}e^t\right) - 3\left(\frac{1}{2}(\sin t - \cos t)\right) = e^t + \frac{3}{2}\cos t - \frac{3}{2}\sin t.$$

EXERCISE 2.104. Solve $x' = x + e^t$ for $x(t) = te^t$ and $x' = x + \cos t$ for $x(t) = (\sin t - \cos t)/2$. Form the linear combination

$$x_p(t) = 3\left(te^t\right) - 4\left(\frac{1}{2}(\sin t - \cos t)\right) = 3te^t + 2\cos t - 2\sin t.$$

EXERCISE 2.106. The equation is hyperbolic if $p = \sin a \neq 0$ or $a \neq n\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. The equation is non-hyperbolic for all other values of a, namely, $a = n\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

EXERCISE 2.108. The equation is hyperbolic if $p = a^2 \neq 0$ or $a \neq 0$. The equation is non-hyperbolic if $a = 0$.

EXERCISE 2.110. Solve $-5x - 7 = 0$ for $x_e = -7/5$, which is an attractor because $p = -5 < 0$.

EXERCISE 2.112. Solve $2x - 10 = 0$ for $x_e = 5$, which is a repeller because $p = 2 > 0$.

EXERCISE 2.114. Since $p = a - 1$, the equilibrium $x_e = -q/p = -2/(a-1)$ is an attractor if $a < 1$ and a repeller if $a > 1$. There is no equilibrium if $a = 1$ since $x' = 2 > 0$ and all solutions increase.

EXERCISE 2.116. Since $p = 0$ the equation is non-hyperbolic. There is an equilibrium if and only if $x' = \sin a = 0$, that is, if and only if $a = n\pi$ for some integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. For each such value of a, every solution is an equilibrium since the equation reduces to $x' = 0$. If $a \neq n\pi$ for any integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, then there is no equilibrium solution since $x' = \sin a \neq 0$. All solutions increase if a is such that $\sin a > 0$ or decrease is a is such that $\sin a < 0$.

EXERCISE 2.118. The equation is hyperbolic and the equilibrium is an attractor if and only if $p < 0$. From the general solution $x(t) = ce^{pt} - q/p$ we see that all non-equilibrium solutions are unbounded as $t \to -\infty$. From the general solution $x(t) = ce^{pt} - q/p$ we see that all non-equilibrium solutions tend to the equilibrium $-p/q$ as $t \to -\infty$.

EXERCISE 2.120. The equation is non-hyperbolic if and only if $p = 0$. If $q \neq 0$ we see from the general solution $x(t) = qt + c$ that all solutions are linearly unbounded as $t \to -\infty$.

EXERCISE 2.122. Since $p = b - d$ and $q = -h$, the phase portraits are

$$\longrightarrow \frac{h}{b-d} \longleftarrow \text{ for } b < d$$
$$\longleftarrow \frac{h}{b-d} \longrightarrow \text{ for } b > d$$

There is a significant change in the asymptotic dynamics (from an attractor to a repeller) as b passes through the bifurcation point d.

EXERCISE 2.124. There is one bifurcation point located at the unique root of the equation $p = a - e^{-a} = 0$, which is $a \approx 0.56714$. For a less than this root the equilibrium

$$x_e = -\frac{1}{a - e^{-a}}$$

is an attractor and for a greater than this root x_e is a repeller.

EXERCISE 2.126.

$$P(t) = \int p(t)dt = \int at\ dt = \frac{1}{2}at^2$$
$$x(t) = ce^{P(t)} = c\exp\left(\frac{1}{2}at^2\right)$$

EXERCISE 2.128.

$$P(t) = \int p(t)dt = \int \left(-te^{-t^2/2}\right) dt = \exp\left(-\frac{1}{2}t^2\right)$$
$$x\left(t\right) = ce^{P(t)} = c\exp\left(\exp\left(-\frac{1}{2}t^2\right)\right)$$

EXERCISE 2.130.

$$P(t) = \int p(t)dt = \int ae^{rt}dt = \frac{a}{r}e^{rt}$$
$$x\left(t\right) = ce^{P(t)} = c\exp\left(\frac{a}{r}e^{rt}\right)$$

EXERCISE 2.132.

$$P(t) = \int p(t)dt = \int \left(\ln t\right) dt = t\ln t - t$$
$$x\left(t\right) = ce^{P(t)} = ct^t e^{-t}$$

EXERCISE 2.134.

$$P(t) = \int p(t)dt = \int \tan t \ dt = -\ln\left(\cos t\right)$$
$$x\left(t\right) = ce^{P(t)} = c\frac{1}{\cos t}$$

EXERCISE 2.136.

$$P(t) = \int \left(1\right) dt = t$$

Using the variation of constants formula, we calculate as follows:

$$\begin{aligned} x\left(t\right) &= ce^t + e^t \int e^{-t}\cos t \ dt \\ &= ce^t + \frac{1}{2}\left(\sin t - \cos t\right). \end{aligned}$$

EXERCISE 2.138. If $\beta \neq 0$, then $P(t) = \int \left(\cos \beta t\right) dt = \frac{1}{\beta}\sin \beta t$. Using the variation of constants formula, we calculate as follows:

$$x\left(t\right) = ce^{\frac{1}{\beta}\sin \beta t} + e^{\frac{1}{\beta}\sin \beta t} \int e^{-\frac{1}{\beta}\sin \beta t}\cos \beta t \ dt = ce^{\frac{1}{\beta}\sin \beta t} - 1.$$

If $\beta = 0$, then $P(t) = \int \left(1\right) dt = t$. Using the variation of constants formula, we calculate as follows: $x\left(t\right) = ce^t + e^t \int e^{-t} \ dt = ce^t - 1$.

EXERCISE 2.140.

$$P(t) = \int \left(\ln t\right) dt = t\ln t - t.$$

Using the variation of constants formula, we calculate as follows:

$$\begin{aligned} x(t) &= ce^{t\ln t-t} + e^{t\ln t-t}\int e^{-t\ln t+t}t^t dt \\ &= ce^{t\ln t}e^{-t} + e^{t\ln t}e^{-t}\int t^{-t}e^t t^t dt \\ &= ct^t e^{-t} + t^t e^{-t}\int e^t dt \\ &= ct^t e^{-t} + t^t. \end{aligned}$$

EXERCISE 2.142.

$$P(t) = \int (1)\, dt = t.$$

Using the variation of constants formula, we calculate as follows:

$$x(t) = ce^t + e^t \int e^{-t}e^{at}dt$$

$$x(t) = \begin{cases} ce^t + \frac{1}{a-1}e^{at} & \text{if } a \neq 1 \\ ce^t + te^t & \text{if } a = 1. \end{cases}$$

EXERCISE 2.144. $x_h(t) = ce^t$ and we search for a particular solution of the form

$$x_p(t) = k_1 \sin 10t + k_2 \cos 10t.$$

Equating

$$x_p' = 10k_1 \cos 10t - 10k_2 \sin 10t$$

and

$$x_p(t) + \sin 10t = (k_1 + 1)\sin 10t + k_2 \cos 10t$$

and algebraically rearranging the result, we obtain

$$(10k_1 - k_2)\cos 10t + (-k_1 - 10k_2 - 1)\sin 10t = 0.$$

Setting the coefficients of $\cos 10t$ and $\sin 10t$ equal to 0, we solve the resulting algebraic equations

$$10k_1 - k_2 = 0, \quad -k_1 - 10k_2 - 1 = 0$$

for $k_1 = -\frac{1}{101}$, $k_2 = -\frac{10}{101}$.Thus,

$$x_p(t) = \left(-\frac{1}{101}\right)\sin 10t + \left(-\frac{10}{101}\right)\cos 10t$$

and

$$x(t) = x_h(t) + x_p(t) = ce^t + \left(-\frac{1}{101}\right)\sin 10t + \left(-\frac{10}{101}\right)\cos 10t.$$

EXERCISE 2.146. $x_h(t) = ce^{-2t}$ and we search for a particular solution of the form

$$x_p(t) = k_1 t^2 e^{-2t} + k_2 t e^{-2t}.$$

Equating

$$x_p' = -2k_1 t^2 e^{-2t} + (2k_1 - 2k_2)\, te^{-2t} + k_2 e^{-2t}$$

with

$$-2x_p(t) + te^{-2t} = -2k_1 t^2 e^{-2t} + (-2k_2 + 1)\, te^{-2t}$$

and algebraically rearranging the result, we obtain

$$(2k_1 - 1)\, te^{-2t} + k_2 e^{-2t} = 0.$$

Setting the coefficients of te^{-2t} and e^{-2t} equal to 0, we solve the resulting algebraic equations

$$\begin{aligned} 2k_1 - 1 &= 0 \\ k_2 &= 0 \end{aligned}$$

for $k_1 = 1/2$, $k_2 = 0$.Thus,

$$x_p(t) = \frac{1}{2} t^2 e^{-2t}$$

and the general solution is

$$x(t) = x_h(t) + x_p(t) = ce^{-2t} + \frac{1}{2} t^2 e^{-2t}.$$

EXERCISE 2.148. The associated homogeneous equation $x' = x$ has general solution $x_h(t) = ce^t$. Repeated differentiations of $t^3 e^{-t} \cos 2t$ yields, up to linear combinations, the functions

$$\begin{aligned} &t^3 e^{-t} \cos 2t,\ t^2 e^{-t} \cos 2t,\ te^{-t} \cos 2t,\ e^{-t} \cos 2t \\ &t^3 e^{-t} \sin 2t,\ t^2 e^{-t} \sin 2t,\ te^{-t} \sin 2t,\ e^{-t} \sin 2t \end{aligned}$$

none of which solve the homogeneous equation. Therefore,

$$\begin{aligned} x(t) = &\left(k_1 t^3 + k_2 t^2 + k_3 t + k_4\right) e^{-t} \cos 2t \\ &+ \left(k_5 t^3 + k_6 t^2 + k_7 t + k_8\right) e^{-t} \sin 2t. \end{aligned}$$

EXERCISE 2.150. A solution of the equation $x' = x + \sin 10t$ is

$$x(t) = -\frac{10}{101} \cos 10t - \frac{1}{101} \sin 10t.$$

A solution of the equation $x' = x + \cos t$ is $x(t) = (\sin t - \cos t)/2$. By the superposition principle a particular solution is a linear combination of these two solutions, namely,

$$\begin{aligned} x_p(t) &= 2\left(-\frac{10}{101} \cos 10t - \frac{1}{101} \sin 10t\right) + 3\left(\frac{1}{2} \sin t - \frac{1}{2} \cos t\right) \\ &= \frac{3}{2} \sin t - \frac{3}{2} \cos t - \frac{20}{101} \cos 10t - \frac{2}{101} \sin 10t. \end{aligned}$$

Thus, the general solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= ce^t + \frac{3}{2}\sin t - \frac{3}{2}\cos t - \frac{20}{101}\cos 10t - \frac{2}{101}\sin 10t. \end{aligned}$$

EXERCISE 2.152. A solution of the equation $x' = 3x + 1$ is $x(t) = -1/3$. A solution of the equation $x' = 3x + t$ is $x(t) = -t/3 - 1/9$. A solution of the equation $x' = 3x + t^2$ is $x(t) = -t^2/3 - 2t/9 - 2/27$. A solution of the equation $x' = 3x + t^3$ is

$$x(t) = -\frac{1}{3}t^2 - \frac{1}{3}t^3 - \frac{2}{9}t - \frac{2}{27}.$$

By the superposition principle a particular solution is a linear combination of these solutions, namely,

$$\begin{aligned} x_p(t) = 2\left(-\frac{1}{3}\right) &+ 3\left(-\frac{1}{3}t - \frac{1}{9}\right) - \left(-\frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}\right) \\ &+ 6\left(-\frac{1}{3}t^2 - \frac{1}{3}t^3 - \frac{2}{9}t - \frac{2}{27}\right) \\ = -\frac{37}{27} &- \frac{19}{9}t - \frac{5}{3}t^2 - 2t^3. \end{aligned}$$

Thus, the general solution is

$$x(t) = x_h(t) + x_p(t) = ce^{3t} - \frac{37}{27} - \frac{19}{9}t - \frac{5}{3}t^2 - 2t^3.$$

EXERCISE 2.154. A solution of the equation $x' = -x + \sin t$ is $x(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t$. A solution of the equation $x' = -x + \sin 2t$ is $x(t) = (\sin 2t - 2\cos 2t)/5$. By the superposition principle a particular solution is a linear combination of these two equations, namely,

$$\begin{aligned} x_p(t) &= 3\left(\frac{1}{2}\sin t - \frac{1}{2}\cos t\right) + 2\left(\frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t\right) \\ &= \frac{3}{2}\sin t - \frac{3}{2}\cos t - \frac{4}{5}\cos 2t + \frac{2}{5}\sin 2t. \end{aligned}$$

Thus, the general solution is

$$x(t) = x_h(t) + x_p(t) = ce^{-t} + \frac{3}{2}\sin t - \frac{3}{2}\cos t - \frac{4}{5}\cos 2t + \frac{2}{5}\sin 2t.$$

EXERCISE 2.156. Using the variation of constants formula, we calculate

$$P(t) = \int_5^t u du = \frac{1}{2}t^2 - \frac{25}{2}$$

and

$$x(t) = x_0 e^{P(t)} = \pi \exp\left(\frac{1}{2}t^2 - \frac{25}{2}\right).$$

EXERCISE 2.158. Using the variation of constants formula, we calculate

$$P(t) = \int_0^t du = t$$

and

$$\begin{aligned} x(t) &= x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^t e^{-P(u)} q(u) du \\ &= 0 + e^t \int_0^t e^{-u} \sin 10t du \\ &= \frac{10}{101} e^t - \frac{10}{101} \cos 10t - \frac{1}{101} \sin 10t. \end{aligned}$$

EXERCISE 2.160. Using the variation of constants formula, we calculate

$$P(t) = \int (-2) dt = -2t$$

and

$$\begin{aligned} x(t) &= x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^t e^{-P(u)} q(u) du \\ &= x_0 e^{-2t} + e^{-2t} \int_0^t e^{2u} u e^{2u} du \\ x(t) &= \left(x_0 + \frac{1}{16} \right) e^{-2t} - \frac{1}{16} e^{2t} + \frac{1}{4} t e^{2t}. \end{aligned}$$

EXERCISE 2.162. Using the variation of constants formula, we calculate

$$P(t) = \int_0^t a u e^{-bu} du = \frac{a}{b^2} \left(1 - (bt+1) e^{-bt} \right)$$

and

$$x(t) = x_0 e^{P(t)} = x_0 \exp\left(\frac{a}{b^2} \left(1 - (bt+1) e^{-bt} \right) \right).$$

EXERCISE 2.164. Using the variation of constants formula, we calculate $P(t) = \int_0^t \frac{a}{1+u} du$ $= a \ln |t+1|$ and

$$x(t) = x_0 e^{P(t)} = x_0 |1+t|^a.$$

EXERCISE 2.166. Using the variation of constants formula, we calculate

$$P(t) = \int_0^t \frac{a}{1+u^2} du = a \arctan t$$

and

$$x(t) = x_0 e^{P(t)} = x_0 \exp(a \arctan t).$$

EXERCISE 2.169. $q(t) = e^{-t}\sin t$ means immigration and emigration alternate periodically (with period 2π), but at an exponentially decreasing rate. We can solve the initial value problem by means of the variation of constants formula with

$$P(t) = \int_0^t r ds = rt$$

to get

$$\begin{aligned} x(t) &= x_0 e^{rt} + e^{rt}\int_0^t e^{-ru}\left(e^{-u}\sin u\right) du \\ &= \left(\frac{1}{(r+1)^2} + x_0\right) e^{rt} - \frac{1}{(r+1)^2}\left(\cos t + (r+1)\sin t\right) e^{-t} \end{aligned}$$

if $r \neq -1$ and $x(t) = (x_0 + 1 - \cos t)\, e^{-t}$ if $r = -1$.

EXERCISE 2.171. $q(t) = 1+2\cos t$ means immigration/emigration oscillates periodically with mean 1 and amplitude 2. Note that it periodically becomes negative at which times emigration occurs. We can solve the initial value problem by means of the variation of constants formula with

$$P(t) = \int_0^t r ds = rt$$

to get

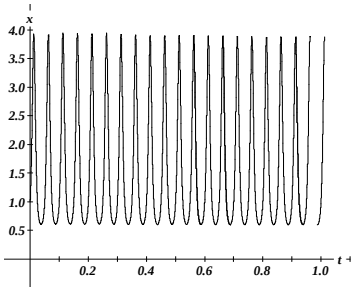
$$\begin{aligned} x(t) &= x_0 e^{rt} + e^{rt}\int_0^t e^{-ru}\left(1 + 2\cos u\right) du \\ &= \left(x_0 + \frac{3r^2+1}{r\left(r^2+1\right)}\right) e^{rt} + \frac{2r}{r\left(r^2+1\right)}\left(\sin t - r\cos t\right) - \frac{1}{r}. \end{aligned}$$

EXERCISE 2.173. **(a)** From the fourth order Runge-Kutta algorithm we obtain the following table of values.

step size	$x(2/3) \approx$
0.0001	2.980710
0.00005	2.998502
0.000025	2.989610
0.000013	2.994057
0.000006	2.991833

It appears that a reasonable estimate is $x \approx 2.99$ at $t = 2/3$.

(b) The graph lies above the t-axis and appears to be periodic with a very short period (i.e., high frequency).



(c) Using the variation of constants formula with

$$P(t) = \int_0^t \left(100e^{\sin 40\pi u}\cos 40\pi u\right) du = \frac{5}{2\pi}e^{\sin 40\pi t} - \frac{5}{2\pi}$$

we obtain the solution formula

$$x(t) = x_0 e^{P(t)} = \exp\left(\frac{5}{2\pi}e^{\sin 40\pi t} - \frac{5}{2\pi}\right)$$

and $x \approx 2.99257$ at $t = 2/3$. The numerical solutions found in (a) give two decimals of accuracy.

(d) $\sin 40\pi t$ is periodic with period equal to $1/20 = 0.05$ (or frequency 20) and hence so is the solution in (c). The solution x is always positive and hence the graph always lies above the t-axis.

EXERCISE 2.175. Since $p = -0.5 < 0$ the equilibrium $x_e = 2$ is an attractor (nonequilibrium solutions $x(t) \to 2$ as $t \to +\infty$). The phase line portrait is $\to 2 \leftarrow$.

EXERCISE 2.177. Since $p = -1 < 0$ the equilibrium $x_e = 2$ is an attractor (nonequilibrium solutions $x(t) \to 2$ as $t \to +\infty$). The phase line portrait is $\to 2 \leftarrow$.

EXERCISE 2.179. Since $p = -\pi < 0$ the equilibrium $x_e = 7/\pi$ is an attractor (nonequilibrium solutions $x(t) \to 7/\pi$ as $t \to +\infty$). The phase line portrait is $\to 7/\pi \leftarrow$.

EXERCISE 2.181. Since $p = 2a - 1$ we have the following possibilities. If $a < 1/2$ then the equilibrium $x_e = (1-2a)^{-1}$ is an attractor, i.e. nonequilibrium solutions $x(t) \to (1-2a)^{-1}$ as $t \to +\infty$. In this case, the phase line portrait is $\to (1-2a)^{-1} \leftarrow$. If $a > 1/2$ then the equilibrium $x_e = (1-2a)^{-1}$ is a repeller, i.e. are unbounded as $t \to +\infty$ and $x(t) \to (1-2a)^{-1}$ as $t \to -\infty$. In this case, the phase line portrait is $\leftarrow (1-2a)^{-1} \to$. If $a = 1/2$ then the equation is non-hyperbolic and nonequilibrium solutions are unbounded. In this case, the phase line portrait is $\to 1/2 \to$.The point $a = 1/2$ is a bifurcation point.

EXERCISE 2.183. Since $p = a^2 - 1$ we have the following possibilities. If $-1 < a < 1$ then the equilibrium $x_e = (1+a)/(1-a^2)$ is an attractor, i.e. nonequilibrium solutions $x(t) \to (1+a)/(1-a^2)$ as $t \to +\infty$. The phase line portrait is $\to x_e \leftarrow$. If $|a| > 1$ then the equilibrium $(1+a)/(1-a^2)$ is a repeller, i.e., nonequilibrium solutions $x(t) \to (1+a)/(1-a^2)$ as $t \to -\infty$. The phase line portrait is $\leftarrow x_e \to$. If $a = +1$ then the equation is non-hyperbolic and nonequilibrium solutions are unbounded. The phase line portrait is a shunt $\to x_e \to$. If $a = -1$ then all solutions are equilibrium points. Bifurcation points are $a = \pm 1$.

EXERCISE 2.186. Use the Method of Undetermined Coefficients and the Superposition Principle. A periodic solution is

$$x_p(t) = -\frac{1}{101}\sin 10t + \left(-\frac{10}{101}\right)\cos 10t.$$

All other solutions

$$x(t) = ce^t - \frac{1}{101}\sin 10t + \left(-\frac{10}{101}\right)\cos 10t$$

are exponentially unbounded.

EXERCISE 2.188. Use the Method of Undetermined Coefficients and the Superposition Principle. A periodic solution is

$$x_p(t) = \frac{1}{101}\sin 10t + \left(-\frac{10}{101}\right)\cos 10t + \frac{1}{2}\sin t + \frac{1}{2}\cos t.$$

All other solutions

$$x(t) = ce^{-t} + \frac{1}{101}\sin 10t + \left(-\frac{10}{101}\right)\cos 10t + \frac{1}{2}\sin t + \frac{1}{2}\cos t$$

tend to the periodic solution.

EXERCISE 2.190. The general solution of the associated homogeneous equation $x' = px$ is $x_h(t) = ce^{pt}$. The general solution is thus $x(t) = ce^{pt} + x_p(t)$ where $x_p(t)$ is any particular solution of $x' = px + 2\sin 2\pi t$. A particular solution can be found by the Method of Undetermined Coefficients by substituting $x_p(t) = k_1 \sin 2\pi t + k_2 \cos 2\pi t$ into the equation to determine the coefficients k_1 and k_2. This substitution leads to the two equations

$$pk_1 + 2k_2 = -2, \quad 2k_1 - pk_2 = 0$$

whose solution is

$$k_1 = -\frac{2p}{p^2 + 4\pi^2}, \quad k_2 = -\frac{4\pi}{p^2 + 4\pi^2}.$$

Thus,

$$x(t) = ce^{pt} - \frac{2p}{p^2 + 4\pi^2}\sin 2\pi t - \frac{4\pi}{p^2 + 4\pi^2}\cos 2\pi t.$$

A.4 Chapter 3: Nonlinear First Order Equations

EXERCISE 3.2. The roots of the equilibrium equation $x^2 + 2x - 3 = 0$ are $x_e = 1, -3$.

EXERCISE 3.4. The root of the equilibrium equation $\ln \frac{2x}{1+x} = 0$ or $\frac{2x}{1+x} = 1$ or $2x = 1 + x$ is $x_e = 1$.

EXERCISE 3.6. Use a computer or calculator to solve the equilibrium equation $x - 2 - e^{-x} = 0$ for $x_e \approx 2.12$.

EXERCISE 3.8. We can write the equilibrium equation as $\ln x = \frac{1}{a^2+x}$. A sketch of the graphs of $\ln x$ and $1/(a^2 + x)$ shows there is one intersection point. Therefore, there is one equilibrium (for each a).

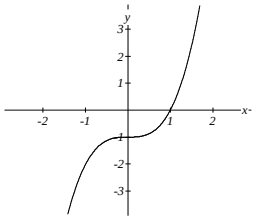
EXERCISE 3.10. We can write the equilibrium equation as $\frac{x^2}{1+x^2} = a - x$. A sketch of the graphs of $x^2/(1 + x^2)$ and $a - x$ for $a > 1$ shows there is one intersection point. Therefore, there is one equilibrium (for each $a > 1$).

EXERCISE 3.12. Solutions are increasing when the sign of $x(1 - x^4)$ is positive and decreasing with the sign is negative. Thus, solutions are increasing if $x_0 < -1$ or $0 < x_0 < 1$ and decreasing if $-1 < x_0 < 0$ or $1 < x$.

EXERCISE 3.14. Solutions are increasing when the sign of $g - cx^2$ is positive and decreasing with the sign is negative. Thus, solutions are increasing if $-\sqrt{g/c} < x_0 < \sqrt{g/c}$ and decreasing if $x_0 < -\sqrt{g/c}$ or $x_0 > \sqrt{g/c}$.

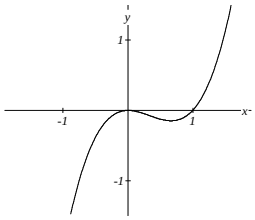
EXERCISE 3.16. Solutions are increasing when the sign of $6x^2 - 5x + 1$ is positive and decreasing with the sign is negative. Thus, solutions are increasing if $x_0 < 1/3$ or $x_0 > 1/2$ and decreasing if $1/3 < x_0 < 1/2$.

EXERCISE 3.18. The equilibrium equation $x^3 - 1 = 0$ has solution $x_e = 1$. Since the graph of $x^3 - 1$ increases through this point, it is a repeller.



The graph of $y = f(x) = x^3 - 1$.

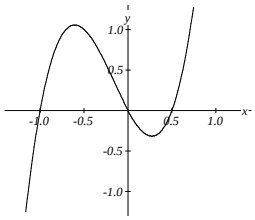
EXERCISE 3.20. The equilibrium equation $x^3 - x^2 = 0$ has solutions $x_e = 0, 1$. Since the graph of $x^3 - x^2$ has a (local) maximum at 0 and increases through 1 , we find that 0 is a (non-hyperbolic) shunt and 1 is a repeller.



The graph of $y = f(x) = x^3 - x^2$.

