Review of Existence and Uniqueness: Section 1.2

January 17, 2019

First Order Equations

Last class we considered a large class of initial value problems with the form:

$$x' = f(t, x) \quad \text{and} \quad x(t_0) = x_0 \tag{1}$$

Our goal was to state a result which tells us when the equation above has one and only one solution. Such results are typically called a results on existence and uniqueness.

Let us denote by

$$\frac{\partial f}{\partial x}(t, x)$$

the partial derivative of f with respect to x.

Main Theorem

Theorem (Existence and Uniqueness for 1st order i.v.p.)

Let $f(t, x)$ be a function that is well-defined for $a < t < b$ and $c < x < d$. Suppose that:

1. *Both $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous in t and continuous in x when $a < t < b$ and $c < x < d$.*
2. *The initial condition lies in these intervals, i.e. $a < t_0 < b$ and $c < x_0 < d$.*

Under these conditions, the initial value problem (1) has a solution on an interval $\alpha < t < \beta$ which contains t_0. Moreover, there is no other solution of (1) on this interval.

Example 1: Consider the following initial value problem

$$x' = tx \quad \text{with} \quad x(0) = \frac{1}{2}$$

Note that

$$f(t, x) = tx \quad \text{and} \quad \frac{\partial f}{\partial x}(t, x) = t$$

are both continuous in both t and x for all real values of t and x. In this case, the theorem applies. Moreover, the function

$$x(t) = \frac{1}{2} e^{\frac{t^2}{2}}$$

is a solution and it is defined for all real t.

Another Example

Example 2: Consider the initial value problem

$$x' = 3x^{2/3} \quad \text{with} \quad x(0) = 0$$

Note that

$$f(t, x) = 3x^{2/3} \quad \text{and} \quad \frac{\partial f}{\partial x}(t, x) = 2x^{-1/3}$$

are both independent of t. Since $\frac{\partial f}{\partial x}$ is undefined at $x = 0$, the theorem does not apply in the case of this particular initial condition.

One readily checks that the functions

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = t^3$$

which are clearly distinct, are both solutions of this IVP.

An Interesting Fact:

Although we will not prove this, the following is an interesting fact.

Fact: In the initial value problem (1) has two distinct solutions, then it has infinitely many solutions.

As a result, we conclude that the initial value problem (1) either has: 0, 1, or infinitely many solutions.

Example 3: Consider the initial value problem

$$x' = 2tx^2 \quad \text{with} \quad x(0) = 1$$

Note that

$$f(t,x) = 2tx^2 \quad \text{and} \quad \frac{\partial f}{\partial x}(t,x) = 4tx$$

are both continuous in both t and x for all real values of t and x. In this case, the theorem applies. Moreover, the function

$$x(t) = \frac{1}{1-t^2}$$

is a solution. Note further, however, that this solution is only defined for $-1 < t < 1$. Thus we only get a local solution, even in cases where f and $\frac{\partial f}{\partial x}$ are very well behaved!