Existence and Uniqueness: Section 1.2

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Some Examples

Example 1: In physics, the differential equation

$$v' = g - k_0 v$$

describes how the velocity $v(t)$ of a falling object changes when the object is assumed to be under the influence of gravity (the constant g) and air resistance (the term proportional to v). The constants g and k_0 are called the coefficients or parameters of this problem. One easily checks that: for any real number c,

$$v(t) = ce^{-k_0 t} + \frac{g}{k_0}$$

is a solution of this differential equation for all real t.

Examples (cont.)

Example 2: The differential equation

$$x' = kx^2$$

where the coefficient $k > 0$ is constant is sometimes used to describe the growth of the world's human population $x(t)$ (in billions) as a function of time t (in years). The function

$$x(t) = \frac{1}{1 - kt}$$

is well-defined on the intervals $(-\infty, k^{-1})$ and (k^{-1}, ∞). One checks that this function satisfies the above differential equation. Find a solution of the initial value problem

$$x' = kx^2 \quad \text{with} \quad x(0) = 1$$

What is the domain of your solution?

First Order Equations

We will consider a large class of first order equations with the form:

$$x' = f(t, x) \quad \text{and} \quad x(t_0) = x_0 \tag{1}$$

Note that the differential equation above is actually an initial value problem since, in addition to the differential equations, we also impose a constraint on the solution. Our goal is to state a result which tells us when the equation above has one and only one solution. Such a result is typically called a result on existence and uniqueness.

Let us denote by

$$\frac{\partial f}{\partial x}(t, x)$$

the partial derivative of f with respect to x.

Main Theorem

Theorem (Existence and Uniqueness for 1st order i.v.p.)

Let $f(t,x)$ be a function that is well-defined for $a < t < b$ and $c < x < d$. Suppose that:

1. *Both $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous in t and continuous in x when $a < t < b$ and $c < x < d$.*
2. *The initial condition lies in these intervals, i.e. $a < t_0 < b$ and $c < x_0 < d$.*

Under these conditions, the initial value problem (1) has a solution on an interval $\alpha < t < \beta$ which contains t_0. Moreover, there is no other solution of (1) on this interval.

Some Comments:

Comment 1: The result of this theorem is called a local existence and uniqueness result because it only guarantees the existence (and uniqueness) of a solution on *some interval* containing t_0. No information is given on the size of this interval; it may be the whole real line or it may be very small . . .

Comment 2: To see if the conditions of the theorem are met, one must first calculate $\frac{\partial f}{\partial x}(t, x)$ and then check the required continuity statements. In calculus, we learned that many of the functions we typically encounter (e.g. polynomials, etc.) are continuous. We also learned that sums, products, and compositions of continuous functions are continuous. You will use these facts when checking whether or not the conditions of this theorem are met.

More Comments:

Comment 3: If $f(t, x)$ does not depend on t, then the continuity conditions (with respect to t) are clearly satisfied.

Comment 4: If the conditions of the theorem are not met, then one cannot conclude anything. In some cases, there will be unique solutions, and in others there will not.

Some Examples

Example 1: Does the above theorem apply to the following initial value problem

$$x' = tx \quad \text{with} \quad x(0) = \frac{1}{2}$$

Consider the function

$$x(t) = \frac{1}{2} e^{\frac{t^2}{2}}$$

which is defined for all real t.

More Examples

Example 2: Let a, r, and K be positive constants with $a < K$. Consider the initial value problem

$$x' = rx\left(1 - \frac{x}{K + a\sin(t)}\right) \quad \text{with} \quad x(0) = x_0$$

This model describes the growth of a population in a periodically fluctuating environment. Note that here x_0 is the initial population size. There is no formula for the solution to this initial value problem, but what does the theorem tell us?

A Last Example

Example 3: Consider the initial value problem

$$x' = 3x^{2/3} \quad \text{with} \quad x(0) = 0$$

Does the theorem apply?
Consider the functions

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = t^3$$

What can you conclude?

An Interesting Fact:

Although we will not prove this, the following is an interesting fact.

Fact: In the initial value problem (1) has two distinct solutions, then it has infinitely many solutions.

As a result, we conclude that the initial value problem (1) either has: 0, 1, or infinitely many solutions.