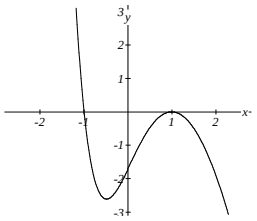
EXERCISE 3.22. Use a computer or a calculator to solve the equilibrium equation $f(x) = -x + \cos x = 0$ for $x_e \approx 0.73909$. The derivative $df/dx = -1 - \sin x$ evaluated at x_e equals $-1 - \sin(0.73909) = -1.674 < 0$. Therefore, x_e is a hyperbolic attractor.

EXERCISE 3.24. Solve the equilibrium equation $f(x) = x(x+1)(x-0.5)^4 = 0$ for $x_e = -1, 0, 0.5$. The derivative df/dx evaluated at these points equals -5.0625, 0.0625, and 0 respectively. The point 0 is non-hyperbolic and the derivative test fails. A sketch of the graph of $f(x)$ shows it has a (local) minimum at 0.5. Therefore, -1 is a hyperbolic attractor, 0 is a hyperbolic repeller, 0.5 is a shunt.



The graph of $y = f(x) = x(x+1)(x-0.5)^4$.

EXERCISE 3.26. Solve the equilibrium equation $f(x) = (1 - x^2)(1 - e^{1-x}) = 0$ for $x_e = -1$ and 1. The derivative df/dx evaluated at these points equals -12.78 and 0 respectively. The derivative test fails at 0. A sketch of the graph of $f(x)$ shows it has a (local) maximum at 0. Therefore, -1 is a hyperbolic attractor and 1 is a non-hyperbolic shunt.



The graph of $y = f(x) = (1 - x^2)(1 - e^{1-x})$.

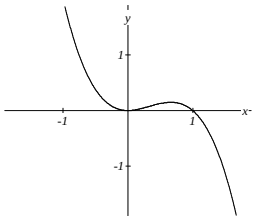
EXERCISE 3.28. The equilibrium equation $f(x) = -x^3 + (1 + a)x^2 - ax = 0$ has roots $x_e = a$, 0 and 1. The derivative $\frac{df(x)}{dx} = -3x^2 + 2(1 + a)x - a$ evaluated at these equilibria equals $a(1 - a)$, $-a$ and $a - 1$ respectively.

If $a < 0$ then a, 0 and 1 are hyperbolic (attractor, repeller, and attractor respectively).

If $0 < a < 1$ then a, 0 and 1 are hyperbolic (source, attractor, attractor respectively).

If $a > 1$ then $a, 0$ and 1 are hyperbolic (attractor, attractor, repeller respectively).

If $a = 0$ then 0 is non-hyperbolic. A graph of $f(x) = -x^3 + x^2$ shows it has a (local) minimum at 0. Therefore, 0 is a shunt and 1 is a hyperbolic attractor.



The graph of $y = f(x) = -x^3 + x^2$.

If $a = 1$ then 0 is non-hyperbolic. A graph of $f(x) = -x^3 + 2x^2 - x$ shows it has a (local) maximum at 1. Therefore, 1 is a shunt and 0 is a hyperbolic attractor.

The following are example answers only (based upon using polynomials). There are infinitely many possible correct answers. Any function $f(x)$ with the specified roots and the appropriate signs between the roots will work. The polynomial answers below are found by multiplying factors determined as follows. If x_e is to be an attractor or a repeller we use the factor $x - x_e$. If x_e is to be a shunt we use the factor $(x - x_e)^2$. After all factors are multiplied together, a sign change might be necessary in order to get the orbit arrows to point in the correct direction.

EXERCISE 3.32. $x' = (x + 3)(x - 3)$

EXERCISE 3.34. $x' = x^2(2 - x)$

EXERCISE 3.36. $x' = x^2(x - 1)^2$

EXERCISE 3.38. $x' = (x - a)(x - b)$

EXERCISE 3.40. $x' = -(x - 1)(x - 2)(x - 3)$

EXERCISE 3.42. $x' = -(x - a)(x - b)(x - c)(x - d)^2$

EXERCISE 3.47. The roots of $f(x) = x^3 - x$ are 0 and ± 1. The derivative $df/dx = 3x^2 - 1$ evaluated at these roots equals -1 and 2 respectively. The linearizations are $u' = -u$ at $x_e = 0$ and $u' = 2u$ at $x_e = \pm 1$.

EXERCISE 3.49. The roots of $f(x) = rx\,(1 - x/K)$ are 0 and K. The derivative $df/dx = r - 2rx/K$ evaluated at these roots equals r and $-r$ respectively. The linearizations are $u' = ru$ at $x_e = 0$ and $u' = -ru$ at $x_e = K$.

EXERCISE 3.51. The root of $f(x) = \frac{x^3}{1+x^2}$ is 0. The derivative

$$\frac{df(x)}{dx} = 3\frac{x^2}{x^2+1} - 2\frac{x^4}{2x^2+x^4+1}$$

evaluated at 0 equals 0 and $-r$ respectively. The linearization at $x_e = 0$ is $u' = 0$.

EXERCISE 3.53 The roots of $f(x) = (1-x)x - h = 0$ are $\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4h}$. The derivative $\frac{df(x)}{dx} = 1 - 2x$ evaluated at these two roots equals $\mp\sqrt{1-4h}$. At $x_e = \left(1 + \sqrt{1-4h}\right)/2$ the linearization is $u' = -\sqrt{1-4h}u$. At $x_e = \left(1 - \sqrt{1-4h}\right)/2$ the linearization is $u' = \sqrt{1-4h}u$.

EXERCISE 3.57. The equations are qualitatively equivalent. Both have phase line portraits of the type

$$\longrightarrow \cdot \longleftarrow$$

EXERCISE 3.59. The equations are qualitatively equivalent. Both have phase line portraits of the type

$$\longrightarrow \cdot \longleftarrow$$

EXERCISE 3.61. The equations are not qualitatively equivalent. $x_e = 0$ is a repeller in the first equation, but an attractor in the second equation.

EXERCISE 3.63. The equations are qualitatively equivalent. Both have phase line portraits of the type

$$\longleftarrow \cdot \longleftarrow \cdot \longrightarrow$$

EXERCISE 3.65. The equations are not qualitatively equivalent. The first equation has two equilibria while the second has one.

EXERCISE 3.67. The equations are qualitatively equivalent. Both have phase line portraits of the type

$$\longleftarrow \cdot \longrightarrow \cdot \longleftarrow$$

EXERCISE 3.69. The equations are qualitatively equivalent. Both have phase line portraits of the type

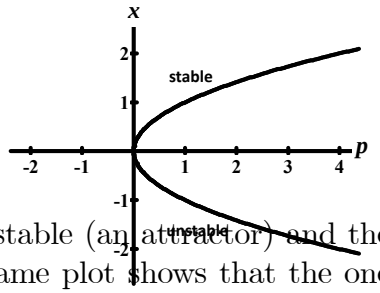
$$\longrightarrow \cdot \longleftarrow$$

EXERCISE 3.71. The equations are not qualitatively equivalent. The first equation has one equilibrium while the second has three.

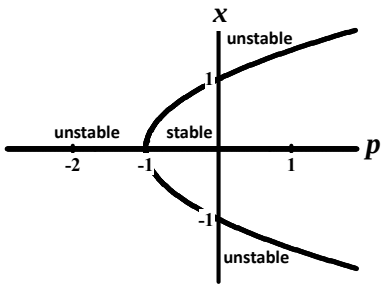
EXERCISE 3.73. The equations are qualitatively equivalent. Both have phase line portraits of the type

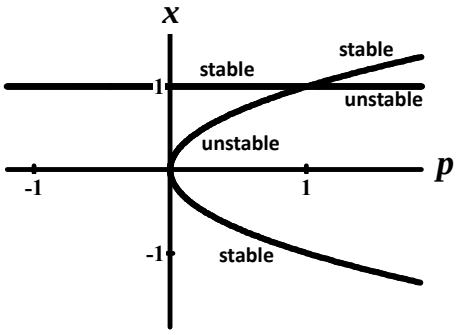
$$\longrightarrow$$

EXERCISE 3.75. Solve the equilibrium equation $0 = p - x^2$ for $p = x^2$ and plot. Reflect the plot through the line $p = x$ to obtain the bifurcation diagram. From the diagram we see there is one bifurcation and it is a *blue-sky bifurcation* that occurs at the point $(p, x) = (0, 0)$. For a fixed value of $p > 0$ a plot of the parabola $y = f(x, p) = p - x^2$ shows, by the Geometric Test criterion, that the positive equilibrium is stable (an attractor) and the negative equilibrium is unstable (a repeller). For $p = 0$ the same plot shows that the one and only equilibrium $x_e = 0$ is a shunt.



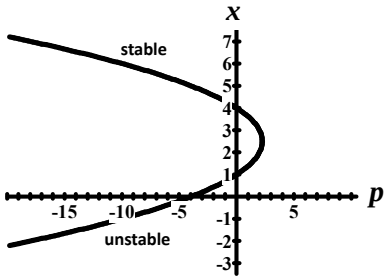
EXERCISE 3.77. The equilibrium equation $x(x^2 - 1 - p) = 0$ has root $x_e = 0$, which plots as the p-axis in the bifurcation diagram. Other equilibria come from the equation $x^2 - 1 - p = 0$. To place these in the bifurcation diagram, plot $p = 1 - x^2$ and reflect the result through the line $p = x$. These two plots result in the bifurcation diagram shown, from which we see that here is one bifurcation and it is a *pitchfork bifurcation* that occurs at the point $(p, x) = (-1, 0)$. To determine the stability properties of the equilibria, you can plot the cubic $y = f(x, p) = x(x^2 - 1 - p)$ for values of $p < -1$ and $p > -1$ and apply the geometric test criterion. The result is shown. Another way is to use the Derivative Test, which involves evaluating the derivative $df(x, p)/dx = x(2x) + (x^2 - 1 - p)$ at the equilibria. At $x_e = 0$ the value of the derivative is $-1 - p$ and the Derivative Test implies that $x_e = 0$ is stable when $p > -1$ and unstable when $p < -1$. At the other equilibria, the value of the derivative is $2x^2 > 0$ and hence these equilibria are unstable.



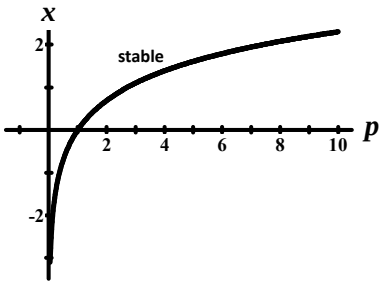


EXERCISE 3.79. The equation $(x - 1)(p - x^2) = 0$ has root $x_e = 1$, which plots as a horizontal straight line in the bifurcation diagram. Other equilibria come from the equation $p - x^2 = 0$. To place these in the bifurcation diagram, plot $p = x^2$ and reflect the result through the line $p = x$. These two plots result in the bifurcation diagram shown, from which we see that are two bifurcations. A *transcritical bifurcation* occurs at the point $(p, x) = (1, 1)$ and a *blue-sky bifurcation* occurs at the point $(p, x) = (0, 0)$. To determine the stability properties of the equilibria, we can use the Derivative Test, which involves evaluating the derivative $df(x, p)/dx = (x - 1)(-2x) + (p - x^2)$ at the equilibria. At $x_e = 1$ the derivative equals $p - 1$ and therefore $x_e = 1$ is stable when $p < 1$ and unstable when $p > 1$. At the equilibrium $x_e = \sqrt{p}$ the derivative equals $(\sqrt{p} - 1)(-2\sqrt{p})$ and therefore this equilibrium is unstable when $p < 1$ and stable when $p > 1$. Finally at the equilibrium $x_e = -\sqrt{p}$ the derivative equals $(-\sqrt{p} - 1)(2\sqrt{p})$ and therefore this equilibrium is is stable for all $p > 0$. All this stability information is indicated in the bifurcation diagram.

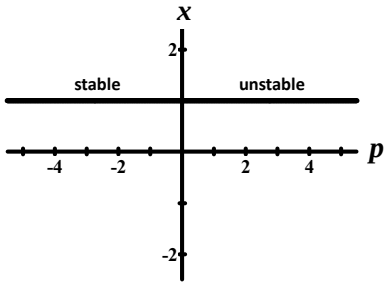
EXERCISE 3.81. Solve the equilibrium equation $px^2+(x-1)(x-4)=0$ for $p=-(x-1)(x-4)/x^2$ and plot. Reflect the plot through the line $p=x$ to obtain the bifurcation diagram. From the diagram we see there is one bifurcation and it is a *blue-sky bifurcation* that occurs at $(p,x)=(9/16,8/5)\approx(0.5625,1.6)$. A plot of $y=f(x,p)=px^2+(x-1)(x-4)$ for $p<9/16$ and an application of the Geometric Test criterion shows that the larger equilibrium is stable and the smaller is unstable.



EXERCISE 3.83. From the equilibrium equation $p-e^x=0$ we see that the only equilibrium is $x_e=\ln p$ whose plot shows there are *no bifurcations*. The derivative $df(x,p)/dx=-e^x$ evaluated at the equilibium $x_e=\ln p$ equals $-p$ which is negative for $p>0$. The Derivative Test criterion implies the equilibrium is stable.

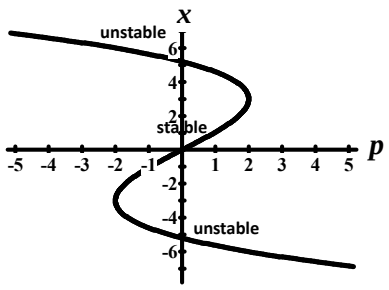


EXERCISE 3.85. The only equilibrium is $x_e=1$ and the bifurcation diagram is a horizontal straight line. First fix $p>0$. Then $f(x,p)=p(x-1)^3$ is positive for $x>1$ and negative for $x<1$ which implies the equilibrium $x_e=1$ is unstable. On the other hand, for $p<0$ we have the opposite: $f(x,p)=p(x-1)^3$ is negative for $x>1$ and positive for $x<1$ which implies the equilibrium $x_e=1$ is stable. Thus, a bifurcation occurs at $(p,x)=(0,1)$, but it is not of any of the three types that we classified.



EXERCISE 3.87.

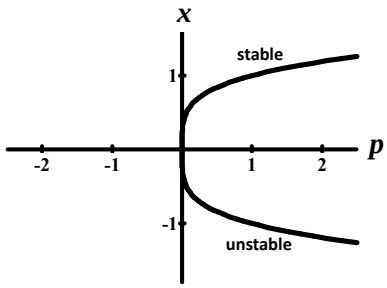
Solve the equilibrium equation $p-x(1-\frac{1}{27}x^2)=0$ for $p=x(1-\frac{1}{27}x^2)$ and plot. Reflect the resulting graph through the line $p=x$ to obtain the bifurcation diagram shown. From the diagram we see that there are *two blue-sky bifurcations*. To determine the stability of the equilibria, we use the Geometric Test by plotting $f(x,p)=p-x(1-\frac{1}{27}x^2)$. First draw a graph of the cubic polynomial $-x(1-\frac{1}{27}x^2)$ obtained when $p=0$. There are three roots and the phase line portrait is $\longleftarrow\ -\sqrt{27}\ \longrightarrow\ 0\ \longleftarrow\ \sqrt{27}\ \longrightarrow$. Adding $p>0$ (<0) to $-x(1-\frac{1}{27}x^2)$ results in a translation of the graph upward (downward). For $p>2$ there is only one (negative) root x_1 and the phase line portrait is $\longleftarrow\ x_1\ \longrightarrow$. For $p<-2$ there is only one (positive) root x_3



and the phase line portrait is $\longleftarrow x_3 \longrightarrow$. For $-2 < p < 2$ there are three roots $x_1 < x_2 < x_3$ and the phase line portrait is $\longleftarrow x_1 \longrightarrow x_2 \longleftarrow x_3 \longrightarrow$. Thus, the two blue-sky bifurcations occur at $p = -2$ and $p = 2$.

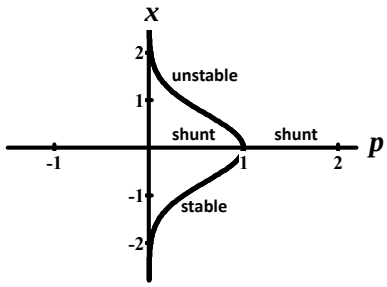
EXERCISE 3.89.

Solve the equilibrium equation $p - x^4 = 0$ for $p = x^4$ and plot. Reflect the resulting graph through the line $p = x$ to obtain the bifurcation diagram shown. From the diagram we see that a blue-sky bifurcation occurs at $(p, x) = (0, 0)$. For $p > 0$ there are two equilibria, $x_e = -\sqrt[4]{p}$ and $+\sqrt[4]{p}$. An evaluation of the derivative $df(x,p)/dx = -4x^3$ at $x_e = -\sqrt[4]{p}$ yields a positive answer so this equilibrium is unstable. An evaluation of the derivative $df(x,p)/dx = -4x^3$ at $x_e = \sqrt[4]{p}$ yields a negative answer so this equilibrium is stable.



EXERCISE 3.91.

The equilibrium equation $x^2\left(p - e^{-x^2}\right) = 0$ has root $x_e = 0$ for all values of p. Other equilibria are roots of $p - e^{-x^2} = 0$. Solving for $p = e^{-x^2}$ and plotting and reflecting the graph through the line $p = x$ yields the bifurcation diagram shown. Note that the equilibrium $x_e = 0$ plots as the p-axis in this diagram. Therefore, there is a pitch-fork bifurcation at $(p, x) = (1, 0)$. For $0 < p < 1$ there are three equilibria: $x_e = -\sqrt{\ln(1/p)}$, $x_e = \sqrt{\ln(1/p)}$, and $x_e = 0$. For $x < -\sqrt{\ln(1/p)}$ and for $x > \sqrt{\ln(1/p)}$, we see that $f(x,p) = x^2\left(p - e^{-x^2}\right)$ is positive. For $-\sqrt{\ln(1/p)} < x < \sqrt{\ln(1/p)}$ we see that $f(x,p) = x^2\left(p - e^{-x^2}\right)$ is negative, except at $x = 0$. Thus, $x_e = -\sqrt{\ln(1/p)}$ is stable, $x_e = \sqrt{\ln(1/p)}$ is unstable and $x_e = 0$ is a shunt. For $p \geq 1$ there is only one equilibrium, namely, $x_e = 0$, and since $f(x,p) = x^2\left(p - e^{-x^2}\right) > 0$ for all x (except $x = 0$) in this case, it follows that $x_e = 0$ is a shunt for all $p > 0$.



EXERCISE 3.99.

$$\int \frac{1}{1+x^2}dx = \int dt$$
$$\arctan x = t + c$$

Since $1 + x^2 = 0$ has no solution, there are no equilibria.

$$x(t) = \tan(t + c),\ c = \text{arbitrary constant.}$$

EXERCISE 3.101.

$$\begin{aligned}
\int \frac{1}{x-\frac{1}{x}}dx &= \int dt \\
\int \frac{x}{x^2-1}dx &= t+k \\
\frac{1}{2}\ln\left|x^2-1\right| &= t+k \\
\left|x^2-1\right| &= e^{2k}e^{2t} \\
x^2-1 &= ce^{2t}, \quad c = \text{ arbitrary nonzero constant}
\end{aligned}$$

The equilibrium equation $x - 1/x = 0$ yields two equilibria $x_e = \pm 1$ both of which can be included in the (implicitly) formula above by allowing c to be 0. The implicit general solution is

$$x^2 - 1 = ce^{2t}, \quad c = \text{arbitrary constant.}$$

EXERCISE 3.103.

$$\begin{aligned}
\int \frac{1}{\cot x}dx &= \int dt \\
-\ln(\cos x) &= t+c, \quad c = \text{arbitrary constant}
\end{aligned}$$

together with the equilibria $x_e = \pi/2 \pm n\pi$, $n = 0, \pm 1, \pm 2, \cdots$ describes (implicitly) the general solution.

EXERCISE 3.105.

$$\begin{aligned}
\int_1^x \frac{1}{-s^4}ds &= \int_1^t ds \\
\frac{1}{3x^3} - \frac{1}{3} &= t-1 \\
x(t) &= \left(\frac{1}{3t-2}\right)^{1/3}
\end{aligned}$$

EXERCISE 3.107.

$$\begin{aligned}
\int_{-1}^x se^s ds &= -\int_0^t ds \\
xe^x - e^x + 2e^{-1} &= -t \\
(x-1)e^x &= -t - 2e^{-1}
\end{aligned}$$

EXERCISE 3.109.

$$\int_{1/2}^{x} \frac{1}{s-\frac{1}{s}} ds = \int_0^t ds$$
$$\int_{\frac{1}{2}}^{x} \frac{s}{s^2-1}\, ds = t$$
$$\frac{1}{2}\ln\left|x^2-1\right| - \frac{1}{2}\ln\left|-\frac{3}{4}\right| = t$$
$$\ln\left|\frac{x^2-1}{3/4}\right| = 2t$$
$$x(t) = \left(1-\frac{3}{4}e^{2t}\right)^{1/2}$$

EXERCISE 3.111.

$$\int_{-1/2}^{x} \frac{1}{s-\frac{1}{s}} ds = \int_0^t ds$$
$$\int_{-1/2}^{x} \frac{s}{s^2-1}\, ds = t$$
$$\frac{1}{2}\ln\left|x^2-1\right| - \frac{1}{2}\ln\left|-\frac{3}{4}\right| = t$$
$$\ln\left|\frac{x^2-1}{3/4}\right| = 2t$$
$$x(t) = -\left(1-\frac{3}{4}e^{2t}\right)^{1/2}$$

EXERCISE 3.116.

$$\int_{x_0}^{x} \frac{1}{s^2-s} ds = \int_0^t dt$$
$$\ln\left|\frac{x-1}{x}\right| - \ln\left|\frac{x_0-1}{x_0}\right| = t$$
$$\left|\frac{x-1}{x}\right| = \left|\frac{x_0-1}{x_0}\right| e^t$$
$$\frac{x-1}{x} = \frac{x_0-1}{x_0}e^t$$

$$x(t) = \frac{x_0}{x_0-(x_0-1)e^t}.$$

For $x_0 > 1$ this solution the maximal interval of existence of this solution is $-\infty < t < \beta$ where

$$\beta = \ln\left(\frac{x_0}{x_0-1}\right).$$

EXERCISE 3.118.

$$\int \frac{1}{x} dx = \int t^2 dt$$
$$ln\,|x| = \frac{1}{3}t^3 + c$$
$$x\,(t) = c \exp\left(\frac{1}{3}t^3\right)$$

$c =$ arbitrary constant. The only equilibrium $x_e = 0$ is included in this formula.

EXERCISE 3.120.

$$\int \frac{1}{x^2} dx = \int \frac{1}{t^2} dt$$
$$-\frac{1}{x} = -\frac{1}{t} - c$$

$c =$ arbitrary constant. The only equilibrium is $x_e = 0$.

$$x\,(t) = \begin{cases} \frac{t}{1+ct}, & c = \text{arbitrary constant} \\ 0 \end{cases}$$

EXERCISE 3.122.

$$\int \frac{1}{1-x^2} dx = \int \frac{1}{t} dt$$
$$\frac{1}{2} \ln \left|\frac{x+1}{x-1}\right| = \ln |t| + k$$

$$\left|\frac{x+1}{x-1}\right| = e^{2k} t^2$$
$$\frac{x+1}{x-1} = ct^2$$

where $c = \pm e^{2k}$ is an arbitrary, nonzero constant. There are two equilibria $x_e = \pm 1$. The equilibrium -1 is contained in the formula if we allow c to equal 0. The general solution is

$$x\,(t) = \begin{cases} \frac{1+c\exp(t^2)}{-1+c\exp(t^2)}, & c = \text{arbitrary constant} \\ 1 \end{cases}$$

EXERCISE 3.124.

$$\begin{aligned}
\int \frac{1}{x^2-3x+2}dx &= \int e^t dt \\
\int \frac{1}{(x-2)(x-1)}dx &= \int e^t dt \\
\ln\left|\frac{x-2}{x-1}\right| &= e^t + k \\
\left|\frac{x-2}{x-1}\right| &= e^k \exp(e^t) \\
\frac{x-2}{x-1} &= \pm e^k \exp(e^t) \\
x &= \frac{2-c\exp(e^t)}{1-c\exp(e^t)}
\end{aligned}$$

where $c = \pm e^k$ is an arbitrary, nonzero constant. The equilibria are the roots of $x^2 - 3x + 2$, i.e., $x_e = 1$ and 2. The equilibrium 2 is contained in the formula if we allow c to equal 0.

$$x(t) = \begin{cases} \frac{2-c\exp(e^t)}{1-c\exp(e^t)}, & c = \text{arbitrary constant} \\ 1 \end{cases}$$

EXERCISE 3.126.

$$\begin{aligned}
\int \frac{1}{a^2-x^2}dx &= \int \cos t dt \\
\int \frac{1}{a-x}\frac{1}{a+x}dx &= \sin t + k \\
\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| &= \sin t + k \\
\left|\frac{a+x}{a-x}\right| &= e^{2ak}e^{2a\sin t} \\
\frac{a+x}{a-x} &= \pm e^{2ak}e^{2a\sin t} \\
x &= -a\frac{1+c\exp(2a\sin t)}{1-c\exp(2a\sin t)}
\end{aligned}$$

where $c = \pm e^{2ak}$ is an arbitrary, nonzero constant. The equilibria are $x_e = \pm a$. The equilibrium $-a$ is included in the formula if we allow c to equal 0. The general solution is

$$x(t) = \begin{cases} -a\frac{1+c\exp(2a\sin t)}{1-c\exp(2a\sin t)}, & c = \text{arbitrary constant} \\ a \end{cases}$$

EXERCISE 3.128.

$$\begin{aligned}\int_1^x \frac{1}{s}ds &= \int_0^t s^2 ds \\ \ln|x| &= \frac{1}{3}t^3 \\ x(t) &= \exp\left(\frac{1}{3}t^3\right)\end{aligned}$$

EXERCISE 3.130.

$$\begin{aligned}\int_{-1}^x s^4 ds &= \int_0^t (s+1)ds \\ \frac{1}{5}x^5 + \frac{1}{5} &= \frac{1}{2}t^2 + t \\ x^5 + 1 &= \frac{5}{2}t^2 + 5t\end{aligned}$$

EXERCISE 3.132.

$$\int_b^x x^a ds = \int_1^t 2s ds$$

If $a \neq -1$

$$\begin{aligned}\int_b^x x^a ds &= \int_1^t 2t ds \\ \frac{1}{a+1}x^{a+1} - \frac{1}{a+1}b^{a+1} &= t^2 - 1 \\ x^{a+1} &= (a+1)(t^2-1) + b^{a+1}\end{aligned}$$

If $a = -1$

$$\begin{aligned}\int_b^x x^a ds &= \int_1^t 2s ds \\ \ln|x| - \ln|b| &= t^2 - 1 \\ |x| &= |b|\exp(t^2-1) \\ x(t) &= b\exp(t^2-1).\end{aligned}$$

EXERCISE 3.136. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{array}{ll} k_0' = k_0 & k_1' = k_1 + e^{-t}k_0 \\ k_0(0) = 2 \text{ and} & k_1(0) = 0. \end{array}$$

The first initial value problem has solution $k_0(t) = 2e^t$ and the second initial value problem becomes

$$\begin{aligned}k_1' &= k_1 + 2 \\ k_1(0) &= 0\end{aligned}$$

whose solution is $k_1(t) = -2 + 2e^t$. Thus, $p_1(t) = 2e^t + (-2 + 2e^t)\,\varepsilon$.

EXERCISE 3.138. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{array}{l} k_0' = 2k_0 \\ k_0(0) = 1 \end{array} \text{ and } \begin{array}{l} k_1' = 2k_1 - \sin t \\ k_1(0) = 0. \end{array}$$

The solutions are $k_0(t) = e^{2t}$ and

$$k_1(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{2t}.$$

Thus,

$$p_1(t) = e^{2t} + \left(\frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{2t}\right)\varepsilon.$$

EXERCISE 3.140. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{array}{l} k_0' = k_0 - k_0^2 \\ k_0(0) = 1 \end{array} \text{ and } \begin{array}{l} k_1' = k_1 - 2k_0k_1 + \sin t \\ k_1(0) = 0. \end{array}$$

The first initial value problem has solution $k_0(t) = 1$ and the second initial value problem becomes

$$\begin{array}{l} k_1' = -k_1 + \sin t \\ k_1(0) = 0. \end{array}$$

whose solution is

$$k_1(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t.$$

Thus,

$$p_1(t) = 1 + \left(\frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t\right)\varepsilon.$$

EXERCISE 3.142. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{array}{l} k_0' = 1 \\ k_0(0) = 1 \end{array} \text{ and } \begin{array}{l} k_1' = k_0(1 - k_0) \\ k_1(0) = 0. \end{array}$$

The first initial value problem has solution $k_0(t) = 1 + t$ and the second initial value problem becomes

$$\begin{array}{l} k_1' = (1 + t)(-t) = -t - t^2 \\ k_1(0) = 0 \end{array}$$

whose solution is $k_1(t) = -t^2/2 - t^3/3$. Thus, $p_1(t) = 1 + t + (-t^2/2 - t^3/3)\,\varepsilon$.

EXERCISE 3.144. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{array}{l} k_0' = 0 \\ k_0(1) = -1 \end{array} \text{ and } \begin{array}{l} k_1' = k_0 \\ k_1(1) = 0. \end{array}$$

The first initial value problem has solution $k_0(t) = -1$ and the second initial value problem becomes

$$\begin{aligned} k_1' &= -1 \\ k_1(1) &= 0 \end{aligned}$$

whose solution is $k_1(t) = 1 - t$. Thus, $p_1(t) = -1 + (1-t)\varepsilon$.

EXERCISE 3.146. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{aligned} k_0' &= -k_0 + 1 \\ k_0(0) &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} k_1' &= -k_1 - k_0^2 \\ k_1(0) &= 0. \end{aligned}$$

The first initial value problem has solution $k_0(t) = -e^{-t} + 1$ and the second initial value problem becomes

$$\begin{aligned} k_1' &= -k_1 + 2e^{-t} - e^{-2t} - 1 \\ k_1(0) &= 0 \end{aligned}$$

whose solution is $k_1(t) = 2te^{-t} + e^{-2t} - 1$. Thus, $p_1(t) = (-e^{-t} + 1) + (2te^{-t} + e^{-2t} - 1)\varepsilon$.

EXERCISE 3.148. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{aligned} k_0' &= -k_0 \\ k_0(0) &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} k_1' &= -k_1 + e^t k_0^3 \\ k_1(0) &= 0. \end{aligned}$$

The first initial value problem has solution $k_0(t) = 2e^{-t}$ and the second initial value problem becomes

$$\begin{aligned} k_1' &= -k_1 + 8e^{-2t} \\ k_1(0) &= 0 \end{aligned}$$

whose solution is $k_1(t) = 8e^{-t} - 8e^{-2t}$. Thus, $p_1(t) = (2e^{-t}) + (8e^{-t} - 8e^{-2t})\varepsilon$.

EXERCISE 3.150. The initial value problems for the coefficients $k_0(t)$ and $k_1(t)$ in the first order perturbation expansion $p_1(t) = k_0(t) + k_1(t)\varepsilon$ are

$$\begin{aligned} k_0' &= k_0 \\ k_0(0) &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} k_1' &= k_2 + \tfrac{1+3e^t}{1+k_0} \\ k_1(0) &= 0. \end{aligned}$$

The first initial value problem has solution $k_0(t) = 3e^t$ and the second initial value problem becomes

$$\begin{aligned} k_1' &= k_2 + 1 \\ k_1(0) &= 0 \end{aligned}$$

whose solution is $k_1(t) = e^t - 1$. Thus, $p_1(t) = (3e^t) + (e^t - 1)\varepsilon$.

EXERCISE 3.153.

$$\begin{cases} k_0' = -k_0, & k_0(t_0) = x_0 \\ k_1' = -k_1 - k_0 \sin t, & k_1(t_0) = 0 \\ k_2' = -k_2 - k_1 \sin t, & k_2(t_0) = 0 \end{cases}$$

The first (linear homogeneous) initial value problem has solution $k_0(t) = x_0 e^{-(t-t_0)}$. Using this solution in the second (linear nonhomogeneous) initial value problem, we solve for

$k_1(t) = x_0(\cos t - \cos t_0)\, e^{-(t-t_0)}$. Finally, solving the last (linear nonhomogeneous) initial value problem we obtain

$$k_2(t) = \frac{1}{2}x_0\left(\cos^2 t - 2\cos t\cos t_0 + \cos^2 t_0\right) e^{-(t-t_0)}.$$

These solutions yield the perturbation approximation.

$$\begin{aligned} p_2(t) = {} & x_0 e^{-(t-t_0)} + x_0(\cos t - \cos t_0)\, e^{-(t-t_0)}\varepsilon \\ & + \frac{1}{2}x_0\left(\cos^2 t - 2\cos t\cos t_0 + \cos^2 t_0\right) e^{-(t-t_0)}\varepsilon^2 \end{aligned}$$

EXERCISE 3.160. $f(x) = 1 - e^{-ax}$ is positive for $x > 0$ and negative for $x < 0$ and therefore solutions increase for $x_0 > 0$ and decrease for $x_0 < 0$.

EXERCISE 3.162. The roots of $f(x) = \cos^2 x$ are $x = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \cdots$. The graph of $f(x)$ has local minima at each of these equilibrium and there they are non-hyperbolic shunts.

$$\cdots \longrightarrow -\frac{3}{2}\pi \longrightarrow -\frac{1}{2}\pi \longrightarrow \frac{1}{2}\pi \longrightarrow \frac{3}{2}\pi \longrightarrow \frac{5}{2}\pi \longrightarrow \cdots$$

EXERCISE 3.164. If $a < 0$ there are no equilibria.

If $a > 0$ there are two equilibria: $x_e = \pm\sqrt{a}$. The derivative $df/dx = -ax^2 - 1$ evaluated at either equilibrium equals $-a^1 - 1 < 0$ and we find that both equilibria are hyperbolic attractors.

If $a = 0$ then the unique equilibrium $x_e = 0$ is an attractor (the equation reduces to $x' = -x$).

$$\begin{array}{lll} a < 0 & \text{implies} & \longrightarrow 0 \longleftarrow \\ a = 0 & \text{implies} & \longrightarrow 0 \longleftarrow \\ a > 0 & \text{implies} & \longrightarrow -\sqrt{a} \longleftarrow 0 \longrightarrow \sqrt{a} \longleftarrow \end{array}$$

Note: 0 is not an equilibrium when $a \neq 0$.

EXERCISE 3.166. The derivative of $f(x) = x^2(1-x)$ is $df/dx = 2x - 3x^2$. The linearization at the equilibrium 0 is $u' = 0$ since df/dx evaluated at 0 equals 0. The linearization at the equilibrium 1 is $u' = -u$ since df/dx evaluated at 1 equals -1.

EXERCISE 3.168. The derivative of $f(x) = b - e^{-ax}$ is $df/dx = ae^{-ax}$. The linearization at the equilibrium $-(\ln b)/a$ is $u' = abu$ since df/dx evaluated at $-(\ln b)/a$ equals ab.

EXERCISE 3.170. **(a)** The equilibria are $x = 0$, $\sqrt{2}$, and $-\sqrt{2}$. The phase line portrait is:

$$\longrightarrow -\sqrt{2} \longleftarrow 0 \longrightarrow \sqrt{2} \longleftarrow .$$

(b) $x_e = 0$ is a repeller. $x_e = \pm\sqrt{2}$ are both attractors.

(c) Let

$$f(x) = x^3\frac{2 - x^2}{1 + x^2}.$$

The derivative df/dx evaluated at 0 equals 0 and $x = 0$ is non-hyperbolic. The derivative df/dx evaluated at $\pm\sqrt{2}$ equals $-8/3$ is nonzero, both $x_e = \pm\sqrt{2}$ are hyperbolic.

(d) The linearization at $x = 0$ is $u' = 0$, which is non-hyperbolic. The linearizations at $x_e = \pm\sqrt{2}$ are both $u' = -\frac{8}{3}u$, which has a hyperbolic attractor.

(e) The linearization theorem does not apply for the non-hyperbolic equilibrium $x_e = 0$. It does apply for the equilibria $x_e = \pm\sqrt{2}$ and it asserts both are attractors.

EXERCISE 3.172.

(a) There are no equilibria if $p > 0$. If $p = 0$, the only equilibrium is $x_e = 0$. If $p < 0$, there are two equilibria: $x = \pm(-p)^{1/4}$. The phase line portrait depends on p as follows:

$$\begin{array}{ll} \longrightarrow & \text{for } p > 0 \\ \longrightarrow 0 \longrightarrow & \text{for } p = 0 \\ \longrightarrow -(-p)^{1/4} \longleftarrow (-p)^{1/4} \longrightarrow & \text{for } p < 0. \end{array}$$

(b) When $p = 0$, $x_e = 0$ is a shunt. When $p < 0$, $x = -(-p)^{1/4}$ is an attractor and $x = (-p)^{1/4}$ is a repeller.

(c) Let $f(x) = p + x^4$. The derivative df/dx evaluated at 0 equals 0 and therefore 0 is non-hyperbolic. The derivative df/dx evaluated at 1 equals4 and 1 is hyperbolic.

(d) The linearization at $x = 0$ is $u' = 0$, which is non-hyperbolic. The linearization at $x_e = 1$ is $u' = 4u$, which is a hyperbolic repeller.

(e) When $p = 0$ the linearization theorem does not apply for the non-hyperbolic equilibrium $x = 0$. When $p < 0$ the theorem does apply for the equilibria $x_e = \pm(-p)^{1/4}$. It asserts $x_e = -(-p)^{1/4}$ is an attractor and $x_e = (-p)^{1/4}$ is a repeller.

The following are example answers only (based upon using polynomials). There are infinitely many possible correct answers. Any function $f(x)$ with the specified roots and the appropriate signs between the roots will work. The polynomial answers below are found by multiplying factors determined as follows. If x_e is to be an attractor or a repeller we use the factor $x - x_e$. If x_e is to be a shunt we use the factor $(x - x_e)^2$. After all factors are multiplied together, a sign change might be necessary in order to get the orbit arrows to point in the correct direction.

EXERCISE 3.174. $x' = x^2(x-1)^2(2-x)$

EXERCISE 3.176. $x' = p(p-x)$

EXERCISE 3.178. The equations are qualitatively equivalent. They have phase line portraits of the form

$$\longleftarrow \bullet \longrightarrow \bullet \longleftarrow$$

EXERCISE 3.180. The equations are not qualitatively equivalent. The first equation has a repeller at $x_e = -1$ and the second equation has an attractor at $x_e = 0$.

EXERCISE 3.182.

$$\begin{cases} \longrightarrow 0 \longleftarrow p \longrightarrow & \text{for } p > 0 \\ \longrightarrow 0 \longrightarrow & \text{for } p = 0 \\ \longrightarrow p \longleftarrow 0 \longrightarrow & \text{for } p < 0 \end{cases}$$

A transcritical bifurcation (with an exchange of stability) occurs at $p = 0$Two saddle-node bifurcations occur at $p = 0$ and another saddle-node bifurcation occurs at $p = -1$.

EXERCISE 3.184.

$$\begin{cases} \longrightarrow & \text{for } p \geq 0 \\ \longleftarrow \bullet \longrightarrow \bullet \longleftarrow & \text{for } -1 < p < 0 \\ \longleftarrow \bullet \longleftarrow & \text{for } p = -1 \\ \longleftarrow & \text{for } p < -1 \end{cases}$$

A saddle-node bifurcation occurs at $p = -1$.

EXERCISE 3.186.

$$\begin{cases} \longrightarrow & \text{for } p < 1 \\ \longrightarrow 1 \longrightarrow & \text{for } p = 1 \\ \longrightarrow \left(1 - \sqrt{p-1}\right) \longleftarrow \left(1 + \sqrt{p-1}\right) \longrightarrow & \text{for } p > 1 \end{cases}$$

A saddle node bifurcation occurs at $p = 1$.

A.5 Chapter 4: Systems and Higher Order Equations

EXERCISE 4.1. Is a solution pair:

$$\begin{aligned} x' &= -6e^{-6t} = -2x + 2y \\ y' &= 12e^{-6t} = 2x - 5y \end{aligned}$$

EXERCISE 4.3. Is not a solution pair:

$$x' = -4e^{-t} - 24e^{-6t} \neq -4e^{-t} + 8e^{-6t} = -2x + 2y$$

EXERCISE 4.5. Is a solution pair:

$$\begin{aligned} x' &= -2\sin t = y \\ y' &= -2\cos t = -x \end{aligned}$$

EXERCISE 4.7. Is not a solution pair:

$$x' = 2\cos 2t \neq y$$

EXERCISE 4.9. Is a solution pair:

$$\begin{aligned} x' &= -\sin t + \cos t = y \\ y' &= -\cos t - \sin t = -x \end{aligned}$$

EXERCISE 4.11. Is a solution pair:

$$\begin{aligned} x' &= c\cos t = y \\ y' &= -c\sin t = -x \end{aligned}$$

EXERCISE 4.14.

$$f(t,x,y) = x(1-x) - \frac{xy}{1+x}$$

and its derivatives and $g(t,x,y) = -y + \frac{xy}{1+x}$ and its derivatives are continuous for all t_0,and y_0 and all $x_0 \neq -1$. For these initial conditions there exists a unique solution pair on an interval containing t_0. For $x_0 = -1$,the fundamental theorem does not apply and no conclusions can be drawn.

EXERCISE 4.16. $f(t,x,y) = ax + by$ and its derivatives and $g(t,x,y) = cx + dy$ and its derivatives are continuous for all t_0, x_0 and y_0 Thus, for all initial conditions there exists a unique solution pair on an interval containing t_0.

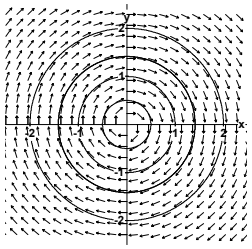
EXERCISE 4.18. $f(t,x,y) = x(1 - \frac{x}{2+\sin t}) - xy$ and its derivatives and $g(t,x,y) = -y + xy$ and its derivatives are continuous for all t_0, x_0 and y_0 Thus, for all initial conditions there exists a unique solution pair on an interval containing t_0.

EXERCISE 4.20. $f(t,x,y) = y$, $g(t,x,y) = -x + \sin t$ and all their derivatives with respect to x and y are continuous for all t_0, x_0 and y_0 Thus, for all initial conditions there exists a unique solution pair on an interval containing t_0.

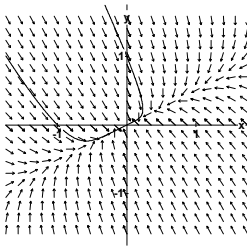
EXERCISE 4.22. $f(t,x,y) = y$, $g(t,x,y) = -t^{-2}x - t^{-1}y$ and all their derivatives with respect to x and y are continuous for all t_0,and y_0 and all $t_0 \neq 0$. For these initial conditions there exists a unique solution pair on an interval containing t_0. For $t_0 = 0$, the fundamental theorem does not apply and no conclusions can be drawn.

EXERCISE 4.24. $f(t,x,y) = y$, $g(t,x,y) = -x - \alpha(x^2-1)y + \beta \sin \theta t$ and all their derivatives with respect to x and y are continuous for all t_0, x_0 and y_0 Thus, for all initial conditions there exists a unique solution pair on an interval containing t_0.

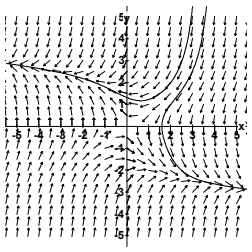
EXERCISE 4.32.



EXERCISE 4.34.

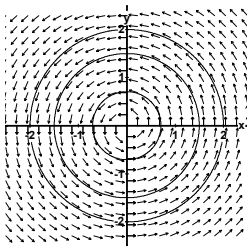


EXERCISE 4.36.

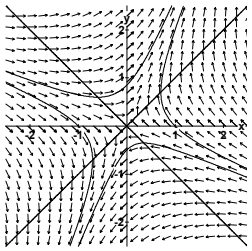


EXERCISE 4.38. (a) and (6), (b) and (2), (c) and (1), (d) and (3), (e) and (5), (f) and (4)

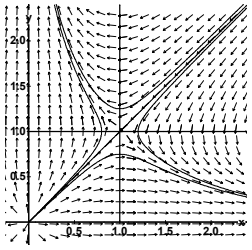
EXERCISE 4.39. The x-nullcline is given by the equation $-y = 0$, i.e., is the x-axis. The y-nullcline is given by the equation $x = 0$, i.e., is the y-axis.



EXERCISE 4.41. The x-nullcline is given by the equation $-x+y = 0$, i.e., is the straight line $y = x$ The y-nullcline is given by the equation $x+y = 0$, i.e., is the straight line $y = -x$.



EXERCISE 4.43. The x-nullcline is given by the equation $x(1 - y) = 0$, i.e., $x = 0$ (the y-axis) and $y = 1$ (a horizontal straight line). The y-nullcline is given by the equation $y(1 - x) = 0$, i.e., $y = 0$ (the x-axis) and $x = 1$ (a vertical straight line).



EXERCISE 4.49.

$$x' = -e^{-t} = -2e^{-t} + e^{-t} = -2x + y$$
$$y' = -e^{-t} = e^{-t} - 2e^{-t} = x - 2y$$

EXERCISE 4.51.

$$x' = -e^{-t} - 3e^{-3t} = -2\left(e^{-t} + e^{-3t}\right) + \left(e^{-t} - e^{-3t}\right) = -2x + y$$
$$y' = -e^{-t} + 3e^{-3t} = \left(e^{-t} + e^{-3t}\right) - 2\left(e^{-t} - e^{-3t}\right) = x - 2y$$

EXERCISE 4.53.

$$x' = -6e^{-3t} = 4\left(2e^{-3t}\right) - 2\left(7e^{-3t}\right) = 4x - 2y$$
$$y' = -21e^{-3t} = 7\left(2e^{-3t}\right) - 5\left(7e^{-3t}\right) = 7x - 5y$$

EXERCISE 4.55.

$$x' = -6e^{-3t} - 2e^{2t} = 4\left(2e^{-3t} - e^{2t}\right) - 2\left(7e^{-3t} - e^{2t}\right) = 4x - 2y$$
$$y' = -21e^{-3t} - 2e^{2t} = 7\left(2e^{-3t} - e^{2t}\right) - 5\left(7e^{-3t} - e^{2t}\right) = 7x - 5y$$

EXERCISE 4.57.

$$x' = -2\sin t - 2\cos t = y$$
$$y' = -2\cos t + 2\sin t = -\left(3 + 2\cos t - 2\sin t\right) + 3 = -x + 3$$

EXERCISE 4.59. $f(t,x,y) = \sin(x+y)$, $g(t,x,y) = \sin(x-y)$ are continuous and continuously differentiable functions for all t, x, and y. The fundamental existence and uniqueness theorem applies to any initial value problem, in particular for the case $x(0) = 1$, $y(0) = 0$. We conclude that there exists a unique solution on an interval containing $t_0 = 0$.

EXERCISE 4.61.

$$f(t,x,y) = \frac{1+x}{1-y}, \quad g(t,x,y) = \frac{1+y}{1-x}$$

are continuous and have continuous derivatives wherever their denominators do not vanish. Thus, we must avoid $x = 1$ and $t = 1$ in initial value problems if the fundamental existence and uniqueness theorem is to be applied. Since $y(0) = 1$ we can conclude nothing from the theorem.

EXERCISE 4.63.

$$f(t,x,y) = \frac{1+x}{1-y}, \quad g(t,x,y) = \frac{1+x}{1-y}$$

are continuous and have continuous derivatives where ever their denominators do not vanish. Thus, for $x_0 \neq 1$ and $y_0 \neq 1$ the fundamental existence and uniqueness theorem applies and for any such initial value problem we conclude there exists a unique solution on an interval containing t_0. For all other initial value problems the theorem does not apply and no conclusions.

EXERCISE 4.65. $f(t,x,y) = \sqrt{t-x-y}$, $g(t,x,y) = \sqrt{t+x+y}$ are continuous and have continuous derivatives at all points for which $t - x - y > 0$ and $t + x + y > 0$. For any initial conditions satisfying these inequalities the fundamental existence and uniqueness

theorem applies. We conclude that if $t_0 - x_0 - y_0 > 0$, $t_0 + x_0 + y_0 > 0$ then the initial value problem will have a unique solution on some interval containing t_0. For all other initial value problems the theorem does not apply and no conclusion can be drawn.

EXERCISE 4.67. Since $a(t) = \cos t$, $b(t) = -2$, $c(t) = 2$, $d(t) = -3\sin t$, and $h_1(t) = h_2(t) = 0$ are all continuous for all t, there exists a unique solution for any initial value problem and it exists for all t.

EXERCISE 4.69. $f(t, x, y) = x^2 + y$ and $g(t, x, y) = x - y$ are continuous for all t, x and y so all initial value problems have a unique solution defined on an interval containing t_0.

EXERCISE 4.71. Since $a(t) = 0$, $b(t) = 1$, $c(t) = -k/m$, $d(t) = 0$ and $h_1(t) = h_2(t) = 0$ in the equivalent first order system are continuous for all t, all initial value problems have unique solutions that exist for all t.

EXERCISE 4.73. Since $a(t) = 0$, $b(t) = 1$, $c(t) = -t^{-2}$, $d(t) = -t^{-1}$ and $h_1(t) = h_2(t) = 0$ in the equivalent first order system, any initial value problem for with $t_0 \neq 0$ will have a unique solution. For $t_0 > 0$ the solution exists for $t > 0$ and for $t_0 < 0$ the solution exists for $t < 0$.

EXERCISE 4.82. For Euler's Algorithm:

s	$x(1)$	$y(1)$
0.10000	2.908672	1.185711
0.05000	3.007559	1.224857
0.02500	3.061206	1.246410
0.01250	3.089191	1.257730
0.00625	3.103490	1.263533

The digit 3 for $x(1)$ and 1.2 for $y(1)$ appear to have stabilized.

A.6 Chapter 5: Linear Systems of First Order Equations

EXERCISE 5.1. linear homogeneous

EXERCISE 5.3. nonlinear (because of the term y^2)

EXERCISE 5.5. linear nonhomogeneous

EXERCISE 5.7. linear homogeneous

EXERCISE 5.9.

$$A(t) = \begin{pmatrix} 5 & -5 \\ -1 & -1 \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 5 & -5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -7 \end{pmatrix}$$

EXERCISE 5.11.

$$A(t) = \begin{pmatrix} c-4 & 3c \\ -1 & d-1 \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} t^2 \\ -t \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} c-4 & 3c \\ -1 & d-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t^2 \\ -t \end{pmatrix}$$

EXERCISE 5.13. The coefficient matrix $A(t)$ and the nonhomogeneous term $\tilde{q}(t)$ are continuous on $-\infty < t < +\infty$ and therefore the (unique) solution exists on $-\infty < t < +\infty$.
EXERCISE 5.15. The coefficient matrix $A(t)$ is continuous on each of the intervals $-\infty < t < -1, -1 < t < 1$ and $1 < t < +\infty$. Since $t_0 = 0$ lies in the interval $-1 < t < 1$, it is only this interval that is relevant. The nonhomogeneous term $\tilde{q}(t)$ is continuous on this interval as well and therefore the (unique) solution exists on $-1 < t < 1$.
EXERCISE 5.17. The coefficient matrix $A(t)$ is continuous on each of the intervals $0 < t < \sqrt{2}$ and $\sqrt{2} < t < +\infty$. Since $t_0 = 1$ lies in the interval $0 < t < \sqrt{2}$, it is only this interval that is relevant. The nonhomogeneous term $\tilde{q}(t)$ is also continuous on this interval (because its only singularities are at $t = 0$ and -1). Therefore the (unique) solution exists on $0 < t < \sqrt{2}$.
EXERCISE 5.20. (a) First solution:

$$\tilde{x}_1'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2t} & -\frac{1}{2} \\ -\frac{1}{2t^2} & \frac{1}{2t} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = A(t)\,\tilde{x}_1(t)$$

Second solution:

$$\tilde{x}_2'(t) = \begin{pmatrix} 2t \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2t} & -\frac{1}{2} \\ -\frac{1}{2t^2} & \frac{1}{2t} \end{pmatrix} \begin{pmatrix} t^2 \\ -t \end{pmatrix} = A(t)\,\tilde{x}_1(t)$$

(b) The two solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are independent on the interval $t > 0$ because the determinant

$$\det \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix} = -2t^2 \neq 0$$

is nonzero on the interval $t > 0$. They can therefore be used to construct a fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix}$$

and the general solution formula

$$\tilde{x}_h(t) = \Phi(t)\,\tilde{c} = \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 t + c_2 t^2 \\ c_1 - c_2 t \end{pmatrix}.$$

(c) Using

$$\Phi^{-1}(t) = \left(-\frac{1}{2t^2}\right) \begin{pmatrix} -t & -t^2 \\ -1 & t \end{pmatrix}$$

we have that

$$\begin{aligned} \tilde{x}_h(t) &= \Phi(t)\,\Phi^{-1}(1)\,\tilde{x}_0 \\ &= \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix} \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t & t^2 \\ 1 & -t \end{pmatrix} \left(-\frac{1}{2}\right) \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^2 + t \\ 1 - t \end{pmatrix}. \end{aligned}$$

EXERCISE 5.22.
(a) On the interval $-\infty < t < +\infty$, we have

$$\tilde{x}_1'(t) = \begin{pmatrix} 4e^{4t} \\ 8e^{4t} \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} e^{4t} \\ 2e^{4t} \end{pmatrix} = A(t)\,\tilde{x}_1(t)$$

$$\tilde{x}_2'(t) = \begin{pmatrix} 9e^{-3t} \\ -3e^{-3t} \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3e^{-3t} \\ e^{-3t} \end{pmatrix} = A(t)\,\tilde{x}_2(t)$$

(b) The solutions in (a) are independent because the determinant

$$\det \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix} = 7e^{t} \neq 0$$

is nonzero on the interval $-\infty < t < +\infty$. They can therefore be used to construct a fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix}$$

and obtain the formula

$$\tilde{x}_h(t) = \Phi(t)\,\tilde{c} = \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} - 3c_2 e^{-3t} \\ 2c_1 e^{4t} + c_2 e^{-3t} \end{pmatrix}$$

for the general solution.
(c)

$$\tilde{x}'(t) = \begin{pmatrix} 8e^{4t} - 9e^{-3t} \\ 16e^{4t} + 3e^{-3t} \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2e^{4t} + 3e^{-3t} \\ 4e^{4t} - e^{-3t} \end{pmatrix} = A(t)\,\tilde{x}(t)$$

(d) We need to find $\tilde{c}$ so that $\Phi(t)\,\tilde{c} = \tilde{x}(t)$, i.e.

$$\begin{aligned} \tilde{c} &= \Phi^{-1}(t)\,\tilde{x}(t) = \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix}^{-1} \begin{pmatrix} 2e^{4t} + 3e^{-3t} \\ 4e^{4t} - e^{-3t} \end{pmatrix} \\ &= \frac{1}{7} \begin{pmatrix} e^{-4t} & 3e^{-4t} \\ -2e^{3t} & e^{3t} \end{pmatrix} \begin{pmatrix} 2e^{4t} + 3e^{-3t} \\ 4e^{4t} - e^{-3t} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \end{aligned}$$

Thus

$$\tilde{x}(t) = \Phi(t)\,\tilde{c} = \begin{pmatrix} e^{4t} & -3e^{-3t} \\ 2e^{4t} & e^{-3t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\tilde{x}_1(t) - \tilde{x}_2(t).$$

EXERCISE 5.24.
(a) A fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} -2e^{-5t} & e^{5t} \\ e^{-5t} & 2e^{5t} \end{pmatrix}$$

Using

$$\Phi^{-1}(t) = \frac{1}{5} \begin{pmatrix} -2e^{5t} & e^{5t} \\ e^{-5t} & 2e^{-5t} \end{pmatrix}$$

we have

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}(0)\\ &= \begin{pmatrix} -2e^{-5t} & e^{5t} \\ e^{-5t} & 2e^{5t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}\\ &= \begin{pmatrix} \frac{3}{5}e^{5t} - \frac{8}{5}e^{-5t} \\ \frac{4}{5}e^{-5t} + \frac{6}{5}e^{5t} \end{pmatrix}.\end{aligned}$$

(b) A fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} -2e^{-5t} & e^{5t} \\ e^{-5t} & 2e^{5t} \end{pmatrix}$$

Using

$$\Phi^{-1}(t) = \frac{1}{5} \begin{pmatrix} -2e^{5t} & e^{5t} \\ e^{-5t} & 2e^{-5t} \end{pmatrix}$$

we have

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}(0)\\ &= \begin{pmatrix} -2e^{-5t} & e^{5t} \\ e^{-5t} & 2e^{5t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}\\ &= \begin{pmatrix} 2e^{-5t} + e^{5t} \\ 2e^{5t} - e^{-5t} \end{pmatrix}.\end{aligned}$$

EXERCISE 5.29. From Example 5.3 we have the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}.$$

(a)

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}_0\\ &= \frac{1}{5} \begin{pmatrix} 4e^{-t} + e^{-6t} & 2e^{-t} - 2e^{-6t} \\ 2e^{-t} - 2e^{-6t} & e^{-t} + 4e^{-6t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\\ &= \begin{pmatrix} \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t} \\ \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t} \end{pmatrix}\end{aligned}$$

(c)

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}_0\\ &= \frac{1}{5} \begin{pmatrix} 4e^{-t} + e^{-6t} & 2e^{-t} - 2e^{-6t} \\ 2e^{-t} - 2e^{-6t} & e^{-t} + 4e^{-6t} \end{pmatrix} \begin{pmatrix} 10 \\ -5 \end{pmatrix}\\ &= \begin{pmatrix} 6e^{-t} + 4e^{-6t} \\ 3e^{-t} - 8e^{-6t} \end{pmatrix}.\end{aligned}$$

EXERCISE 5.30. The coefficient matrix of the equivalent linear homogeneous system is

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From Example 5.4 we have the fundamental solution matrix and its inverse:

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad \Phi^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

To find solution formulas for each initial value problem we use the formula $\tilde{x}(t) = \Phi(t)\Phi^{-1}(t_0)\tilde{x}(t_0)$.

(a)

$$\begin{aligned}\tilde{x}(t) = \Phi(t)\Phi^{-1}(0)\tilde{x}(0) &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\cos t + \sin t \\ \sin t + \cos t \end{pmatrix}.\end{aligned}$$

(c)

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(\pi)\tilde{x}(\pi) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t - \sin t \\ -\cos t - \sin t \end{pmatrix}.\end{aligned}$$

EXERCISE 5.31. (a) A fundamental matrix is

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

So

$$\Phi^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and

$$\begin{aligned}\int \Phi^{-1}(t)\tilde{h}(t)dt &= \int \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} dt \\ &= \begin{pmatrix} \sin t - \cos t \\ -\cos t - \sin t \end{pmatrix}.\end{aligned}$$

Thus,

$$\Phi(t)\int \Phi^{-1}(t)\tilde{h}(t)dt = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \sin t - \cos t \\ -\cos t - \sin t \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

and the general solution

$$\tilde{x}(t) = \Phi(t)\tilde{c} + \Phi(t)\int \Phi^{-1}(t)\tilde{h}(t)dt$$

is

$$\begin{aligned}\tilde{x}(t) &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} c_1\cos t + c_2\sin t - 1 \\ -c_1\sin t + c_2\cos t - 1 \end{pmatrix}.\end{aligned}$$

(b)

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\tilde{h}(s)ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} ds \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} -\cos t + \sin t + 1 \\ -\cos t - \sin t + 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t + \sin t - 1 \\ -\sin t + \cos t - 1 \end{pmatrix}.\end{aligned}$$

EXERCISE 5.33. (a)

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\tilde{c} + \Phi(t)\int^t \Phi^{-1}(s)\tilde{q}(s)ds \\ &= \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\int^t \begin{pmatrix} e^s & 2e^{3s} \\ e^s & 3e^{3s} \end{pmatrix}^{-1}\begin{pmatrix} e^{-s} \\ 1 \end{pmatrix} ds \\ &= \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\int^t \begin{pmatrix} 3e^{-s} & -2e^{-s} \\ -e^{-3s} & e^{-3s} \end{pmatrix}\begin{pmatrix} e^{-s} \\ 1 \end{pmatrix} ds \\ &= \begin{pmatrix} c_1e^t + 2c_2e^{3t} \\ c_1e^t + 3c_2e^{3t} \end{pmatrix} + \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\int^t \begin{pmatrix} 3e^{-2s} - 2e^{-s} \\ e^{-3s} - e^{-4s} \end{pmatrix} ds\end{aligned}$$

$$\begin{aligned}&= \begin{pmatrix} c_1e^t + 2c_2e^{3t} \\ c_1e^t + 3c_2e^{3t} \end{pmatrix} + \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\begin{pmatrix} -\frac{3}{2}e^{-2t} + 2e^{-t} \\ \frac{1}{4}e^{-4t} - \frac{1}{3}e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} c_1e^t + 2c_2e^{3t} + \frac{4}{3} - e^{-t} \\ c_1e^t + 3c_2e^{3t} + 1 - \frac{3}{4}e^{-t} \end{pmatrix}\end{aligned}$$

(b)

$$\begin{aligned}\tilde{x}(t) &= \Phi(t)\Phi^{-1}(0)\tilde{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\tilde{q}(s)ds \\ &= \tilde{0} + \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}\int_0^t \begin{pmatrix} e^s & 2e^{3s} \\ e^s & 3e^{3s} \end{pmatrix}^{-1}\begin{pmatrix} e^{-s} \\ 1 \end{pmatrix} ds\end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{3}{2}e^{-2t} + 2e^{-t} - \frac{1}{2} \\ \frac{1}{4}e^{-4t} - \frac{1}{3}e^{-3t} + \frac{1}{12} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{6}e^{3t} + \frac{4}{3} - e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{4}e^{3t} + 1 - \frac{3}{4}e^{-t} \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} \tilde{x}\,(t) &= \Phi(t)\Phi^{-1}\,(0)\,\tilde{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\tilde{q}(s)ds \\ &= \begin{pmatrix} e^t & 2e^{3t} \\ e^t & 3e^{3t} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{6}e^{3t} + \frac{4}{3} - e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{4}e^{3t} + 1 - \frac{3}{4}e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} 3e^t - 2e^{3t} & 2e^{3t} - 2e^t \\ 3e^t - 3e^{3t} & 3e^{3t} - 2e^t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{6}e^{3t} + \frac{4}{3} - e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{4}e^{3t} + 1 - \frac{3}{4}e^{-t} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} 4e^t - 2e^{3t} \\ 4e^t - 3e^{3t} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}e^t + \frac{1}{6}e^{3t} + \frac{4}{3} - e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{4}e^{3t} + 1 - \frac{3}{4}e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{7}{2}e^t - \frac{11}{6}e^{3t} + \frac{4}{3} - e^{-t} \\ \frac{7}{2}e^t - \frac{11}{4}e^{3t} + 1 - \frac{3}{4}e^{-t} \end{pmatrix} \end{aligned}$$

EXERCISE 5.36. (a) The second order equation is equivalent to the system

$$\begin{aligned} x' &= y \\ y' &= -x + \tan t. \end{aligned}$$

From

$$\Phi(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}, \quad \Phi^{-1}(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}$$

we calculate

$$\begin{aligned} \int \Phi^{-1}(t)\tilde{h}(t)dt &= \int \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} 0 \\ \tan t \end{pmatrix} dt = \int \begin{pmatrix} \sin t \\ -\frac{\sin^2 t}{\cos t} \end{pmatrix} dt \\ &= \int \begin{pmatrix} \sin t \\ \frac{\cos^2 t - 1}{\cos t} \end{pmatrix} dt = \int \begin{pmatrix} \sin t \\ \cos t - \sec t \end{pmatrix} dt = \begin{pmatrix} -\cos t \\ \sin t + \frac{1}{2}\ln\left|\frac{1-\sin t}{1+\sin t}\right| \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \Phi(t)\int \Phi^{-1}(s)\tilde{h}(s)ds &= \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t + \frac{1}{2}\ln\left|\frac{1-\sin t}{1+\sin t}\right| \end{pmatrix} \\ &= \begin{pmatrix} -\sin t\cos t + \cos t\sin t + \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \\ -\cos^2 t - \sin^2 t + \frac{1}{2}(-\sin t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \\ -1 - \frac{1}{2}(\sin t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \end{pmatrix} \end{aligned}$$

and the general solution

$$\tilde{x} = \Phi(t)\tilde{c} + \Phi(t)\int \Phi^{-1}(s)\tilde{h}(s)ds$$

is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \\ -1-\frac{1}{2}(\sin t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \end{pmatrix}$$
$$= \begin{pmatrix} c_1\sin t + c_2\cos t + \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \\ c_1\cos t - c_2\sin t - 1 - \frac{1}{2}(\sin t)\ln\left|\frac{1-\sin t}{1+\sin t}\right| \end{pmatrix}.$$

The general solution of the second order differential equations is the first component

$$x(t) = c_1\sin t + c_2\cos t + \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right|.$$

(b) The solution formula calculated in Exercise **5.36.** The initial conditions yield $c_2 = 0, c_1 - 1 = 0$ or $c_1 = 1, c_2 = 0$ which yield the solution formula

$$x(t) = \sin t + \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right|.$$

(c) The solution formula calculated in Exercise **5.36**, together with the initial conditions, yield $c_2 = 1, c_1 - 1 = 0$ or $c_1 = 1, c_2 = 1$ which yields the solution formula

$$x = \sin t + \cos t + \frac{1}{2}(\cos t)\ln\left|\frac{1-\sin t}{1+\sin t}\right|.$$

EXERCISE 5.37.

(a)

$$\Phi^{-1}(t) = \begin{pmatrix} 2e^{\frac{1}{2}t} & -e^{\frac{1}{2}t} \\ -e^{-\frac{1}{2}t} & e^{-\frac{1}{2}t} \end{pmatrix}$$

and

$$\int \Phi^{-1}(t)\tilde{q}(t)dt = \int \begin{pmatrix} 2e^{\frac{1}{2}t} & -e^{\frac{1}{2}t} \\ -e^{-\frac{1}{2}t} & e^{-\frac{1}{2}t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} dt = \begin{pmatrix} -4e^{\frac{1}{2}t} \\ -3e^{-\frac{1}{2}t} \end{pmatrix}$$

so that

$$\Phi(t)\int \Phi^{-1}(t)\tilde{q}(t)dt = \begin{pmatrix} e^{-\frac{1}{2}t} & e^{\frac{1}{2}t} \\ e^{-\frac{1}{2}t} & 2e^{\frac{1}{2}t} \end{pmatrix} \begin{pmatrix} -4e^{\frac{1}{2}t} \\ -3e^{-\frac{1}{2}t} \end{pmatrix} = \begin{pmatrix} -7 \\ -10 \end{pmatrix}.$$

The general solution $\tilde{x} = \Phi(t)\tilde{c} + \Phi(t)\int \Phi^{-1}(s)\tilde{q}(s)ds$ is

$$\tilde{x}(t) = \begin{pmatrix} e^{-\frac{1}{2}t} & e^{\frac{1}{2}t} \\ e^{-\frac{1}{2}t} & 2e^{\frac{1}{2}t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -7 \\ -10 \end{pmatrix}$$
$$= \begin{pmatrix} c_1e^{-\frac{1}{2}t} + c_2e^{\frac{1}{2}t} - 7 \\ c_1e^{-\frac{1}{2}t} + 2c_2e^{\frac{1}{2}t} - 10 \end{pmatrix}$$

or

$$x(t) = c_1e^{-\frac{1}{2}t} + c_2e^{\frac{1}{2}t} - 7$$
$$y(t) = c_1e^{-\frac{1}{2}t} + 2c_2e^{\frac{1}{2}t} - 10.$$

(c)

$$\Phi^{-1}(t) = \begin{pmatrix} -\sin t & -\frac{1}{2}\cos t - \frac{3}{2}\sin t \\ \cos t & \frac{3}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix}$$

and

$$\begin{aligned}\int \Phi^{-1}(t)\tilde{q}(t)dt &= \int \begin{pmatrix} -\sin t & -\frac{1}{2}\cos t - \frac{3}{2}\sin t \\ \cos t & \frac{3}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} dt \\ &= \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix}\end{aligned}$$

so that

$$\Phi(t)\int \Phi^{-1}(t)\tilde{q}(t)dt =$$

$$\begin{pmatrix} 3\cos t - \sin t & \cos t + 3\sin t \\ -2\cos t & -2\sin t \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The general solution

$$\tilde{x} = \Phi(t)\tilde{c} + \Phi(t)\int \Phi^{-1}(s)\tilde{q}(s)ds$$

is

$$\begin{aligned}\tilde{x}(t) &= \begin{pmatrix} 3\cos t - \sin t & \cos t + 3\sin t \\ -2\cos t & -2\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1(3\cos t - \sin t) + c_2(\cos t + 3\sin t) - 1 \\ -2c_1\cos t - 2c_2\sin t + 1 \end{pmatrix}\end{aligned}$$

or

$$\begin{aligned} x(t) &= c_1(3\cos t - \sin t) + c_2(\cos t + 3\sin t) - 1 \\ y(t) &= -2c_1\cos t - 2c_2\sin t + 1. \end{aligned}$$

(e)

$$\Phi^{-1}(t) = \begin{pmatrix} -\frac{1}{2}e^{t} & -e^{t} \\ \frac{3}{2}e^{-t} & 2e^{-t} \end{pmatrix}$$

and

$$\int \Phi^{-1}(t)\tilde{q}(t)dt = \int \begin{pmatrix} -\frac{1}{2}e^{t} & -e^{t} \\ \frac{3}{2}e^{-t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{2}re^{t} \\ -\frac{3}{2}re^{-t} \end{pmatrix}$$

so that

$$\Phi(t)\int \Phi^{-1}(t)\tilde{q}(t)dt = \begin{pmatrix} 4e^{-t} & 2e^{t} \\ -3e^{-t} & -e^{t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}re^{t} \\ -\frac{3}{2}re^{-t} \end{pmatrix} = \begin{pmatrix} -5r \\ 3r \end{pmatrix}.$$

The general solution

$$\tilde{x} = \Phi(t)\tilde{c} + \Phi(t)\int \Phi^{-1}(s)\tilde{q}(s)ds$$

is

$$\begin{aligned}\tilde{x}(t) &= \begin{pmatrix} 4e^{-t} & 2e^{t} \\ -3e^{-t} & -e^{t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -5r \\ 3r \end{pmatrix} \\ &= \begin{pmatrix} 4c_1e^{-t} + 2c_2e^{t} - 5r \\ -3c_1e^{-t} - c_2e^{t} + 3r \end{pmatrix}\end{aligned}$$

or

$$\begin{aligned}x\,(t) &= 4c_1e^{-t} + 2c_2e^{t} - 5r \\ y\,(t) &= -3c_1e^{-t} - c_2e^{t} + 3r.\end{aligned}$$

EXERCISE 5.38. To solve the initial value problems, use the general solutions calculated in Exercises 5.37.

(a) The initial conditions yield the equations

$$\begin{aligned}c_1 + c_2 - 7 &= 1 \\ c_1 + 2c_2 - 10 &= -1\end{aligned}$$

to be solved for

$$c_1 = 7, \quad c_2 = 1.$$

A substitution of these into the general solution gives the solution formulas

$$\begin{aligned}x &= 7e^{-\frac{1}{2}t} + e^{\frac{1}{2}t} - 7 \\ y &= 7e^{-\frac{1}{2}t} + 2e^{\frac{1}{2}t} - 10.\end{aligned}$$

(c). The initial conditions yield the equations

$$\begin{aligned}3c_1 + c_2 - 1 &= 1 \\ -2c_1 + 1 &= -1\end{aligned}$$

to be solved for

$$c_1 = 1, \quad c_2 = -1.$$

A substitution of these into the general solution gives the solution formulas

$$\begin{aligned}x\,(t) &= 2\cos t - 4\sin t - 1 \\ y\,(t) &= -2\cos t + 2\sin t + 1.\end{aligned}$$

(e) The initial conditions yield the equations

$$\begin{aligned}4c_1 + 2c_2 - 5r &= 1 \\ -3c_1 - c_2 + 3r &= -1\end{aligned}$$

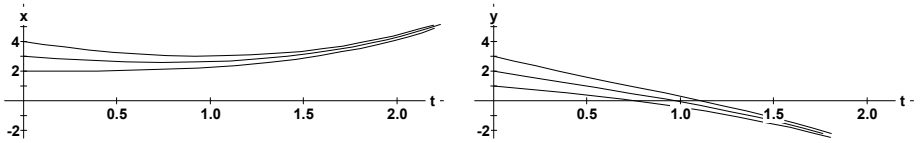
to be solved for

$$c_1 = \frac{1}{2}(1 - r), \quad c_2 = -\frac{1}{2}(3r + 1).$$

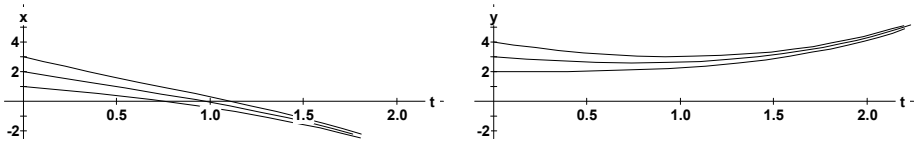
A substitution of these into the general solution gives the solution formulas

$$x(t) = 2(1-r)e^{-t} - (3r+1)e^{t} + 5r$$
$$y(t) = \frac{3}{2}(r-1)e^{-t} + \frac{1}{2}(3r+1)e^{t} - 3r.$$

EXERCISE 5.39. (a) The army with the smallest initial strength loses.



$c = d = 1$
$x(0) = 4, 3, 2$ and $y(0) = 3, 2, 1$ respectively



$c = d = 1$
$x(0) = 3, 2, 1$ and $y(0) = 4, 3, 2$ respectively

(b) Using

$$\Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{t} & e^{t} \end{pmatrix}$$

and the Variation of Constants Formula for initial value problems (5.21), we calculate

$$\tilde{x}(t) = \Phi(t)\Phi^{-1}(0)\tilde{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\tilde{q}(s)ds$$

$$\tilde{x}(t) = \begin{pmatrix} e^{t} & e^{-t} \\ -e^{t} & e^{-t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} e^{t} & e^{-t} \\ -e^{t} & e^{-t} \end{pmatrix}\int_0^t \frac{1}{2}\begin{pmatrix} e^{-s} & -e^{-s} \\ e^{s} & e^{s} \end{pmatrix}\begin{pmatrix} e^{-cs} \\ e^{-cs} \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{t} & e^{-t} \\ -e^{t} & e^{-t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} x_0 - y_0 \\ x_0 + y_0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} e^{t} & e^{-t} \\ -e^{t} & e^{-t} \end{pmatrix}\int_0^t \begin{pmatrix} 0 \\ 2e^{(1-c)s} \end{pmatrix} ds$$

$$= \frac{1}{2}\begin{pmatrix} (x_0 - y_0)e^{t} + (x_0 + y_0)e^{-t} \\ -(x_0 - y_0)e^{t} + (x_0 + y_0)e^{-t} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} e^{t} & e^{-t} \\ -e^{t} & e^{-t} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{2}{1-c}e^{(1-c)t} - \frac{2}{1-c} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (x_0 - y_0)e^{t} + (x_0 + y_0)e^{-t} \\ -(x_0 - y_0)e^{t} + (x_0 + y_0)e^{-t} \end{pmatrix} + \frac{1}{1-c}\begin{pmatrix} e^{-ct} - e^{-t} \\ e^{-ct} - e^{-t} \end{pmatrix}$$

or, component-wise

$$x(t) = \frac{1}{2}(x_0 - y_0)e^{t} + \frac{1}{2}(x_0 + y_0)e^{-t} + \frac{1}{1-c}\left(e^{-ct} - e^{-t}\right)$$
$$y(t) = \frac{1}{2}(-x_0 + y_0)e^{t} + \frac{1}{2}(x_0 + y_0)e^{-t} + \frac{1}{1-c}\left(e^{-ct} - e^{-t}\right).$$

This formula works if $c \neq 1$. If $c = 1$ then the integral is different:

$$\int_0^t \begin{pmatrix} 0 \\ 2e^{(1-c)s} \end{pmatrix} ds = \int_0^t \begin{pmatrix} 0 \\ 2 \end{pmatrix} ds = \begin{pmatrix} 0 \\ 2t \end{pmatrix}$$

and solution becomes

$$\begin{aligned} \tilde{x}(t) &= \begin{pmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} x_0 - y_0 \\ x_0 + y_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{pmatrix} \begin{pmatrix} 0 \\ 2t \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} x_0 - y_0 \\ x_0 + y_0 \end{pmatrix} + \begin{pmatrix} te^{-t} \\ te^{-t} \end{pmatrix} \end{aligned}$$

or, component-wise

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) = \frac{1}{2}(x_0 - y_0)e^t + \frac{1}{2}(x_0 + y_0)e^{-t} + te^{-t} \\ y(t) &= y_h(t) + y_p(t) = -\frac{1}{2}(x_0 - y_0)e^t + \frac{1}{2}(x_0 + y_0)e^{-t} + te^{-t}. \end{aligned}$$

(c) If $x_0 > y_0$ then using the formulas from (b) we see that $\lim_{t\to+\infty} x(t) = +\infty$ and $\lim_{t\to+\infty} y(t) = -\infty$. This means the y-army drops to 0 (loses) in finite time. The reason is that $y(t)$ starts positive $y_0 > 0$ at time $t = 0$ and the intermediate value theorem from calculus implies $y(t)$ must therefore equal 0 for some time $t > 0$. The situation is reversed if $y_0 > x_0$, in which case the x-army loses.

A.7 Chapter 6: Autonomous Linear Homogeneous Systems

EXERCISE 6.1.

(a) The eigenvalues of the coefficient matrix are $\lambda_1 = 2$, $\lambda_2 = -3$. The Putzer Formula (6.5) yields

$$\Phi(t) = \begin{pmatrix} \frac{7}{5}e^{2t} - \frac{2}{5}e^{-3t} & \frac{2}{5}e^{-3t} - \frac{2}{5}e^{2t} \\ \frac{7}{5}e^{2t} - \frac{7}{5}e^{-3t} & \frac{7}{5}e^{-3t} - \frac{2}{5}e^{2t} \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\tilde{x}(0)$ or

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \frac{7}{5}e^{2t} - \frac{2}{5}e^{-3t} & \frac{2}{5}e^{-3t} - \frac{2}{5}e^{2t} \\ \frac{7}{5}e^{2t} - \frac{7}{5}e^{-3t} & \frac{7}{5}e^{-3t} - \frac{2}{5}e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{9}{5}e^{2t} - \frac{4}{5}e^{-3t} \\ \frac{9}{5}e^{2t} - \frac{14}{5}e^{-3t} \end{pmatrix}. \end{aligned}$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\tilde{x}(0)$ or

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \frac{7}{5}e^{2t} - \frac{2}{5}e^{-3t} & \frac{2}{5}e^{-3t} - \frac{2}{5}e^{2t} \\ \frac{7}{5}e^{2t} - \frac{7}{5}e^{-3t} & \frac{7}{5}e^{-3t} - \frac{2}{5}e^{2t} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{8}{5}e^{2t} + \frac{2}{5}e^{-3t} \\ \frac{8}{5}e^{2t} + \frac{7}{5}e^{-3t} \end{pmatrix}. \end{aligned}$$

EXERCISE 6.3.

(a) The eigenvalues of the coefficient matrix are $\lambda = \frac{1}{2} \pm \frac{3}{2}i$. The Putzer Formula (6.5) yields

$$\Phi(t) = \begin{pmatrix} e^{\frac{1}{2}t}\cos\frac{3}{2}t & -e^{\frac{1}{2}t}\sin\frac{3}{2}t \\ e^{\frac{1}{2}t}\sin\frac{3}{2}t & e^{\frac{1}{2}t}\cos\frac{3}{2}t \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}t}\cos\frac{3}{2}t & -e^{\frac{1}{2}t}\sin\frac{3}{2}t \\ e^{\frac{1}{2}t}\sin\frac{3}{2}t & e^{\frac{1}{2}t}\cos\frac{3}{2}t \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}t}\cos\frac{3}{2}t + e^{\frac{1}{2}t}\sin\frac{3}{2}t \\ e^{\frac{1}{2}t}\sin\frac{3}{2}t - e^{\frac{1}{2}t}\cos\frac{3}{2}t \end{pmatrix}.$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}t}\cos\frac{3}{2}t & -e^{\frac{1}{2}t}\sin\frac{3}{2}t \\ e^{\frac{1}{2}t}\sin\frac{3}{2}t & e^{\frac{1}{2}t}\cos\frac{3}{2}t \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^{\frac{1}{2}t}\cos\frac{3}{2}t - 3e^{\frac{1}{2}t}\sin\frac{3}{2}t \\ 3e^{\frac{1}{2}t}\cos\frac{3}{2}t + 2e^{\frac{1}{2}t}\sin\frac{3}{2}t \end{pmatrix}.$$

EXERCISE 6.5.

(a) The eigenvalues of the coefficient matrix are $\lambda_1 = -2.77$, $\lambda_2 = -0.39$. The Putzer Formula (6.5) yields

$$\Phi(t) = \begin{pmatrix} 1.159\,6e^{-0.39t} - 0.159\,6e^{-2.77t} & 0.191\,98e^{-2.77t} - 0.191\,98e^{-0.39t} \\ 0.963\,80e^{-0.39t} - 0.963\,80e^{-2.77t} & 1.159\,6e^{-2.77t} - 0.159\,6e^{-0.39t} \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} 1.159\,6e^{-0.39t} - 0.159\,6e^{-2.77t} & 0.191\,98e^{-2.77t} - 0.191\,98e^{-0.39t} \\ 0.963\,80e^{-0.39t} - 0.963\,80e^{-2.77t} & 1.159\,6e^{-2.77t} - 0.159\,6e^{-0.39t} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1.351\,6e^{-0.39t} - 0.351\,58e^{-2.77t} \\ 1.123\,4e^{-0.39t} - 2.123\,4e^{-2.77t} \end{pmatrix}. \end{aligned}$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} 1.159\,6e^{-0.39t} - 0.1596e^{-2.77t} & 0.19198e^{-2.77t} - 0.19198e^{-0.39t} \\ 0.96380e^{-0.39t} - 0.96380e^{-2.77t} & 1.1596e^{-2.77t} - 0.1596e^{-0.39t} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 0.256\,74e^{-2.77t} + 1.743\,3e^{-0.39t} \\ 1.551\,2e^{-2.77t} + 1.448\,8e^{-0.39t} \end{pmatrix}. \end{aligned}$$

EXERCISE 6.7.

(a) The coefficient matrix has complex eigenvalues $\lambda_1 = 1 \pm i$. The Putzer Formula (6.6) yields

$$\Phi(t) = \begin{pmatrix} e^t(\cos t + \sin t) & e^t \sin t \\ -2e^t \sin t & e^t \cos t - e^t \sin t \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t(\cos t + \sin t) & e^t \sin t \\ -2e^t \sin t & e^t \cos t - e^t \sin t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^t \cos t \\ -e^t(\cos t + \sin t) \end{pmatrix}.$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t(\cos t + \sin t) & e^t \sin t \\ -2e^t \sin t & e^t \cos t - e^t \sin t \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^t \cos t + 5e^t \sin t \\ 3e^t \cos t - 7e^t \sin t \end{pmatrix}.$$

EXERCISE 6.9.

(a) The eigenvalues of the coefficient matrix are $\lambda_1 = 3$, $\lambda_2 = -2$. The Putzer Formula (6.5) yields

$$\Phi(t) = \begin{pmatrix} \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t} & \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \\ \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} + \frac{4}{5}e^{3t} \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t} & \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \\ \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} + \frac{4}{5}e^{3t} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}.$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t} & \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \\ \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} + \frac{4}{5}e^{3t} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{-2t} + e^{3t} \\ 4e^{3t} - e^{-2t} \end{pmatrix}.$$

EXERCISE 6.11.

(a) The eigenvalues of the coefficient matrix are $\lambda_1 = -6.1$, $\lambda_2 = -1.5$. The Putzer Formula (6.5) yields

$$\Phi(t) = \begin{pmatrix} 1.0101e^{-6.1t} - 0.0101e^{-1.5t} & -0.042736e^{-6.1t} + 0.042736e^{-1.5t} \\ 0.23952e^{-6.1t} - 0.23952e^{-1.5t} & 1.0101e^{-1.5t} - 0.0101e^{-6.1t} \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} 1.0101e^{-6.1t} - 0.0101e^{-1.5t} & -0.042736e^{-6.1t} + 0.042736e^{-1.5t} \\ 0.23952e^{-6.1t} - 0.23952e^{-1.5t} & 1.0101e^{-1.5t} - 0.0101e^{-6.1t} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1.052\,8e^{-6.1t} - 0.052\,8e^{-1.5t} \\ 0.249\,62e^{-6.1t} - 1.249\,6e^{-1.5t} \end{pmatrix}. \end{aligned}$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1.0101e^{-6.1t} - 0.0101e^{-1.5t} & -0.042736e^{-6.1t} + 0.042736e^{-1.5t} \\ 0.23952e^{-6.1t} - 0.23952e^{-1.5t} & 1.0101e^{-1.5t} - 0.0101e^{-6.1t} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0.108e^{-1.5t} + 1.892e^{-6.1t} \\ 2.551\,3e^{-1.5t} + 0.448\,7e^{-6.1t} \end{pmatrix}.$$

EXERCISE 6.13.

(a) The eigenvalues of the coefficient matrix are $\lambda = \pm 5i$. The Putzer Formula (6.5) yields

$$\Phi(t) = \begin{pmatrix} \cos 5t + \frac{1}{5}\sin 5t & \frac{13}{5}\sin 5t \\ -\frac{2}{5}\sin 5t & \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 5t + \frac{1}{5}\sin 5t & \frac{13}{5}\sin 5t \\ -\frac{2}{5}\sin 5t & \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \cos 5t - \frac{12}{5}\sin 5t \\ -\cos 5t - \frac{1}{5}\sin 5t \end{pmatrix}.$$

EXERCISE 6.15.

(a) The coefficient matrix has a double eigenvalue $\lambda_1 = 1$. The Putzer Formula (6.6) yields

$$\Phi(t) = \begin{pmatrix} e^t - \frac{3}{2}te^t & \frac{3}{4}te^t \\ -3te^t & e^t + \frac{3}{2}te^t \end{pmatrix}.$$

(b) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is $\tilde{x}(t) = \Phi(t)\,\tilde{x}(0)$ or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t - \frac{3}{2}te^t & \frac{3}{4}te^t \\ -3te^t & e^t + \frac{3}{2}te^t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^t - \frac{9}{4}te^t \\ -e^t - \frac{9}{2}te^t \end{pmatrix}.$$

(c) Since the fundamental solution matrix $\Phi(t)$ in (a) is normalized at $t = 0$, the solution of the initial value problem is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t - \frac{3}{2}te^t & \frac{3}{4}te^t \\ -3te^t & e^t + \frac{3}{2}te^t \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^t - \frac{3}{4}te^t \\ 3e^t - \frac{3}{2}te^t \end{pmatrix}.$$

EXERCISE 6.19. The eigen-pairs

$$\lambda_1 = 2, \quad \tilde{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -3, \quad \tilde{w} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

of the coefficient matrix $A = \begin{pmatrix} 4 & -2 \\ 7 & -5 \end{pmatrix}$ produce two exponential solutions $\tilde{v}e^{\lambda t}$ which for the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^{2t} & 2e^{-3t} \\ e^{2t} & 7e^{-3t} \end{pmatrix}.$$

EXERCISE 6.21. The complex eigen-pair

$$\lambda = \frac{1}{2} + \frac{3}{2}i, \quad \tilde{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

of the coefficient matrix

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

produce a complex exponential solution $\tilde{v}e^{\lambda t}$ whose real and imaginary parts form the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^{\frac{1}{2}t}\cos\frac{3}{2}t & e^{\frac{1}{2}t}\sin\frac{3}{2}t \\ e^{\frac{1}{2}t}\sin\frac{3}{2}t & -e^{\frac{1}{2}t}\cos\frac{3}{2}t \end{pmatrix}.$$

EXERCISE 6.23. The eigen-pairs

$$\lambda_1 = -2.7736, \quad \tilde{v} = \begin{pmatrix} 0.45 \\ 2.761\,6 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -0.388\,41, \quad \tilde{w} = \begin{pmatrix} 0.45 \\ 0.376\,41 \end{pmatrix}$$

of the coefficient matrix

$$A = \begin{pmatrix} -0.012 & -0.45 \\ 2.31 & -3.15 \end{pmatrix}$$

produce two exponential solutions $\tilde{v}e^{\lambda t}$ which for the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 0.45e^{-2.7736t} & 0.45e^{-0.388\,41t} \\ 2.761\,6e^{-2.7736t} & 0.376\,41e^{-0.388\,41t} \end{pmatrix}.$$

EXERCISE 6.25 The complex eigen-pair

$$\lambda_1 = 1 + i, \quad \tilde{v} = \begin{pmatrix} 1+i \\ -2 \end{pmatrix}$$

of the coefficient matrix

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$$

produces a complex exponential solution $\tilde{v}e^{\lambda t}$ whose real and imaginary parts form the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^t\cos t - e^t\sin t & e^t\cos t + e^t\sin t \\ -2e^t\cos t & -2e^t\sin t \end{pmatrix}.$$

EXERCISE 6.27. The eigen-pairs

$$\lambda_1 = -2, \quad \tilde{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 3, \quad \tilde{w} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

of the coefficient matrix

$$A = \begin{pmatrix} -1 & 1 \\ 4 & 2 \end{pmatrix}$$

produce two exponential solutions $\tilde{v}e^{\lambda t}$ which for the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^{-2t} & e^{3t} \\ -e^{-2t} & 4e^{3t} \end{pmatrix}.$$

EXERCISE 6.29. The eigen-pairs

$$\lambda_1 = -1.5483, \quad \tilde{v} = \begin{pmatrix} 0.2 \\ -4.5517 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -6.0517, \quad \tilde{w} = \begin{pmatrix} 0.2 \\ -0.0483 \end{pmatrix}$$

of the coefficient matrix

$$A = \begin{pmatrix} -6.1 & 0.2 \\ -1.1 & -1.5 \end{pmatrix}$$

produce two exponential solutions $\tilde{v}e^{\lambda t}$ which for the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 0.2e^{-1.5483t} & 0.2e^{-6.0517t} \\ -4.5517e^{-1.5483t} & -0.0483e^{-6.0517t} \end{pmatrix}.$$

EXERCISE 6.31. The complex eigen-pair

$$\lambda = 5i, \quad \tilde{v} = \begin{pmatrix} 1+5i \\ -2 \end{pmatrix}$$

of the coefficient matrix

$$A = \begin{pmatrix} 1 & 13 \\ -2 & -1 \end{pmatrix}$$

produce a complex exponential solution $\tilde{v}e^{\lambda t}$ whose real and imaginary parts form the columns of the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \cos 5t - 5\sin 5t & 5\cos 5t + \sin 5t \\ -2\cos 5t & -2\sin 5t \end{pmatrix}.$$

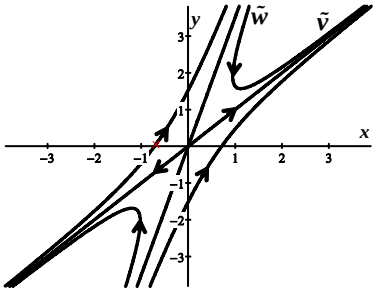
EXERCISE 6.33. The eigenvalues and some associated eigenvectors of the coefficient matrix

$$A = \begin{pmatrix} 4 & -2 \\ 7 & -5 \end{pmatrix}$$

are

$$\lambda_1 = 2, \quad \hat{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -3, \quad \hat{w} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

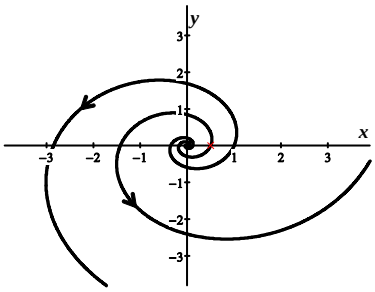


The phase portrait is a saddle (since one eigenvalue is positive and the other is negative) and orbits are asymptotic to $\hat{v}$ as $t \to +\infty$ and to $\hat{w}$ as $t \to -\infty$.

EXERCISE 6.35. The eigenvalues $\lambda = \frac{1}{2} \pm \frac{3}{2}i$ of the coefficient matrix

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

are complex with a positive real part $\alpha = 1/2$ and therefore the phase plane portrait is an unstable spiral. To determine the orientation of the spiral orbits we calculate that at the point $(x, y) = (1, 0)$ the direction field points in the direction of $(x', y') = \left(\frac{1}{2}, \frac{3}{2}\right)$ and therefore the orientation is counter clockwise.



EXERCISE 6.37. The eigenvalues and some associated eigenvectors of the coefficient matrix

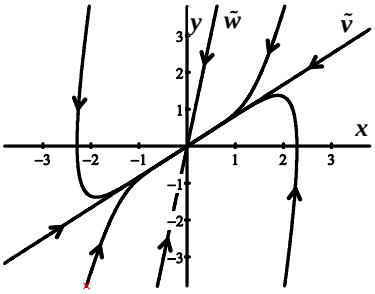
$$A = \begin{pmatrix} -0.012 & -0.450 \\ 2.310 & -3.150 \end{pmatrix}$$

are (to three decimals accuracy)

$$\lambda_1 = -0.388, \quad \hat{v} = \begin{pmatrix} 0.45 \\ 0.376 \end{pmatrix}$$

$$\lambda_2 = -2.774, \quad \hat{w} = \begin{pmatrix} 0.45 \\ 2.762 \end{pmatrix}.$$

The phase plane portrait is a stable node and orbits approach the origin tangentially to $\hat{v}$ as $t \to +\infty$.

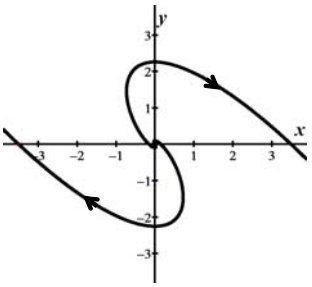


EXERCISE 6.39. The eigenvalues $\lambda = 1 \pm i$ of the coefficient matrix

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$$

are complex with positive real part 1.

The phase plane portrait is an unstable spiral. At the test point $(x, y) = (1, 0)$ in the phase plane, the direction field arrow col $(2, -2)$ points to the SE and therefore the sprial rotates clockwise.



EXERCISE 6.41. The eigenvalues and some associated eigenvectors of the coefficient matrix

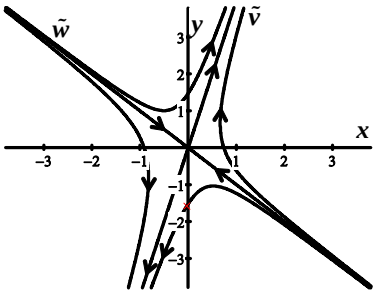
$$A = \begin{pmatrix} -1 & 1 \\ 4 & 2 \end{pmatrix}$$

are

$$\lambda_1 = 3, \quad \hat{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\lambda_2 = -2, \quad \hat{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The phase portrait is a saddle (since one eigenvalue is positive and the other is negative) and orbits are asymptotic to $\hat{v}$ as $t \to +\infty$ and to $\hat{w}$ as $t \to -\infty$.



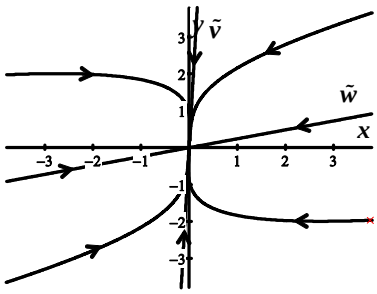
EXERCISE 6.43. The eigenvalues and some associated eigenvectors of the coefficient matrix

$$A = \begin{pmatrix} -6.1 & 0.2 \\ -1.1 & -1.5 \end{pmatrix}$$

are

$$\lambda_1 = -1.548, \quad \hat{v} = \begin{pmatrix} 0.2 \\ 4.552 \end{pmatrix}$$

$$\lambda_2 = -6.052, \quad \hat{w} = \begin{pmatrix} 0.2 \\ 0.048\,3 \end{pmatrix}.$$

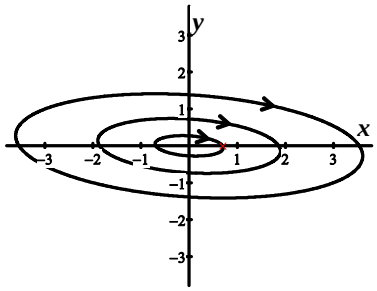


The phase plane portrait is a stable node and orbits approach the origin tangentially to $\hat{v}$ as $t \to +\infty$.

EXERCISE 6.45. The eigenvalues $\lambda = \pm 5i$ of the coefficient matrix

$$A = \begin{pmatrix} 1 & 13 \\ -2 & -1 \end{pmatrix},$$

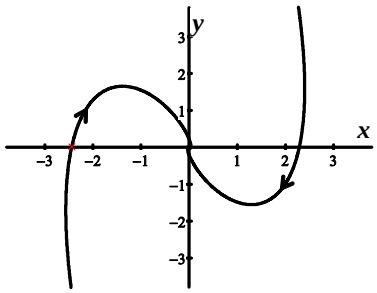
are complex with zero real part $\alpha = 0$ and therefore the phase plane portrait is center. To determine the orientation of the orbits we calculate that at the point $(x, y) = (1, 0)$ the direction field points in the direction of $(x', y') = (1, -2)$ and therefore the orientation is clockwise.



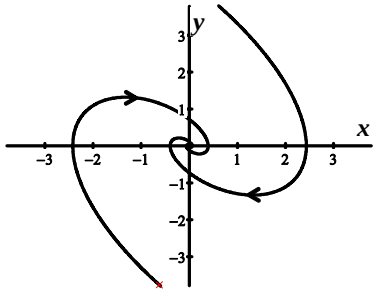
EXERCISE 6.47. The eigenvalues $\lambda = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$ of the coefficient matrix

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{3}{4} \\ -3 & -\frac{5}{2} \end{pmatrix}$$

are complex with a negative real part $\alpha = -3/2$ and therefore the phase plane portrait is a stable spiral. To determine the orientation of the spiral orbits we calculate that at the point $(x, y) = (1, 0)$ the direction field points in the direction of $(x', y') = (-1/2, -3)$ and therefore the orientation is clockwise.



EXERCISE 6.49. The roots $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ of the characteristic polynomial $\lambda^2 + \lambda + 1$ are complex with a negative real part $\alpha = -\frac{1}{2}$ and therefore the phase plane portrait is a stable spiral. To determine the orientation of the spiral orbits we calculate that at the point $(x, y) = (1, 0)$ the direction field points in the direction of $(x', y') = (y, -x - y) = (0, -1)$ and therefore the orientation is clockwise.



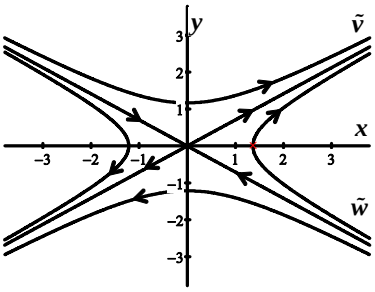
EXERCISE 6.51. The roots $\lambda = \pm\frac{\sqrt{2}}{2}$ of the characteristic polynomial $2\lambda^2 - 1$ imply the phase plane portrait is a saddle. Eigenvectors of the coefficient matrix for the equivalent linear system

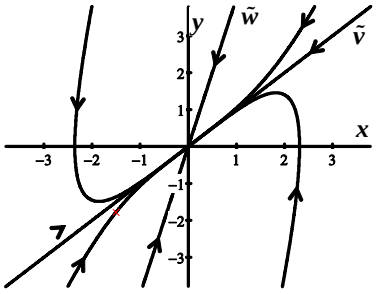
$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

associated with these eigenvalues are

$$\lambda_1 = \tfrac{\sqrt{2}}{2},\ \hat{v} = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \text{ and } \lambda_2 = -\tfrac{\sqrt{2}}{2},\ \hat{w} = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Orbits are asymptotic to $\hat{v}$ as $t \to +\infty$ and to $\hat{w}$ as $t \to -\infty$.





EXERCISE 6.53. The roots $\lambda = 1, 4$ of the characteristic polynomial $\lambda^2 - 5\lambda + 4$ imply the phase plane portrait is an unstable node. Eigenvectors of the coefficient matrix for the equivalent linear system

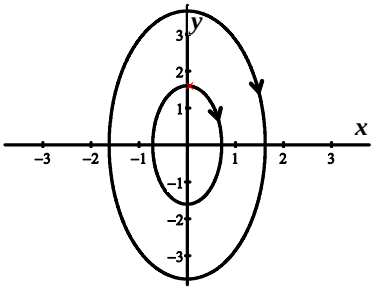
$$A = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix}$$

associated with these eigenvalues are

$$\lambda_1 = 1,\ \hat{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \lambda_2 = 4,\ \hat{w} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Orbits are asymptotic to $\hat{v}$ as $t \to +\infty$.

EXERCISE 6.56. The roots $\lambda = \pm\sqrt{5}i$ of the characteristic polynomial $\lambda^2 + 5$ imply the phase plane portrait is a center (hence neutrally stable). To determine the orientation of the orbits we calculate that at the point $(x, y) = (1, 0)$ the direction field points in the direction of $(x', y') = (y, -5x) = (0, -5)$ and therefore the orientation is clockwise.



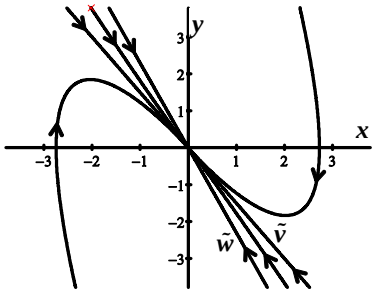
EXERCISE 6.57. The roots $\lambda = -1.5, -2.3$ of the characteristic polynomial $\lambda^2 + 3.8\lambda + 3.45$ imply the phase plane portrait is a stable node. Eigenvectors of the coefficient matrix for the equivalent linear system

$$A = \begin{pmatrix} 0 & 1 \\ -3.45 & -3.8 \end{pmatrix}$$

associated with these eigenvalues are

$$\lambda_1 = -1.5,\ \hat{v} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} \text{ and } \lambda_2 = -2.3,\ \hat{w} = \begin{pmatrix} 1 \\ -2.3 \end{pmatrix}.$$

Orbits approach the origin tangentially to $\hat{v}$ as $t \to +\infty$.



EXERCISE 6.64. The matrix

$$A = \begin{pmatrix} 3 & -6 & -2 \\ 2 & 3 & 2 \\ -2 & 6 & 3 \end{pmatrix}$$

has eigen-pairs

$$\lambda_1 = 3, \quad \tilde{v} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}; \quad \lambda_2 = 1, \quad \tilde{v} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}; \quad \lambda_3 = 5, \quad \tilde{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

which yield the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} 3e^{3t} & e^{t} & -e^{5t} \\ e^{3t} & e^{t} & 0 \\ -3e^{3t} & -2e^{t} & e^{5t} \end{pmatrix}.$$

EXERCISE 6.66. The matrix

$$A = \begin{pmatrix} 7 & 4 & 6 \\ -5 & -3 & -4 \\ -5 & -2 & -5 \end{pmatrix}$$

has eigenvalues and eigenvectors

$$\lambda_1 = -1, \quad \tilde{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}; \quad \lambda_2 = i, \quad \tilde{v} = \begin{pmatrix} -7 - i \\ 5 \\ 5 \end{pmatrix}$$

which yield the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} -2e^{-t} & 7\cos t - \sin t & \cos t + 7\sin t \\ e^{-t} & -5\cos t & -5\sin t \\ 2e^{-t} & -5\cos t & -5\sin t \end{pmatrix}.$$

EXERCISE 6.68. The matrix

$$A = \begin{pmatrix} -5 & 6 & -3 & -2 \\ -5 & 6 & -1 & -2 \\ 1 & -1 & 2 & 0 \\ -5 & 5 & 0 & -1 \end{pmatrix}$$

has eigenvalues and eigenvectors

$$\lambda_1 = 1, \quad \tilde{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \lambda_2 = -1, \quad \tilde{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \quad \lambda = 1 + i, \quad \tilde{v} = \begin{pmatrix} -i \\ 1 - 2i \\ 1 \\ 1 - 3i \end{pmatrix}$$

which yield the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} & e^t \sin t & -e^t \cos t \\ e^t & e^{-t} & e^t \cos t + 2e^t \sin t & -2e^t \cos t + e^t \sin t \\ 0 & 0 & e^t \cos t & e^t \sin t \\ 0 & e^{-t} & e^t \cos t + 3e^t \sin t & -3e^t \cos t + e^t \sin t \end{pmatrix}.$$

EXERCISE 6.70. For each initial value problem the solution is

$$\widetilde{x}(t) = \Phi(t)\Phi(0)^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Use the fundamental solution matrix $\Phi(t)$ from Exercises 6.64-6.67.

Exercise 6.64

$$\widetilde{x}(t) = \begin{pmatrix} e^{5t} \\ 0 \\ -e^{5t} \end{pmatrix}$$

Exercise 6.66

$$\widetilde{x}(t) = \begin{pmatrix} -\cos t + 3\sin t + 2e^{-t} \\ -2\sin t - e^{-t} + \cos t \\ -2\sin t - 2e^{-t} + \cos t \end{pmatrix}$$

EXERCISE 6.71. For each initial value problem the solution is

$$\widetilde{x}(t) = \Phi(t)\Phi(0)^{-1} \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix}.$$

Use the fundamental solution matrix $\Phi(t)$ from Exercises 6.68 and 6.69.

*Exercise*6.68

$$\widetilde{x}(t) = \begin{pmatrix} 6e^t - 8e^{-t} + 3e^t \cos t - 2e^t \sin t \\ 6e^t - 8e^{-t} + 4e^t \cos t - 7e^t \sin t \\ -3e^t \sin t - 2e^t \cos t \\ -8e^{-t} + 7e^t \cos t - 9e^t \sin t \end{pmatrix}$$

EXERCISE 6.73. A calculation shows $\mathrm{tr}A = -1$ and $\det A = 4$. By Theorem 6.3(a) the phase plane portrait is *stable*. Since phase plane portrait lies in the 2nd (NW) quadrant of the map and $\det A = 4 > \frac{1}{4} = \frac{1}{4}(\mathrm{tr}A)^2$, it follows that the phase plane portrait is a *stable spiral*.

EXERCISE 6.75. A calculation shows $\mathrm{tr}A = 13/7$ and $\det A = 20/7$. By Theorem 6.3(c) the phase plane portrait is *unstable*. Since the phase plane portrait lies in the 1st (NE) quadrant of the map and $\det A = 20/7 > 169/196 = (\mathrm{tr}A)^2/4$, it follows that the phase plane portrait is an *unstable spiral*.

EXERCISE 6.77. A calculation shows $\mathrm{tr}A = 1/12$ and $\det A = -1/10$. By Theorem 6.3(b) the phase plane portrait is *unstable.* Since the phase plane portrait lies in the lower half plane of the map, it follows that the phase plane portrait is a *saddle*.

EXERCISE 6.79. A calculation shows $\mathrm{tr}A = -7$ and $\det A = 12$. By Theorem 6.3(a) the phase plane portrait is *stable.* Since the phase plane portrait lies in the 2nd (NW) quadrant of the map and $\det A = 12 < 49/4 = (\mathrm{tr}A)^2/4$, it follows that the phase plane portrait is an *stable node.*

EXERCISE 6.81. A calculation shows $\mathrm{tr}\ A = -1$ and $\det A = -27$. By Theorem 6.3(b) the phase plane portrait is *unstable.* Since the phase plane portrait lies in the lower half plane of the map, it follows that the phase plane portrait is a *saddle.*

EXERCISE 6.83. A calculation shows $\mathrm{tr}\ A = p - 1$ and $\det A = -2p$. If $p < 0$ then $\mathrm{tr}\ A < 0$ and $\det A > 0$ and the phase portrait is stable by Theorem 6.3(a). If $p = 0$ then $\det A = 0$ and Theorem 6.3 is inapplicable. If $0 < p$ then $\det A < 0$ and the phase portrait is unstable by Theorem 6.3(b).

EXERCISE 6.85. A calculation shows $\mathrm{tr}\ A = 2(1+p)$ and $\det A = 1 + p^2$. If $p < -1$ then $\mathrm{tr}\ A < 0$ and $\det A > 0$ and the phase portrait is stable by Theorem 6.3(a). If $p = -1$ then $\mathrm{tr}\ A = 0$ and Theorem 6.3 is inapplicable. If $-1 < p$ then $\mathrm{tr}\ A > 0$ and $\det A > 0$ the phase portrait is unstable by Theorem 6.3(c).

EXERCISE 6.87. A calculation shows $\mathrm{tr}\ A = 2p^2 - 11$ and $\det A = -11p^2$. If $p \neq 0$ then $\det A < 0$ and the phase portrait is unstable by Theorem 6.3(b). If $p = 0$ then $\det A = 0$ and Theorem 6.3 is inapplicable.

EXERCISE 6.89. A calculation shows $\mathrm{tr}\ A = -1$ and $\det A = p^2/4$. For $p = 0$, $\det A = 0$ and the phase plane portrait is unclassified. For $p \neq 0$, $\det A > 0$ and the portrait lies in the upper half plane of the tr-det map. The portrait lies below the parabola $\det A = (\mathrm{tr}A)^2/4$ and is therefore a node if $p^2/4 < 1/4$ or $p^2 < 1$ or $-1 < p < 1$. It is a stable node because $\mathrm{tr}\ A = -1$. On the other hand, if $p^2 > 1$ then the portrait lies above the parabola and is therefore a spiral (stable because $\mathrm{tr}\ A = -1$). Thus, the portrait is a stable spiral if $p < -1$ or if $p > 1$. Finally, if $p = \pm 1$ the portrait lies on the parabola and is a stable degenerate node.

EXERCISE 6.91. A calculation shows $\mathrm{tr}\ A = 2p$ and $\det A = p$.

If $p < 0$, then $\det A < 0$ and the phase portrait lies in the lower half plane of the map and is therefore a saddle.

If $p > 0$, then $\det A > 0$ and $\mathrm{tr}\ A > 0$ and the phase portrait is in the first (NE) quadrant of the map. We need to determine when the point $(trA, \det A)$ lies above or below the parabola $\det A = (\mathrm{tr}A)^2/4$. The point lies above the parabola if $p > (2p)^2/4 = p^2$ or $0 < p < 1$. In this case the phase portrait is an unstable spiral. The point lies below the parabola if $p < (2p)^2/4 = p^2$ or in other words if $1 < p$. In this case the phase portrait is an unstable node

Finally, suppose $p = 0$, the phase plane portrait lies on the parabola and is therefore an unstable degenerate node.

If $p = 0$, $\det A = 0$ and the phase plane portrait is unclassified.

EXERCISE 6.92. Each equation can be solved independently.

$$x(t) = c_1 e^{-5t}$$
$$y(t) = c_2 e^{-5t}$$

EXERCISE 6.94. The coefficient matrix has a complex eigenvalue root $\lambda = -1 + i\sqrt{3}$

with an eigenvector

$$\tilde{v} = \begin{pmatrix} 1 \\ -2 - \sqrt{3}i \end{pmatrix}.$$

From a linear combination of the real and imaginary parts of the resulting complex solution $\tilde{v}e^{\lambda t}$ we obtain a fundamental solution matrix from which we in turn get the general solution formula:

$$\begin{aligned} x(t) &= c_1 e^{-t} \cos\sqrt{3}t + c_2 e^{-t} \sin\sqrt{3}t \\ y(t) &= c_1 e^{-t} \left(-2\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t\right) \\ &\quad -c_2 e^{-t} \left(\sqrt{3}\cos\sqrt{3}t + 2\sin\sqrt{3}t\right) \end{aligned}$$

EXERCISE 6.96. The coefficient matrix has a complex eigenvalue $\lambda = 5.98i$ with an eigenvector

$$\tilde{v} = \begin{pmatrix} -0.59 - 0.41i \\ i \end{pmatrix}.$$

From a linear combination of the real and imaginary parts of the resulting complex solution $\tilde{v}e^{\lambda t}$ we obtain a fundamental solution matrix from which we in turn get the general solution formula:

$$\begin{aligned} x(t) &= c_1 \left(-0.59\cos 5.98t + 0.41\sin 5.98t\right) \\ &\quad +c_2 \left(-0.41\cos 5.98t - 0.59\sin 5.98t\right) \\ y(t) &= -c_1 \sin 5.98t + c_2 \cos 5.98t \end{aligned}$$

EXERCISE 6.98. The coefficient matrix has a complex eigenvalue $\lambda = -0.39 + 1.53i$ with an eigenvector

$$\tilde{v} = \begin{pmatrix} -0.31 + 0.08i \\ i \end{pmatrix}.$$

From a linear combination of the real and imaginary parts of the resulting complex solution $\tilde{v}e^{\lambda t}$ we obtain a fundamental solution matrix from which we in turn get the general solution formula:

$$\begin{aligned} x(t) &= c_1 e^{-0.39t} \left(-0.31\cos 1.53t - 0.08\sin 1.53t\right) \\ &\quad +c_2 e^{-0.39t} \left(0.08\cos 1.53t - 0.31\sin 1.53t\right) \\ y(t) &= -c_1 e^{-0.39t} \sin 1.53t + c_2 e^{-0.39t} \cos 1.53t. \end{aligned}$$

EXERCISE 6.100. The coefficient matrix has a double eigenvalue $\lambda = e^{-a}$. From the Putzer Formula (6.6) we obtain the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^{\lambda t} + 2\frac{\lambda^2}{3\lambda - 2} te^{\lambda t} & -2\frac{\lambda^2}{3\lambda - 2} te^{\lambda t} \\ 2\frac{\lambda}{3\lambda - 2} te^{\lambda t} & e^{\lambda t} - 2\frac{\lambda}{3\lambda - 2} te^{\lambda t} \end{pmatrix}$$

and, in turn, the general solution

$$\begin{aligned} x(t) &= c_1 e^{\lambda t} + 2(c_1 - c_2)\frac{\lambda^2}{3\lambda - 2} te^{\lambda t} \\ y(t) &= c_2 e^{\lambda t} + 2(c_1 - c_2)\frac{\lambda}{3\lambda - 2} te^{\lambda t} \end{aligned}$$

EXERCISE 6.102. The coefficient matrix has a complex eigenvalue $\lambda = -0.71 + 2.79i$ with an eigenvector

$$\tilde{v} = \begin{pmatrix} 0.85 \\ -0.18 + 0.49i \end{pmatrix}.$$

From a linear combination of the real and imaginary parts of the resulting complex solution we obtain the general solution formula:

$$\begin{aligned} x(t) &= 0.85c_1 e^{-0.71t}\cos 2.79t + 0.85c_2 e^{-0.71t}\sin 2.79t \\ y(t) &= c_1\left(-0.18e^{-0.71t}\cos 2.79t - 0.49e^{-0.71t}\sin 2.79t\right) \\ &\quad + c_2\left(0.49e^{-0.71t}\cos 2.79t - 0.18e^{-0.71t}\sin 2.79t\right) \end{aligned}$$

EXERCISE 6.104. These exercises use the general solutions calculated in Exercises **6.92-6.103**. The given initial conditions yield two linear algebraic equations for the arbitrary constants c_1 and c_2. Numerical values for these constants are found by solving these algebraic equations.

Exercise **6.92**

$$\begin{aligned} x(t) &= e^{-5t} \\ y(t) &= -e^{-5t} \end{aligned}$$

Exercise 6.94

$$\begin{aligned} x(t) &= e^{-t}\cos\sqrt{3}t - \tfrac{\sqrt{3}}{3}e^{-t}\sin\sqrt{3}t \\ y(t) &= -e^{-t}\cos\sqrt{3}t + \tfrac{5\sqrt{3}}{3}e^{-t}\sin\sqrt{3}t \end{aligned}$$

Exercise 6.96

$$\begin{aligned} x(t) &= \cos 5.98t + 0.18\sin 5.98t \\ y(t) &= -\cos 5.98t - \sin 5.98t \end{aligned}$$

Exercise 6.98

$$\begin{aligned} x(t) &= e^{-0.39t}\cos 1.53t + 0.59e^{-0.39t}\sin 1.53t \\ y(t) &= -e^{-0.39t}\cos 1.53t + 3.48e^{-0.39t}\sin 1.53t \end{aligned}$$

Exercise 6.100

$$\begin{aligned} x(t) &= 2e^{\lambda t} + 2(2-3\lambda)\lambda t e^{\lambda t} \\ y(t) &= 3e^{t\lambda} + 2(2-3\lambda)te^{t\lambda} \end{aligned}$$

Exercise 6.102

$$\begin{aligned} x(t) &= e^{-1.47t}\cos 2.79t - 15.04e^{-1.47t}\sin 2.79t \\ y(t) &= -0.30e^{-1.47t}\sin 2.79t - e^{-1.47t}\cos 2.79t \end{aligned}$$

EXERCISE 6.105. These exercises use the general solutions calculated in Exercises **6.92-6.103**. The given initial conditions yield two linear algebraic equations for the arbitrary constants c_1 and c_2. Numerical values for these constants are found by solving these algebraic equations.

Exercise 6.92

$$\begin{aligned} x(t) &= 2e^{-5t} \\ y(t) &= 3e^{-5t} \end{aligned}$$

Exercise 6.94

$$\begin{aligned} x(t) &= 2e^{-t}\cos\sqrt{3}t - \tfrac{7}{3}e^{-t}\sqrt{3}\sin\sqrt{3}t \\ y(t) &= \tfrac{20}{3}e^{-t}\sqrt{3}\sin\sqrt{3}t + 3e^{-t}\cos\sqrt{3}t \end{aligned}$$

Exercise 6.96

$$\begin{aligned} x &\approx -3.98\sin 5.98t + 2\cos 5.98t \\ y &\approx 3\cos 5.98t + 5.44\sin 5.98t \end{aligned}$$

Exercise 6.98

$$x(t) = 2e^{-0.39t}\cos 1.53t - 0.47e^{-0.39t}\sin 1.53t$$
$$y(t) = 3e^{-0.39t}\cos 1.53t + 5.76e^{-0.39t}\sin 1.53t$$

Exercise 6.100

$$x(t) = 2e^{t\lambda} + 2(2-3\lambda)\lambda te^{t\lambda}$$
$$y(t) = 3e^{t\lambda} + 2(2-3\lambda)te^{t\lambda}$$

Exercise 6.102

$$x(t) = 5.89e^{-0.71t}\sin 2.79t + 2e^{-0.71t}\cos 2.79t$$
$$y(t) = -2.42e^{-0.71t}\sin 2.79t + 3e^{-0.71t}\cos 2.79t$$

EXERCISE 6.106.

(a) The characteristic polynomial $L\lambda^2 + R\lambda + 1/C$ has roots

$$\lambda_1 = \left(-RC + \sqrt{R^2C^2 - 4LC}\right)/2LC$$
$$\lambda_2 = \left(-RC - \sqrt{R^2C^2 - 4LC}\right)/2LC$$

and the general solution is

$$x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}.$$

Both eigenvalues are negative so the phase plane portrait is a stable node.

EXERCISE 6.110. Using the eigenvalue-eigenvector method, we calculate a fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} e^{-t} & -e^{t} & 2e^{2t} \\ e^{-t} & 2e^{t} & 0 \\ e^{-t} & e^{t} & e^{2t} \end{pmatrix}$$

of the associated homogeneous system. Then

$$\Phi^{-1}(t) = \begin{pmatrix} 2e^{t} & 3e^{t} & -4e^{t} \\ -e^{-t} & -e^{-t} & 2e^{-t} \\ -e^{-2t} & -2e^{-2t} & 3e^{-2t} \end{pmatrix}$$

and

$$\int \Phi^{-1}(t)\widetilde{h}(t)dt = \int \begin{pmatrix} 2e^{t} & 3e^{t} & -4e^{t} \\ -e^{-t} & -e^{-t} & 2e^{-t} \\ -e^{-2t} & -2e^{-2t} & 3e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} dt = \begin{pmatrix} -2e^{t} \\ -e^{-t} \\ -e^{-2t} \end{pmatrix}$$

so that

$$\Phi(t)\int \Phi^{-1}(t)\widetilde{h}(t)dt = \begin{pmatrix} e^{-t} & -e^{t} & 2e^{2t} \\ e^{-t} & 2e^{t} & 0 \\ e^{-t} & e^{t} & e^{2t} \end{pmatrix} \begin{pmatrix} -2e^{t} \\ -e^{-t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ -4 \end{pmatrix}.$$

the general solution

$$\tilde{x} = \Phi(t)\tilde{c} + \Phi(t)\int \Phi^{-1}(s)\widetilde{h}(s)ds$$

is

$$\widetilde{x}(t) = \begin{pmatrix} e^{-t} & -e^{t} & 2e^{2t} \\ e^{-t} & 2e^{t} & 0 \\ e^{-t} & e^{t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ -4 \end{pmatrix} = \begin{pmatrix} c_1e^{-t} - c_2e^{t} + 2c_3e^{2t} - 3 \\ c_1e^{-t} + 2c_2e^{t} - 4 \\ c_1e^{-t} + c_2e^{t} + c_3e^{2t} - 4 \end{pmatrix}.$$

A.8 Chapter 7: 2nd Order Linear Differential Equations

EXERCISE *7.1* The coefficients $c_2(t) = 1$, $c_1(t) = 0$, and $c_0(t) = 1$ and the nonhomogeneous term $q(t) = \sin t$ are continuous for all values of t and the leading coefficient $c_2(t) = 1$ is never equal to 0 for all values of t. Therefore, Theorem 7.1 applies for any initial conditions t_0, x_0 and y_0 and, as a result, any initial value problem has a unique solution that exists for all t.

EXERCISE *7.3* The coefficients $c_2(t) = t^2$, $c_1(t) = t$, and $c_0(t) = 1$ and the nonhomogeneous term $q(t) = 0$ are continuous for all values of t. The leading coefficient $c_2(t) = t^2$ equals 0 only at $t = 0$. Therefore, Theorem 7.1 applies for any initial conditions $t_0 \neq 0$, x_0 and y_0. If $t_0 > 0$ then the initial value problem has a unique solution that exists for all $t > 0$. If $t_0 < 0$ then the initial value problem has a unique solution that exists for all $t < 0$.

EXERCISE *7.5* The coefficients $c_2(t) = 1$, $c_1(t) = \alpha$, and $c_0(t) = 1$ and the nonhomogeneous term $q(t) = \beta \sin \theta t$ are continuous for all values of t and the leading coefficient $c_2(t) = 1$ is never equal to 0 for all values of t. Therefore, Theorem 7.1 applies for any initial conditions t_0, x_0 and y_0 and, as a result, any initial value problem has a unique solution that exists for all t.

EXERCISE 7.7. The roots of the characteristic polynomial $\lambda^2+\lambda+1$ are $\lambda = \left(-1 \pm \sqrt{3}i\right)/2$. By Table 8.1 the general solution is

$$x(t) = e^{-\frac{1}{2}t}\left(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t\right).$$

EXERCISE 7.9. The roots of the characteristic polynomial $2\lambda^2 - 1$ are $\lambda = \pm\sqrt{2}/2$. By Table 8.1 the general solution is

$$x(t) = c_1 e^{\sqrt{2}t/2} + c_2 e^{-\sqrt{2}t/2}.$$

EXERCISE 7.11. The roots of the characteristic polynomial $\lambda^2 + 3\lambda - 4$ are $\lambda_1 = 1$ and $\lambda_2 = -4$. By Table 8.1 the general solution is

$$x(t) = c_1 e^t + c_2 e^{-4t}.$$

EXERCISE 7.13. The roots of the characteristic polynomial $\lambda^2 + 5$ are $\lambda = \pm i\sqrt{5}$. By Table 8.1 the general solution is

$$x(t) = c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t.$$

EXERCISE 7.15. The roots of the characteristic polynomial $\lambda^2 - 6\lambda + 9$ are $\lambda_1 = \lambda_2 = 3$. By Table 8.1 the general solution is

$$x = c_1 e^{3t} + c_2 t e^{3t}.$$

EXERCISE 7.21. A substitution of $x = t^m$ into the differential equations yields

$$m(m-1)t^m - 2mt^m + 2t^m = 0$$

or $(m-1)(m-2)t^m = 0$. Thus, $m = 1$ or $m = 2$ and we obtain the two independent solutions $x_1(t) = t$ and $x_2(t) = t^2$.

EXERCISE 7.25. Since the nonhomogeneous term is a multiple of $e^{-t}\sin t$ we look for a solution of the form $x_p(t) = k_1 e^{-t}\cos t + k_2 e^{-t}\sin t$. A substitution into the differential equation yields $k_1 = 2/5$, $k_2 = 1/5$ and

$$x_p(t) = \frac{2}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t.$$

EXERCISE 7.27. Since the nonhomogeneous term is a multiple of e^t we would normally look for a solution of the form $x_p(t) = ke^t$. However e^t is a solution of the associated homogeneous equation and therefore we look for a solution of the form $x_p(t) = kte^t$. A substitution into the differential equation yields $k = 1/2$ and

$$x_p(t) = \frac{1}{2}te^t.$$

EXERCISE 7.29.

Exercise 7.25. The general solution of the associated homogeneous equation is $x_h(t) = c_1\sin t + c_2\cos t$. Therefore, the general solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= c_1\sin t + c_2\cos t + \frac{2}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t. \end{aligned}$$

The initial conditions yield the equations

$$c_2 + \frac{2}{5} = 0, \quad c_1 - \frac{1}{5} = 0$$

to solve for $c_1 = 1/5, c_2 = -2/5$. These yield the solution formula

$$x(t) = \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{2}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t.$$

Exercise 7.27. The general solution of the associated homogeneous equation is $x_h(t) = c_1e^{-t} + c_2e^t$. Therefore, the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1e^{-t} + c_2e^t + \frac{1}{2}te^t.$$

The initial conditions yield the equations

$$c_1 + c_2 = 0, \quad -c_1 + c_2 + \frac{1}{2} = 0$$

to solve for $c_1 = 1/4, c_2 = -1/4$. These yield the solution formula

$$x(t) = \frac{1}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t.$$

EXERCISE 7.33. The equilibrium solution is $x_p(t) = CE_0 = 10^{-5}E_0$. The roots of the characteristic polynomial are $\lambda_1 = -500$ and $\lambda_2 = -2000$. Therefore, the general solution of the associated homogeneous equation is $x_h(t) = c_1e^{-500t} + c_2e^{-2000t}$ and a formula for the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1e^{-500t} + c_2e^{-2000t} + 10^{-5}E_0.$$

The initial conditions yield the equations

$$\begin{aligned} c_1 + c_2 + 10^{-5}E_0 &= 0 \\ -500c_1 - 2000c_2 &= 0 \end{aligned}$$

to be solved for

$$c_1 = -\frac{1}{75} \times 10^{-3}, \quad c_2 = \frac{1}{3} \times 10^{-5}$$

which give the solution formula

$$x(t) = -\left(\frac{1}{75} \times 10^{-3}\right) E_0e^{-500t} + \left(\frac{1}{3} \times 10^{-5}\right) E_0e^{-2000t} + 10^{-5}E_0.$$

EXERCISE 7.35. The equilibrium solution is $x_p(t) = CE_0 = 10^{-5}E_0$. The roots of the characteristic polynomial are $\lambda = \pm 1000i$. Therefore, the general solution of the associated homogeneous equation is $x_h(t) = c_1 \cos 1000t + c_2 \sin 1000t$ and a formula for the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1 \cos 1000t + c_2 \sin 1000t + 10^{-5}E_0.$$

The initial conditions yield the equations

$$\begin{aligned} c_1 + 10^{-5}E_0 &= 0 \\ 1000c_2 &= 0 \end{aligned}$$

to be solved for $c_1 = -10^{-5}E_0, c_2 = 0$ which give the solution formula $x(t) = -10^{-5}E_0 \cos 1000t + 10^{-5}E_0$.

In Exercises 7.37-7.40 the associated homogeneous equation has general solution $x_h(t) = c_1e^{-t} + c_2te^{-t}$. This is because the characteristic polynomial $\lambda^2 + 2\lambda + 1$ has a repeated root $\lambda = -1$.

EXERCISE 7.37. Using the Method of Undetermined Coefficients we look for a solution of the form $x_p(t) = k_1 \sin t + k_2 \cos t$. A substitution into the nonhomogeneous equation yields $k_1 = 0$, $k_2 = -1/2$ and $x_p(t) = -\frac{1}{2} \cos t$. A formula for the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1e^{-t} + c_2te^{-t} - \frac{1}{2} \cos t.$$

The initial conditions yield the equations

$$\begin{aligned} c_1 - \frac{1}{2} &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}$$

to be solved for $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ which yield the solution formula $x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{2}\cos t$.

EXERCISE 7.39. Using the Method of Undetermined Coefficients we would look for a solution of the form $x_p(t) = ke^{-t}$. However, e^{-t} and te^{-t} are solutions of the associated homogeneous equation and therefore we look for a solution of the form $x_p(t) = kt^2e^{-t}$. A substitution into the nonhomogeneous equation yields $k = 1/2$ and $x_p(t) = t^2e^{-t}/2$. A formula for the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}.$$

The initial conditions yield the equations

$$c_1 = 0, \quad -c_1 + c_2 = 0$$

to be solved for $c_1 = 0, c_2 = 0$ which yield the solution formula $x(t) = t^2e^{-t}/2$.

EXERCISE 7.41. For the equivalent first order system of the equation we have

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}, \quad \tilde{q}(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

and using the Variation of Constants formula

$$\tilde{x}_p(t) = \Phi(t) \int^t \Phi^{-1}(u)\, \tilde{q}(u)\, du$$

we have

$$\begin{aligned}
\begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix} &= \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \int^t \begin{pmatrix} e^u & e^{-u} \\ e^u & -e^{-u} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^u \end{pmatrix} du \\
&= \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \int^t \frac{1}{2} \begin{pmatrix} e^{-u} & e^{-u} \\ e^u & -e^u \end{pmatrix} \begin{pmatrix} 0 \\ e^u \end{pmatrix} du \\
&= \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \int^t \frac{1}{2} \begin{pmatrix} 1 \\ -e^{2u} \end{pmatrix} du \\
&= \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} t \\ -\frac{1}{2}e^{2t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}te^t - \frac{1}{4}e^t \\ \frac{1}{4}e^t + \frac{1}{2}te^t \end{pmatrix}.
\end{aligned}$$

Hence $x_p(t) = \frac{1}{2}te^t - \frac{1}{4}e^t$ and $x(t) = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t - \frac{1}{4}e^t$. More concisely, we can combine the two e^t terms and re-label $c_1 - 1/4$ as c_1 to get

$$x(t) = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t.$$

EXERCISE 7.43. Using the formula

$$x(t) = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t$$

from Exercise 7.41 and the initial conditions, we obtain the equations

$$c_1 + c_2 = 2 \quad \text{and} \quad c_1 - c_2 + \frac{1}{2} = -2$$

to be solved for $c_1 = -1/4, c_2 = 9/4$, which yield the solution formula

$$x(t) = -\frac{1}{4}e^t + \frac{9}{4}e^{-t} + \frac{1}{2}te^t.$$

EXERCISE 7.45. This autonomous equation has the equilibrium solution $x_p(t) = -2$. Since the general solution of the associated homogeneous equation is $x_h(t) = c_1e^{-3t} + c_2e^t$, we have the general solution

$$x(t) = c_1e^{-3t} + c_2e^t - 2.$$

EXERCISE 7.47. Using the Method of Undetermined Coefficients, find a solution in the form $x_p(t) = k_1 \sin t + k_2 \cos t$. The results is

$$x_p(t) = \left(-\frac{2}{5}\right)\sin t + \left(-\frac{1}{5}\right)\cos t.$$

Since the general solution of the associated homogeneous equation is $x_h(t) = c_1e^{-3t} + c_2e^t$, we have the general solution

$$x(t) = c_1e^{-3t} + c_2e^t + \left(-\frac{2}{5}\right)\sin t + \left(-\frac{1}{5}\right)\cos t.$$

EXERCISE 7.49. Using the Method of Undetermined Coefficients, find a solution in the form $x_p(t) = k_1 + k_2t$. The results is

$$x_p(t) = -\frac{2}{9}k - \frac{1}{3}kt.$$

Since the general solution of the associated homogeneous equation is $x_h(t) = c_1e^{-3t} + c_2e^t$, we have the general solution

$$x(t) = c_1e^{-3t} + c_2e^t - \frac{2}{9}k - \frac{1}{3}kt.$$

EXERCISE *7.51* The characteristic polynomial $\lambda^2 + 2\lambda + 2$ has roots $\lambda = -1 \pm i$. By Table 8.1 the $x_h(t) = c_1e^{-t}\cos t + c_2e^{-t}\sin t$. By the Method of Undetermined Coefficients we guess $x_p(t) = k_1 \cos t + k_2 \sin t$, which when substituted into the differential equation, gives

$$(k_1 + 2k_2)\cos t + (-2k_1 + k_2)\sin t = \cos t$$

and hence the equations $k_1 + 2k_2 = 1$, $-2k_1 + k_2 = 0$. Thus $k_1 = 1/5$, $k_2 = 2/5$ and

$$x_p(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t$$

which gives the general solution

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t.$$

EXERCISE *7.53* The characteristic polynomial $\lambda^2 + 2\lambda + 2$ has roots $\lambda = -1 \pm i$. By Table 8.1 the $x_h(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$. By the Method of Undetermined Coefficients we guess $x_p(t) = k_1\cos t + k_2\sin t + k_3 t e^{-t}\cos t + k_4 t e^{-t}\sin t$, which when substituted into the differential equation, gives

$$(k_1 + 2k_2)\cos t + (-2k_1 + k_2)\sin t + 2k_4 e^{-t}\cos t - 2k_3 e^{-t}\sin t = 2\cos t - e^{-t}\sin t$$

and hence the equations $k_1 + 2k_2 = 2$, $-2k_1 + k_2 = 0$, $2k_4 = 0$, $-2k_3 = -1$. Thus $k_1 = 2/5$, $k_2 = 4/5$, $k_3 = 1/2$, $k_4 = 0$ and

$$x_p(t) = \frac{2}{5}\cos t + \frac{4}{5}\sin t + \frac{1}{2}te^{-t}\cos t$$

which gives the general solution

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{2}{6}\cos t + \frac{4}{5}\sin t + \frac{1}{2}te^{-t}\cos t.$$

EXERCISE *7.55* The characteristic polynomial $\lambda^2 + 6\lambda + 5 = (\lambda + 5)(\lambda + 1)$ has roots $\lambda_1 = -1$ and $\lambda_2 = -5$. By Table 8.1 the $x_h(t) = c_1 e^{-t} + c_2 e^{-5t}$. By the Method of Undetermined Coefficients we guess

$$x_p(t) = \begin{cases} ke^{at} & \text{if } a \neq -1 \text{ or } -5 \\ kte^{at} & \text{if } a = -1 \text{ or } -5 \end{cases}$$

which when substituted into the differential equation, gives

$$\begin{aligned} k(a+5)(a+1)e^{at} &= e^{at} \quad \text{if } a \neq -1 \text{ or } -5 \\ 2k(a+3)e^{at} &= e^{at} \quad \text{if } a = -1 \text{ or } -5 \end{aligned}$$

and hence

$$\begin{aligned} k &= \tfrac{1}{(a+5)(a+1)} \quad \text{if } a \neq -1 \text{ or } -5 \\ k &= \tfrac{1}{2(a+3)} \quad \text{if } a = -1 \text{ or } -5 \end{aligned}$$

$$x_p(t) = \begin{cases} \frac{1}{(a+5)(a+1)}e^{at} & \text{if } a \neq -1 \text{ or } -5 \\ \frac{1}{2(a+3)}te^{at} & \text{if } a = -1 \text{ or } -5 \end{cases}$$

which gives the general solution

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= \begin{cases} c_1 e^{-t} + c_2 e^{-5t} + \frac{1}{(a+5)(a+1)}e^{at} & \text{if } a \neq -1 \text{ or } -5 \\ c_1 e^{-t} + c_2 e^{-5t} + \frac{1}{2(a+3)}te^{at} & \text{if } a = -1 \text{ or } -5 \end{cases} \end{aligned}$$

EXERCISE *7.57* The characteristic polynomial $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$ has double root $\lambda = -2$. By Table 8.1 the $x_h(t) = c_1 e^{-2t} + c_2 te^{-2t}$. By the Method of Undetermined

Coefficients we guess $x_p(t) = k_1 + k_2 t$ which when substituted into the differential equation, gives

$$(4k_1 + 4k_2) + 4k_2 t = t$$

and hence $4k_1 + 4k_2 = 0$ and $4k_2 = 1$. Thus $k_1 = -1/4$, $k_2 = 1/4$ and

$$x_p(t) = -\frac{1}{4} + \frac{1}{4}t$$

which gives the general solution

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-2t} + c_2 t e^{-2t} - \frac{1}{4} + \frac{1}{4}t.$$

EXERCISE *7.59* The characteristic polynomial $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$ has double root $\lambda = -2$. By Table 8.1 the $x_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. By the Method of Undetermined Coefficients we guess $x_p(t) = k_1 + k_2 t + k_3 e^{-t} + k_4 t^2 e^{-2t}$ which when substituted into the differential equation, gives

$$(4k_1 + 4k_2) + 4k_2 t + k_3 e^{-t} + 2k_4 e^{-2t} = 3t - e^{-t} + 2e^{-2t}$$

and hence $4k_1 + 4k_2 = 0$, $4k_2 = 3$, $k_3 = -1$ and $2k_4 = 2$. Thus $k_1 = -3/4$, $k_2 = 3/4$, $k_3 = -1$, $k_4 = 1$ and

$$x_p(t) = -\frac{3}{4} + \frac{3}{4}t - e^{-t} + t^2 e^{-2t}$$

which gives the general solution

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-2t} + c_2 t e^{-2t} - \frac{3}{4} + \frac{3}{4}t - e^{-t} + t^2 e^{-2t}.$$

EXERCISE *7.61* The characteristic polynomial $\lambda^2 + k^2$ has roots $\lambda = \pm ik$. By Table 8.1 the $x_h(t) = c_1 \cos kt + c_2 \sin kt$. By the Method of Undetermined Coefficients we guess

$$x_p(t) = \begin{cases} k_1 \sin t + k_2 \cos t & \text{if } k \neq 1 \\ k_1 t \sin t + k_2 t \cos t & \text{if } k = 1 \end{cases}$$

which when substituted into the differential equation, gives

$$\begin{aligned} k_1 (k^2 - 1) \sin t + k_2 (k^2 - 1) \cos t = \sin t & \quad \text{if } k \neq 1 \\ -2k_2 \sin t + 2k_1 \cos t = \sin t & \quad \text{if } k = 1 \end{aligned}$$

and hence

$$\begin{aligned} k_1 = 1/(k^2 - 1),\ k_2 = 0 & \quad \text{if } k \neq 1 \\ k_1 = 0,\ k_2 = -1/2 & \quad \text{if } k = 1 \end{aligned}$$

$$x_p(t) = \begin{cases} \frac{1}{k^2 - 1} \cos t & \text{if } k \neq 1 \\ -\frac{1}{2} t \cos t & \text{if } k = 1 \end{cases}$$

which gives the general solution

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= \begin{cases} c_1 \cos kt + c_2 \sin kt + \frac{1}{k^2 - 1} \sin t & \text{if } k \neq 1 \\ c_1 \cos kt + c_2 \sin kt + -\frac{1}{2} t \cos t & \text{if } k = 1 \end{cases} \end{aligned}$$

A.9 Chapter 8: Nonlinear Systems

EXERCISE 8.1.

(a) From the second of the two equilibrium equations

$$\begin{aligned} x - e^{-y} &= 0 \\ x - y &= 0 \end{aligned}$$

we have $y = x$, which when used in the first equilibrium equation yields the equation $x - e^{-x} = 0$ for x. Using a computer or calculator we obtain the solution $x(t) = x_e \approx 0.5671$. Thus, the only equilibrium point is $\tilde{x}_e \approx \text{col}(0.5671, 0.5671)$.

(c) From the second of the two equilibrium equations

$$\begin{aligned} \ln\left(\frac{1}{1+2x^2}\right) - y &= 0 \\ -3x - 4y &= 0 \end{aligned}$$

we have $y = -3x/4$, and hence from the first equilibrium equation we get the equation

$$\ln\left(\frac{1}{1+2x^2}\right) + \frac{3}{4}x = 0$$

for x. One roots is $x(t) = 0$. A plot of the left hand side indicates there are two other roots, which we find from a computer to be $x \approx 0.4452$ and 5.4848. Thus, the equilibria are

$$\begin{aligned} &\tilde{x}_e = \text{col}(0, 0) \\ &\tilde{x}_e \approx \text{col}\,(0.4452, -0.3339) \quad \text{and} \quad \text{col}\,(5.4848, -4.1136)\,. \end{aligned}$$

(e) Using a computer or calculator to solve the equilibrium equation $xe^{-x} = 1/4$ we find $x_e \approx 0.3574$ and 2.1533.

EXERCISE 8.2.

(a) The first equilibrium equation $x^2 + y^2 - r^2 = 0$ is a circle of radius r centered at the origin. The second equilibrium equation $(x-3)^2 + y^2 - 4 = 0$ is a circle of radius 2 centered at the point $(x, y) = (3, 0)$. These two circles intersect at two points if $1 < r < 5$ and at one point if $r = 1$ or $r = 5$. They do not intersect for other values of r. Therefore, there are no equilibria if $r < 1$ or $r > 5$, one equilibrium if $r = 1$ or 5, and two equilibria if $1 < r < 5$.

(c) The first equilibrium equation $rx - y = 0$ is a straight line with slope $r > 0$ passing through the origin. The second equilibrium equation $6x + y - 8x^2 + 2x^3 = 0$, or $y = -2x\,(x-1)\,(x-3)$, is a cubic polynomial passing through the origin. Graphs of the cubic and the line show, besides the intersection point at the origin, two additional intersection points if $r < 2$ and no other intersection point if $r > 2$. Therefore, there are three equilibria if $r < 2$ and one equilibrium if $r > 2$. When $r = 2$ the line is tangent to the cubic and there are exactly two equilibria.

Another approach is to solve the first equilibrium equation for $y = rx$ and substitute this answer into the second equilibrium equation. The result is the polynomial equation

$$\begin{aligned} 6x + rx - 8x^2 + 2x^3 &= 0 \\ x\left(6 + r - 8x + 2x^2\right) &= 0 \end{aligned}$$

whose roots are $= 0$ and $x = 2 \pm \sqrt{2}\sqrt{2-r}/2$.

(e) The equilibrium equation is $xe^{-x} = r$. From a graph of xe^{-x} (whose maximum e^{-1} occurs at $x = 1$) we see that if $r > e^{-1}$ there are no equilibria, if $r < e^{-1}$ there are two equilibria (one less than $x = 1$ and the other greater than $x = 1$) and if $r = e^{-1}$ there is exactly one equilibrium ($x = 1$).

EXERCISE 8.4.

(a) The equilibrium equations are

$$x - y^2 = 0$$
$$x - y = 0.$$

The second equation implies $y = x$ which, when substituted into the first equation, yields $x - x^2 = 0$. Thus, $x = 0$ and $x = 1$. The equilibria are

$$\tilde{x}_e = \text{col}(0,0) \quad \text{and} \quad \text{col}(1,1).$$

(b) The Jacobian

$$J(x,y) = \begin{pmatrix} 1 & -2y \\ 1 & -1 \end{pmatrix}$$

evaluated at the equilibria gives

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad J(1,1) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

(c) One of the eigenvalues 1 and -1 of $J(0,0)$ is positive and therefore the equilibrium $\tilde{x}_e = \text{col}(0,0)$ is unstable. The eigenvalues of $J(1,1)$ are $\pm i$. Theorem 8.1 does not apply and no conclusion can be drawn from it.

(d) $\text{tr}J(0,0) = 0$ and $\text{tr}J(1,1) = 0$. Therefore Theorem 8.2 does not apply to either equilibrium and no conclusion can be drawn from it.

EXERCISE 8.6.

(a) The equilibrium equations are

$$x(1 - x - y) = 0$$
$$y(2 - x - 4y) = 0.$$

Consider the first equation. There are two alternatives: $x = 0$ or $1 - x - y = 0$, which we consider one at a time.

If $x = 0$, the second equation becomes

$$y(2 - 4y) = 0$$

which has two solutions $y = 0$ and $y = 1/2$. Thus, the first alternative $x = 0$ yields two equilibria $\tilde{x}_e = \text{col}(0,0)$ and $\text{col}(0,1/2)$.

The second alternative is $1 - x - y = 0$ or $x = 1 - y$. Using this in the second equation, we obtain

$$y(1 - 3y) = 0$$

which has two solutions $y = 0$ and $y = 1/3$. The first gives $x = 1$ and the second gives $x = 2/3$. Thus, the second alternative $1 - x - y = 0$ yields two equilibrium $\tilde{x}_e = \text{col}\,(1, 0)$ and $\text{col}\,(2/3, 1/3)$.

In summary, we have equilibria

$$\tilde{x}_e = \text{col}\,(0, 0)\,, \quad \text{col}\left(0, \frac{1}{2}\right), \quad \text{col}\,(1, 0)\,, \quad \text{col}\left(\frac{2}{3}, \frac{1}{3}\right).$$

(b) The Jacobian

$$J(x, y) = \begin{pmatrix} 1 - 2x - y & -x \\ -y & 2 - 8y - x \end{pmatrix}$$

evaluated at these equilibria gives

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad J\left(0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & -2 \end{pmatrix}$$

$$J(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad J\left(\frac{2}{3}, \frac{1}{3}\right) = \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{4}{3} \end{pmatrix}.$$

(c) The eigenvalues 1 and 2 *of* $J\,(0, 0)$ are positive and therefore the equilibrium $\tilde{x}_e = \text{col}(0, 0)$ is unstable. One of the eigenvalues $1/2$ and -2 of $J\,(0, 1/2)$ is positive and therefore the equilibrium $\tilde{x}_e = \text{col}(0, 1/2)$ is unstable. One of the eigenvalues -1 and 1 of $J\,(1, 0)$ is positive and therefore the equilibrium $\tilde{x}_e = \text{col}(1, 0)$ is unstable. Both eigenvalues $-1 \pm \sqrt{3}/3$ of $J\,(2/3, 1/3)$ are negative and therefore the equilibrium $\tilde{x}_e = \text{col}\,(2/3, 1/3)$ is stable.

(d) $\text{tr}J(0, 0) = 3 > 0$ and $\det J\,(0, 0) = 2 > 0$ imply the equilibrium $\tilde{x}_e = \text{col}\,(0, 0)$ is unstable. $\text{tr}J(0, 1/2) = -2/3 < 0$ and $\det J\,(0, 1/2) = -1 < 0$ imply the equilibrium $\tilde{x}_e = \text{col}\,(0, 1/2)$ is unstable. $\text{tr}J(1, 0) = 0$ and $\det J\,(1, 0) = -1 < 0$ imply Theorem 8.2 is not applicable and nothing about the equilibrium $\tilde{x}_e = \text{col}\,(1, 0)$ can be drawn from it. $\text{tr}J\,(2/3, 1/3) = -2 < 0$ and $\det J\,(2/3, 1/3) = 2/3 > 0$ imply the equilibrium $\tilde{x}_e = \text{col}\,(2/3, 1/3)$ is stable.

EXERCISE 8.8

(a) The equilibrium equations are

$$\begin{aligned} 1 - x^2 - y^2 &= 0 \\ x - y &= 0. \end{aligned}$$

The second equation implies $y = x$ which, when substituted into the first equation, yields

$$1 - 2x^2 = 0$$

or $x = \pm\sqrt{2}/2$. The equilibria are

$$\tilde{x}_e = \text{col}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad \text{col}\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

(b) The Jacobian

$$J(x, y) = \begin{pmatrix} -2x & -2y \\ 1 & -1 \end{pmatrix}$$

evaluated at these equilibria gives

$$J\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=\begin{pmatrix}-\sqrt{2} & -\sqrt{2}\\ 1 & -1\end{pmatrix}$$

$$J\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ 1 & -1\end{pmatrix}.$$

(c) The eigenvalues $-1.2071 \pm 1.171i$ of $J\left(\sqrt{2}/2,\sqrt{2}/2\right)$ have negative real parts and therefore the equilibrium $\tilde{x}_e = \text{col}\left(\sqrt{2}/2,\sqrt{2}/2\right)$ is stable. One of the eigenvalues 1.9016 and -1.4874 of $J\left(-\sqrt{2}/2,-\sqrt{2}/2\right)$ is positive and therefore the equilibrium $\tilde{x}_e = \text{col}\left(-\sqrt{2}/2,-\sqrt{2}/2\right)$ is unstable.

(d) $\text{tr}J\left(\sqrt{2}/2,\sqrt{2}/2\right) = -1-\sqrt{2} < 0$ and $\det J\left(\sqrt{2}/2,\sqrt{2}/2\right) = 2\sqrt{2} > 0$ imply the equilibrium $\tilde{x}_e = \text{col}\left(\sqrt{2}/2,\sqrt{2}/2\right)$ is stable. $\text{tr}J\left(-\sqrt{2}/2,-\sqrt{2}/2\right) = \sqrt{2}-1 > 0$ and $\det J\left(-\sqrt{2}/2,-\sqrt{2}/2\right) = -2\sqrt{2} < 0$ imply the equilibrium $\tilde{x}_e = \text{col}\left(-\sqrt{2}/2,-\sqrt{2}/2\right)$ is unstable.

EXERCISE 8.10.

(a) An equilibrium is a constant solution whose derivatives are zero. The equilibrium equation is, therefore, $\sin x = 0$ and there are infinitely many equilibria $x_e = n\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

(b) The Jacobian of the equivalent system

$$\begin{aligned} x' &= y\\ y' &= -\sin x - y\end{aligned}$$

is

$$J(x,y)=\begin{pmatrix}0 & 1\\ -\cos x & -1\end{pmatrix}.$$

(c) The Jacobian evaluated at an equilibrium $\tilde{x}_e = \text{col}(n\pi, 0)$ is (note that $\cos n\pi = (-1)^n$)

$$J(n\pi,0)=\begin{cases}\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix} & \text{if } n \text{ is even}\\ \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix} & \text{if } n \text{ is odd}\end{cases}$$

The eigenvalues are

$$\lambda=\begin{cases}-\frac{1}{2}\pm\frac{1}{2}i\sqrt{3} & \text{if } n \text{ is even}\\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{5} & \text{if } n \text{ is odd}\end{cases}$$

Thus, $\tilde{x}_e = \text{col}(n\pi, 0)$ is stable if n is even and unstable if n is odd.

(d)

$$\text{tr}J(n\pi,0) = -1$$

$$\det J(n\pi,0)=\begin{cases}1 \text{ if } n \text{ is even}\\ -1 \text{ if } n \text{ is odd}\end{cases}$$

Therefore, $\tilde{x}_e = \text{col}(n\pi, 0)$ is stable if n is even and unstable if n is odd.

EXERCISE 8.12

(a) An equilibrium is a constant solution whose derivatives are zero. The equilibrium equation is, therefore, $qx^3 = 0$ and there is one equilibrium $x_e = 0$.

(b) The Jacobian of the equivalent system

$$\begin{aligned} x' &= y \\ y' &= -qx^3 - py \end{aligned}$$

is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -3qx^2 & -p \end{pmatrix}.$$

(c) The eigenvalue $\lambda = 0$ of

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is a double eigenvalue. Theorem 8.1 does not apply and no conclusion about the equilibrium $\tilde{x}_e = \text{col}(0, 0)$ can be drawn from it.

(d) $\text{tr}J(0, 0) = 0$ and $\det J(0, 0) = -1 < 0$ and therefore Theorem 8.2 does not apply and no conclusion can be about the equilibrium $\tilde{x}_e = \text{col}(0, 0)$ drawn from it.

EXERCISE 8.14.

(a) The equilibrium equations are

$$\begin{aligned} \left(\frac{3}{2} - x - 2y\right) x = 0 \\ \left(-\frac{1}{4} + x\right) y = 0. \end{aligned}$$

The second equation implies either $y = 0$ or $x = 1/4$. In the first case the first equation implies $x = 0$ or $x = 3/2$. Thus, two equilibria are $(x_e, y_e) = (0, 0)$ and $(3/2, 0)$. In the second case the first equation implies $y = 5/8$. Thus, there are three equilibrium points:

$$(x_e, y_e) = (0, 0), \left(\frac{3}{2}, 0\right) \quad \text{and} \quad \left(\frac{1}{4}, \frac{5}{8}\right).$$

(b) The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} \frac{3}{2} - 2x - 2y & -2x \\ y & -\frac{1}{4} + x \end{pmatrix}.$$

Thus,

$$J(0, 0) = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}, \quad J\left(\frac{3}{2}, 0\right) = \begin{pmatrix} -\frac{3}{2} & -3 \\ 0 & \frac{5}{4} \end{pmatrix}$$

$$J\left(\frac{1}{4}, \frac{5}{8}\right) = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{5}{8} & 0 \end{pmatrix}.$$

(c). The eigenvalues of $J(0, 0)$ are $\lambda_1 = 3/2$, $\lambda_2 = -1/4$ and $\tilde{x}_e = \text{col}(0, 0)$ is unstable since $\lambda_1 > 0$.

The eigenvalues of $J(3/2,0)$ are $\lambda_1 = -3/2$, $\lambda_2 = 5/4$ and $\tilde{x}_e = \mathrm{col}\,(3/2,0)$ is unstable since $\lambda_2 > 0$.

The eigenvalues of $J(1/4,5/8)$ are $\lambda = -1/8 \pm i\sqrt{19}/8$ and $\tilde{x}_e = \mathrm{col}\,(1/4,5/8)$ is stable since the real part $-1/8$ of both roots is negative.

EXERCISE 8.18. The equilibria are $\tilde{x}_e = \mathrm{col}(0,0)$ and $\mathrm{col}(1,1)$. (See Exercise 8.4) The eigenvalues of the Jacobian

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

are $\lambda = \pm 1$ and therefore $(0,0)$ is an hyperbolic saddle.

The eigenvalues of the Jacobian

$$J(1,1) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

are $\lambda = \pm i$. Therefore, $(1,1)$ is non-hyperbolic and the Hartman-Grobman Theorem 8.3.is not applicable.

EXERCISE 8.20. The equilibria are

$$(x_e, y_e) = \mathrm{col}(0,0), \quad \mathrm{col}\left(0, \frac{1}{2}\right), \quad \mathrm{col}(1,0), \quad \mathrm{col}\left(\frac{2}{3}, \frac{1}{3}\right).$$

(See Exercise 8.6.) The eigenvalues of the Jacobian

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

are $\lambda = 1, 2$ and therefore $\tilde{x}_e = \mathrm{col}(0,0)$ is an hyperbolic, unstable node.

The eigenvalues of the Jacobian

$$J\left(0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & -2 \end{pmatrix}$$

are $\lambda = 1/2, -2$ and therefore $\tilde{x}_e = \mathrm{col}\left(0, \frac{1}{2}\right)$ is an hyperbolic saddle.

The eigenvalues of the Jacobian

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

are $\lambda = -1, 1$ and therefore $\tilde{x}_e = \mathrm{col}(1,0)$ is an hyperbolic saddle.

The eigenvalues of the Jacobian

$$J\left(\frac{2}{3}, \frac{1}{3}\right) = \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{4}{3} \end{pmatrix}$$

are $\lambda = -1 \pm \sqrt{3}/3 < 0$ and therefore $\tilde{x}_e = \mathrm{col}\,(2/3, 1/3)$ is an hyperbolic stable node.

EXERCISE 8.22. The equilibria are

$$\tilde{x}_e = \mathrm{col}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad \mathrm{col}\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

(See Exercise 8.8.) The eigenvalues of the Jacobian

$$J\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=\begin{pmatrix}-\sqrt{2} & -\sqrt{2}\\ 1 & -1\end{pmatrix}$$

are $\lambda\approx -1.207\,1\pm 1.171i$ and therefore $\tilde{x}_e=\operatorname{col}\left(\sqrt{2}/2,\sqrt{2}/2\right)$is an hyperbolic, stable spiral.

The eigenvalues of the Jacobian

$$J\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ 1 & -1\end{pmatrix}$$

are $\lambda\approx 1.902$ and -1.487 and therefore $\tilde{x}_e=\operatorname{col}\left(-\sqrt{2}/2,-\sqrt{2}/2\right)$ is an hyperbolic saddle.

EXERCISE 8.26. The equilibrium equations are

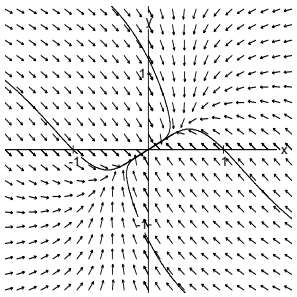
$$x_{in}-x-\frac{2x}{1+x}y=0$$
$$\left(\frac{2x}{1+x}-1\right)y=0$$

For the second equation there are two choices: $y=0$, which implies by the first equation that $x=x_{in}$, and

$$\frac{2x}{1+x}-1=0$$

or $x=1$. In the later case, the first equation implies $y=x_{in}-1$. The equilibria are $\tilde{x}_e=\operatorname{col}\left(x_{in},0\right)$ and $\operatorname{col}\left(1,x_{in}-1\right)$.

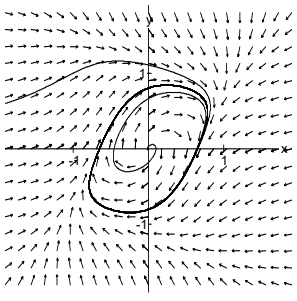
EXERCISE 8.28. The only equilibrium point is $\tilde{x}_e=\operatorname{col}(0,0)$. A sketch of the direction field shows that all orbits are bounded and that no cycle can encircle the equilibrium. This rules out the existence of a cycle.

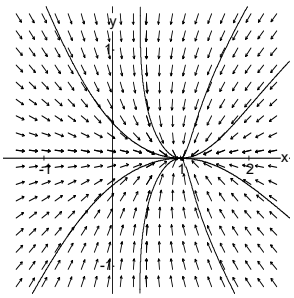


The first version of the Poincaré-Bendixson Theorem implies the forward (omega) limit set S^+ (of every orbit) contains $\tilde{x}_e=\operatorname{col}(0,0)$. An application of the Linearization Principle shows $\tilde{x}_e=\operatorname{col}(0,0)$ is a stable node and therefore S^+ consists solely of $\tilde{x}_e=\operatorname{col}(0,0)$. It follows that all orbits tend to the equilibrium $\tilde{x}_e=\operatorname{col}(0,0)$ as $t\to+\infty$.

Alternatively, we can use the second version of the Poincaré-Bendixson Theorem. Since there are no saddles, there can be no cycle chain. It follows that S^+ is an equilibrium. Since the origin is the only equilibrium, it follows that all orbits tend to the equilibrium $\tilde{x}_e=\operatorname{col}(0,0)$ as $t\to+\infty$.

EXERCISE 8.30. The only equilibrium point is $\tilde{x}_e=\operatorname{col}(0,0)$. A sketch of the direction field shows that all orbits are bounded. An application of the Linearization Principle shows $\tilde{x}_e=\operatorname{col}(0,0)$ is an unstable spiral and therefore cannot be in the forward (omega) limit set S^+ of any orbit. The first version of the Poincaré-Bendixson Theorem implies each orbit approaches a limit cycle. The second version of the Poincaré-Bendixson Theorem implies the same result, since there can be no cycle chain (there are no saddles).





EXERCISE 8.32. The only equilibrium point is $\tilde{x}_e = \text{col}(1,0)$. A sketch of the direction field shows that all orbits are bounded and that no cycle can encircle $\tilde{x}_e = \text{col}(1,0)$. This rules out the existence of a cycle.

The first version of the Poincaré-Bendixson Theorem implies that the forward (omega) limit set S^+ (of every orbit) contains the equilibrium $\tilde{x}_e = \text{col}(1,0)$. An application of the Linearization Principle shows $\tilde{x}_e = \text{col}(1,0)$ is a stable node and therefore S^+ consists solely of $\tilde{x}_e = \text{col}(1,0)$. It follows that all orbits tend to the equilibrium $\tilde{x}_e = \text{col}(1,0)$ as $t \to +\infty$.

Alternatively, we can use the second version of the Poincaré-Bendixson Theorem. Since there are no saddles, there can be no cycle chain. It follows that S^+ is an equilibrium. Since the origin is the only equilibrium, it follows that all orbits tend to the equilibrium $\tilde{x}_e = \text{col}(1,0)$ as $t \to +\infty$.

EXERCISE 8.38. The Jacobian is

$$J(x,y) = \begin{pmatrix} 2xy & x^2 - 1 \\ 1 - y^2 & -2xy - \frac{9}{10}y^2 + \frac{3}{10} \end{pmatrix}.$$

The eigenvalues of

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & \frac{3}{10} \end{pmatrix}$$

are

$$\lambda = \frac{3}{20} \pm \frac{\sqrt{391}}{20} i \approx 0.15 \pm 0.99i$$

and the origin is an hyperbolic, unstable spiral point.

Each of the Jacobians

$$J(1,1) = J(-1,-1) = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{13}{5} \end{pmatrix}$$

$$J(-1,1) = J(1,-1) = \begin{pmatrix} -2 & 0 \\ 0 & \frac{7}{5} \end{pmatrix}$$

has a positive and a negative eigenvalue and therefore all of the corner equilibria are hyperbolic saddles.

The characteristic roots of the Jacobians

$$J\left(1, -\frac{10}{3}\right) = J\left(-1, \frac{10}{3}\right) = \begin{pmatrix} -\frac{20}{3} & 0 \\ -\frac{91}{9} & -\frac{91}{30} \end{pmatrix}$$

are $\lambda = -20/3$ and $-91/30$and these two equilibria are hyperbolic stable nodes.

There are no non-hyperbolic equilibria.

EXERCISE 8.39.

(a) $xf(x,y) + yg(x,y) = -x^2 - 2y^2 - 2x^2y^2 < 0$ for $r > r_0 = 0$.

(c) $xf(x,y) + yg(x,y) = -\left(x^2 + y^2 - 1\right)\left(x^2 + y^2\right) < 0$ for all $r > r_0 = 1$.

EXERCISE 8.40.

(a) Take D to be the first quadrant: $x > 0$, $y > 0$. In D, a calculation shows

$$\frac{d}{dx}(\mu f) + \frac{d}{dy}(\mu g) = -\frac{1}{y} - \frac{1}{x} < 0.$$

(c) Take D to be the inside of the circle $x^2 + y^2 < 4$. In D a calculation shows

$$\frac{df}{dx} + \frac{dg}{dy} = 4 - y^2 - x^2 > 0.$$

EXERCISE 8.41.

(a) With $\mu = 1$ calculate

$$\frac{d}{dx}(\mu f) + \frac{d}{dy}(\mu g) = -3 - y^2 - x^2 < 0$$

for all (x, y).

(c) With $\mu = y$ calculate

$$\frac{d}{dx}(\mu f) + \frac{d}{dy}(\mu g) = 1 + x^2 + 6x^2y^2 > 0$$

for all (x, y).

(e) With $\mu = 1$ calculate

$$\frac{d}{dx}(\mu f) + \frac{d}{dy}(\mu g) = 2 + 4x^2 > 0$$

for all (x, y). This is the desired contradiction.

EXERCISE 8.43. The second of the equilibrium equations

$$\begin{aligned} px - x^3 &= 0 \\ -y &= 0 \end{aligned}$$

implies $y = 0$ and the first equation implies $x = 0$ or $\pm\sqrt{p}$ (provided $p > 0$).

Thus, for p less than the critical value $p_0 = 0$ there is one equilibrium, namely, $\tilde{x} = \text{col}(0, 0)$. (An application of the Linearization Principle shows that it is a stable node.)

For p greater than $p_0 = 0$ there are three equilibria, $\tilde{x} = \text{col}((0, 0)$ and $\text{col}(\pm\sqrt{p}, 0)$. This is characteristic of a pitchfork bifurcation. (The Linearization Principle shows that the equilibrium $\tilde{x} = \text{col}(0, 0)$ is a saddle for $p > 0$ while both equilibria $\tilde{x} = \text{col}(\pm\sqrt{p}, 0)$ are stable nodes.)

EXERCISE 8.45. The equilibrium equations are

$$\begin{aligned} p - x^2 - y^2 &= 0 \\ 1 - x - y &= 0. \end{aligned}$$

Solve the second equation for $y = 1 - x$ and substitute this answer into the first equation. The result equation

$$p - x^2 - (1 - x)^2 = -2x^2 + 2x + p - 1 = 0$$

has roots

$$x = \frac{1}{2}\left(1 \pm \sqrt{2p-1}\right).$$

The equilibria are

$$\tilde{x} = \text{col}\left(\frac{1}{2}\left(1+\sqrt{2p-1}\right), \frac{1}{2}\left(1-\sqrt{2p-1}\right)\right)$$
$$\tilde{x} = \text{col}\left(\frac{1}{2}\left(1-\sqrt{2p-1}\right), \frac{1}{2}\left(1+\sqrt{2p-1}\right)\right)$$

provided $p \geq p_0 = 1/2$. A saddle node bifurcation occurs at $p_0 = 1/2$.

EXERCISE 8.47. The equilibrium equations are

$$\left(p - 2x^2 - y^2\right)\left[(x-1)^2 + y^2\right] = 0$$
$$(x-1)\left[(x-1)^2 + y^2\right] = 0.$$

Setting the square bracketed terms equal to 0 gives the solution $x = 1$, $y = 0$. Otherwise, the second equation implies $x = 1$ and the first equation $p - 2 - y^2 = 0$. The equilibria are $\tilde{x} = \text{col}(1, 0)$ and, if $p > p_0 = 2$,

$$\tilde{x} = \text{col}\left(1, \sqrt{p-2}\right) \quad \text{and} \quad \text{col}\left(1, -\sqrt{p-2}\right).$$

A pitchfork bifurcation occurs at $p_0 = 2$.

EXERCISE 8.49. The equilibrium equations are

$$(p-y)x = 0$$
$$(p-x)y = 0.$$

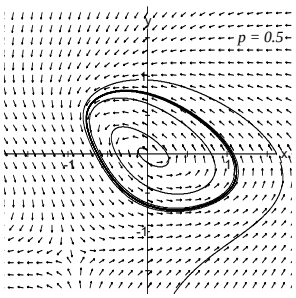
The second equation implies either $x = p$ or $y = 0$. In the first case, the first equation implies $y = p$. In the second case, the first equation implies $x = 0$. The equilibria are $\tilde{x} = \text{col}(0, 0)$ and $\text{col}(p, p)$. These coincide for $p = 0$. A transcritical bifurcation occurs at $p_0 = 0$.

EXERCISE 8.51. The equilibrium equations are

$$y - \ln x = 0$$
$$px - y = 0.$$

From the second equation $y = px$ which, when substituted into the first equation, yields the equation $px = \ln x$ for the x component. A plot of the right and left hand sides shows that these two curves intersect in exactly two points if $p < p_0$ were p_0 is the value of p at which the graphs of $\ln x$ and the straight line px are tangent. The point of tangency occurs at $p_0 = e^{-1}$. Thus, there are no equilibria for $p > p_0$ and two equilibria for $p < p_0$ and a saddle node bifurcation occurs at p_0.

EXERCISE 8.56. The Jacobian



$$J(x,y) = \begin{pmatrix} p - ye^{xy} - 1 & p - xe^{xy} - 2 \\ 1 & -3y^2 + 1 \end{pmatrix}$$

at the equilibrium $\tilde{x}_e = \text{col}(0,0)$, namely,

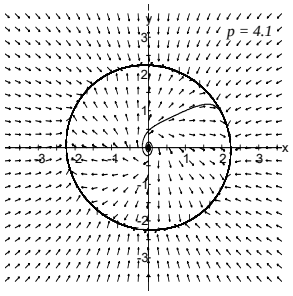
$$J(0,0) = \begin{pmatrix} p-1 & p-2 \\ 1 & 1 \end{pmatrix}$$

has eigenvalues $\lambda = \frac{1}{2}p \pm \frac{1}{2}i\sqrt{4-p^2}$ and the Hopf bifurcation criteria hold at $p_0 = 0$ where $\alpha = 0$, $\beta = 1$ and $d\alpha/dp = 1/2 \neq 0$. For $p < 0$ the origin is a stable spiral. For $p > 0$ the origin is an unstable spiral. A computer sketch of the phase portrait indicate a stable limit cycle exists for small values of $p > 0$. A Hopf bifurcation occurs at $p_0 = 0$. See the accompanying figure for an example when $p = 0.5$

EXERCISE 8.58. The Jacobian

$$J(x,y) = \begin{pmatrix} p + 15x^2 + 5y^2 - 5x^4 & 10xy - 4xy^3 - 4x^3y + 1 \\ -y^4 - 6x^2y^2 - 4 & . \\ . & . \\ 10xy - 4xy^3 & p + 5x^2 + 15y^2 - x^4 \\ -4x^3y - 3 & -5y^4 - 6x^2y^2 - 4 \end{pmatrix}$$

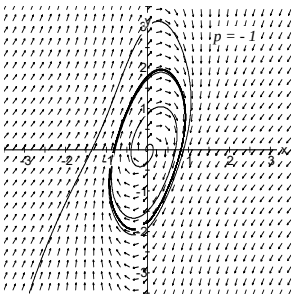
at the equilibrium $\tilde{x}_e = \text{col}(0,0)$, namely,



$$J(0,0) = \begin{pmatrix} p-4 & 1 \\ -3 & p-4 \end{pmatrix}$$

has eigenvalues $\lambda = p - 4 \pm i\sqrt{3}$ and the Hopf bifurcation criteria hold at $p_0 = 4$ where $\alpha = 0$, $\beta = \sqrt{3}$ and $d\alpha/dp = 1 \neq 0$. For $p < 4$ the origin is a stable spiral and computer sketches of the phase plane portrait show there is also an unstable limit cycle encircling $\tilde{x}_e = \text{col}(0,0)$. There is also a stable limit cycle encircling the unstable limit cycle! For $p > 4$ the unstable cycle disappears (although the stable cycle remains), and the origin becomes unstable. See figure for an example when $p = 4.1$.

EXERCISE 8.60. The Jacobian



$$J(x,y) = \begin{pmatrix} -3x^2 & 1 \\ -3x^2 - y^2 - 2 & -p - 2xy \end{pmatrix}$$

at the equilibrium $\tilde{x}_e = \text{col}(0,0)$, namely,

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -2 & -p \end{pmatrix}$$

has eigenvalues $\lambda = -\frac{1}{2}p \pm \frac{1}{2}i\sqrt{8-p^2}$ and the Hopf bifurcation criteria hold at $p_0 = 0$ where $\alpha = 0$, $\beta = \sqrt{2}$ and $d\alpha/dp = -1/2 \neq 0$..

For $p > 0$ the origin is a stable spiral. For $p < 0$ the origin is an unstable spiral. A computer sketch of the phase portrait indicate a stable limit cycle exists for small values of $p < 0$. A Hopf bifurcation of limit cycles occurs at $p_0 = 0$. See figure for an example when $p = -1$.

EXERCISE 8.62.

(a) The Jacobian

$$J(x,y,z) = \begin{pmatrix} 1-2x-y-z & -x & -x \\ y & -1+x & 0 \\ z & 0 & -1+x \end{pmatrix}$$

evaluated at the origin

$$J(0,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has eigenvalues $\lambda = 1$ and -1 and the Linearization Principle implies the origin is unstable (because 1 is positive).

(c) The Jacobian

$$J(x,y,z) = \begin{pmatrix} -2x-y^2-z & -2xy & -x \\ -y & 1-x-2y-z^2 & -2yz \\ -2zx & -z & -1-x^2-y-2z \end{pmatrix}$$

evaluated at the origin

$$J(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has eigenvalues $\lambda = 0$, 1 and -1. The origin is non-hyperbolic (because of the eigenvalue 0) and the Linearization Hartman-Grobman Theorem does not apply. The Fundamental Theorem of Stability, however, implies the origin is unstable (because of the positive eigenvalue 1).

(e) The Jacobian

$$J(x,y,z) = \begin{pmatrix} -2e^{y-2x} & e^{y-2x} & 0 \\ 0 & 0 & -1 \\ 0 & e^{y-2z} & -2e^{y-2z} \end{pmatrix}$$

evaluated at the origin

$$J(0,0,0) = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$$

has eigenvalues $\lambda = -2$ and -1 and the Linearization Principle implies the origin is (locally asymptotically) stable (since all eigenvalues are negative).

EXERCISE 8.63.

(a) There are three equilibria: $\tilde{x}_e = \text{col}(0,0,0)$, $\text{col}(1,2,1)$ and $\text{col}(-5,5,-5)$. The Jacobian is

$$J(x,y,z) = \begin{pmatrix} 1-y-2x & 1-x & 0 \\ -y & -1-x & 4 \\ 1 & 0 & -1 \end{pmatrix}.$$

The eigenvalues of

$$J(0,0,0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

are $\lambda \approx 1.594\,3$ and $-1.297\,2 \pm 1.205\,6i$. The Linearization Principle implies $\tilde{x}_e = \text{col}(0,0,0)$ is unstable (because of the positive eigenvalue 1.5943).

The eigenvalues of

$$J(1,2,1) = \begin{pmatrix} -3 & 0 & 0 \\ -2 & -2 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

are $\lambda = -1, -2$, and -3. The Linearization Principle implies $\tilde{x}_e = \text{col}(1,2,1)$ is stable (since all eigenvalues are negative).

The eigenvalues of

$$J(-5,5,-5) = \begin{pmatrix} 6 & 6 & 0 \\ -5 & 4 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

are $\lambda \approx -0.602\,58$ and $4.8013 \pm 5.1705i$. The Linearization Principle implies $\tilde{x}_e = \text{col}(-5,5,-5)$ is unstable (since the real part of the complex eigenvalues is positive).

EXERCISE 8.65. The characteristic roots of the Jacobian

$$J(0,0,0) = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & p \\ \frac{1}{5} & 0 & -\frac{1}{5} \end{pmatrix}$$

satisfy the equation

$$\lambda^3 - \frac{4}{5}\lambda^2 - \frac{11}{5}\lambda - \left(\frac{2}{5}p + \frac{2}{5}\right) = 0$$

or

$$(\lambda - 2)(\lambda + 1)\left(\lambda + \frac{1}{5}\right) = \frac{2}{5}p.$$

View the solving of this equation as finding the intersection points of the cubic polynomial on the left hand side of the equation with the horizontal straight line located at level $2p/5$. A sketch of the cubic shows that for all $p > 0$ there is an intersection point in the right half plane, i.e., there is a positive characteristic root $\lambda > 0$ for all $p > 0$.

EXERCISE 8.69.

(a) The equilibria are $\tilde{x}_e = \text{col}(0,0)$, $\text{col}(-1,0)$ and $\text{col}(-1,-1)$. The Jacobian is

$$J(x,y) = \begin{pmatrix} 1+2x & 1+2y \\ y & 1+x \end{pmatrix}.$$

The characteristic roots of the Jacobian

$$J(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are both equal to 1 and therefore $(0,0)$ is an unstable node. The eigenvalues of the Jacobian

$$J(-1,0) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

are $\lambda = 0$ and -1. and the Linearization Principle does not apply (because of the eigenvalue 0).

The eigenvalues of the Jacobian

$$J(-1,-1) = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$$

are $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ and the equilibrium $\tilde{x}_e = \text{col}(-1,-1)$ is a saddle (because one eigenvalues is positive and one is negative).

(c) The equilibria are $\tilde{x}_e = \text{col}\,(0,0)$, $\text{col}\,(1,-1)$ and $\text{col}\,(-1,-1)$. The Jacobian is

$$J(x,y) = \begin{pmatrix} -2x & -1 \\ -1+y^2 & 2xy \end{pmatrix}.$$

The eigenvalues of the Jacobian

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

are $\lambda = 1$ and -1 and the equilibrium $(0,0)$ is a saddle (because one eigenvalue is positive and one is negative).

The eigenvalues of the Jacobian

$$J(1,-1) = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$$

are both equal to -2 and the equilibrium $(1,-1)$ is a stable node (since both eigenvalues are negative).

The eigenvalues of the Jacobian

$$J(-1,-1) = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

are both equal to 2 and the equilibrium $\tilde{x}_e = \text{col}(-1,-1)$ is an unstable node (since both eigenvalues are positive).

(e) The equilibrium is $\tilde{x}_e = \text{col}\,(0,0)$. The Jacobian is

$$J(x,y) = \begin{pmatrix} -1-y^2 & -2xy \\ 1 & -4y \end{pmatrix}.$$

The eigenvalues of the Jacobian

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$$

are $\lambda = -1$ and 0 and the Linearization Principle does not apply.

EXERCISE 8.70.

(a) The equilibrium equations are

$$\begin{aligned} x + py + x^2 + y^2 &= 0 \\ (1+x)\, y &= 0. \end{aligned}$$

The second equation implies either $y = 0$ or $x = -1$. In the first case the first equation implies $x = 0$ or -1. In the second case the first equation implies $y = 0$ or $-p$. The equilibria are

$$\tilde{x}_e = \operatorname{col}(0,0), \quad \operatorname{col}(-1,0), \quad \operatorname{col}(-1,-p).$$

There is a transcritical bifurcation at $p_0 = 0$ where the second and third equilibria coincide.

(c) The equilibrium equations are

$$\begin{aligned} x\left(1 - y^2\right) &= 0 \\ p + x - 2y^2 &= 0. \end{aligned}$$

The first equation implies $x = 0$ or $y = \pm 1$. In the first case, the second equation implies $y = \pm\sqrt{2p}/2$ provided $p \geq 0$. In the second case, the second equation implies $x = 2 - p$. There are four equilibria:

$$\tilde{x}_e = \operatorname{col}(2-p, 1), \quad \operatorname{col}(2-p, -1), \quad \operatorname{col}\left(0, \pm\frac{1}{2}\sqrt{2p}\right).$$

The is a saddle-node bifurcation at $p_0 = 0$ where the last two equilibria come into existence. There are two simultaneous transcritical bifurcations at $p_0 = 2$ where the first two equilibria coincide with one of the last equilibria.

(e) The equilibrium equations are

$$\begin{aligned} x + y - y^2 &= 0 \\ x - y + p^2 y^2 &= 0. \end{aligned}$$

The first equation implies $x = y^2 - y$ which, when substituted into the second equation, yields the equation

$$y^2 - y - y + p^2 y^2 = \left(\left(1 + p^2\right) y - 2\right) y = 0$$

whose solutions are $y = 0$ and $y = 2/\left(p^2 + 1\right)$. The equilibria are

$$\tilde{x}_e = \operatorname{col}(0,0) \quad \text{and} \quad \operatorname{col}\left(2\frac{1-p^2}{\left(p^2+1\right)^2}, 2\frac{1}{p^2+1}\right).$$

Both equilibria exist for all p and are never equal to each other. There are no equilibrium bifurcations.

(g) The equilibrium equations are

$$\begin{aligned} x\,(1-y) &= 0 \\ p + x - 2y &= 0. \end{aligned}$$

The first equation implies $x = 0$ or $y = 1$. In the first case, the second equation implies $y = p/2$. In the second case, the second equation implies $x = 2 - p$. The equilibria are

$$\tilde{x}_e = \mathrm{col}\left(0, \frac{1}{2}p\right), \quad \mathrm{col}\,(2-p, 1)\,.$$

There is a transcritical bifurcation at $p_0 = 2$ where these two equilibria coincide.

EXERCISE 8.71.

(a) The Jacobian

$$J(0,0) = \begin{pmatrix} p & -2 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\lambda = \frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 8}$ which are complex for $-\sqrt{8} < p < \sqrt{8}$. The real parts $p/2$ vanish and have nonzero derivative with respect to p at $p_0 = 0$. Therefore, criteria hold. The figures below show that a Hopf bifurcation of a limit cycle occurs at $p_0 = 0$.