MATH 129:
TEST 4

FALL 2015

Name	
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

(1) Find the Taylor polynomial of degree 4, i.e. $P_4(x)$, for the function $f(x) = \cos(x)$ centered at $a = \pi/2$.

$f(x) = \cos(x)$	$f(\pi/2) = 0$
$f'(x) = -\sin(x)$	$f'(\pi/2) = -1$
$f''(x) = -\cos(x)$	$f''(\pi/2) = 0$
$f'''(x) = \sin(x)$	$f'''(\pi/2) = 1$
$f^{(4)}(x) = \cos(x)$	$f^{(4)}(\pi/2) = 0$

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(\pi/2)}{k!}(x-\pi/2)^k$$

$$= f(\pi/2) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \frac{f'''(\pi/2)}{3!}(x-\pi/2)^3 + \frac{f^{(4)}(\pi/2)}{4!}(x-\pi/2)^4$$

$$= 0 - (x-\pi/2) + 0 + \frac{1}{3!}(x-\pi/2)^3 + 0$$

$$= -(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3$$

(2) Consider the function

$$f(x) = \frac{1}{1-2x}.$$

a) Find the Taylor series centered at $a = 0$ for this function.

This is a geometric series

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

Also $f(x) = (1-2x)^{-1}$

$$f'(x) = (-1)(1-2x)^{-2}(-2)$$

$$f''(x) = (-1)(-2)(1-2x)^{-3}(-2)^2$$

$$f'''(x) = (-1)(-2)(-3)(1-2x)^{-4}(-2)^3$$

$\vdots$

$$f^{(n)}(x) = (-1)\cdots(-n)(1-2x)^{-(n+1)}(-2)^n$$

$$\Rightarrow f^{(n)}(0) = n!\, 2^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 2^n x^n$$

as above!

b) Use your result in a) to find the Taylor series for

$$g(x) = \frac{x}{1+2x^3}$$

again centered at $a = 0$.

$$g(x) = \frac{x}{1+2x^3} = x \cdot f(-x^3)$$

$$= x \cdot \sum_{n=0}^{\infty} 2^n (-x^3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n\, 2^n\, x^{3n+1}$$

(3) Consider the differential equation

$$\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} = 9\,.$$

For what values of a is the function

$$y(x) = \sin(ax)$$

a solution of this differential equation? Show all work to receive any credit.

$y(x) = \sin(ax)$

$y'(x) = a \cdot \cos(ax)$

$y''(x) = -a^2 \sin(ax)$

$(y')^2 - y \cdot y''$

$= (a\cos(ax))^2 - \sin(ax) \cdot (-a^2 \sin(ax))$

$= a^2 (\cos(ax)^2 + \sin(ax)^2)$

$= a^2$

$\Rightarrow \quad (y')^2 - y y'' = 9 \quad \Leftrightarrow \quad a^2 = 9$

$\boxed{a = \pm 3}$

(4) Solve the following differential equation

$$\frac{dy}{dx} = y^2 e^{3x}; \quad y(0) = 1\,.$$

This is separable.

STEP 1: $y^2 = 0 \Rightarrow y = 0$ Since $y(0) = 1$, this is NOT the solution we want.

STEP 2: $\frac{1}{y^2}\frac{dy}{dx} = e^{3x}$

STEP 3: $\int \frac{1}{y^2} dy = \int e^{3x}\, dx + C$

$$-\frac{1}{y} = \frac{1}{3}e^{3x} + C$$

Using that $y(0) = 1$

$$\Rightarrow -\frac{1}{1} = \frac{1}{3}e^{3(0)} + C$$

$$C = -1 - \frac{1}{3} = -\frac{4}{3}$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{3}e^{3x} - \frac{4}{3}$$

$$y(x) = \frac{-1}{\frac{1}{3}e^{3x} - \frac{4}{3}} = \frac{3}{4 - e^{3x}}$$

(5) An egg, initially at 15°C, is put in a pot of boiling water (at 100°C).

a) Write a differential equation which describes the temperature $T(t)$ of the egg as a function of time t.

$$\frac{dT}{dt} = -k(T-100)$$

b) Find all solutions of the differential equation you found above. For credit, show your work.

This is separable.

STEP 1: $T-100=0$

$\Rightarrow T=100$ is a constant solution.

$$\boxed{T(t)=100}$$

STEP 2: $\frac{1}{T-100}\frac{dT}{dt} = -k$

STEP 3: $\int \frac{1}{T-100}\,dT = \int -k\,dt + C$

$\ln|T-100| = -kt + C$

$|T-100| = e^{C}\cdot e^{-kt}$ $\Rightarrow$ $\boxed{T(t) = 100 + Ae^{-kt}}$

c) Suppose that after 1 minute the egg is 35°C. At what time will the egg reach 90°C? Give an exact answer and then approximate this value with two decimal place accuracy.

Note: $15 = T(0) = 100 + A \Rightarrow A = 15 - 100 = -85$

$35 = T(1) = 100 - 85e^{-k} \Rightarrow e^{-k} = \frac{35-100}{-85}$

$\Rightarrow -k = \ln\left(\frac{65}{85}\right)$

$k = \ln(17/13)$

Let t_* be the time when $90 = T(t_*)$

$90 = T(t_*) = 100 - 85e^{-kt_*}$

$e^{-kt_*} = \frac{90-100}{-85} \Rightarrow -kt_* = \ln\left(\frac{10}{85}\right)$

$t_* = \frac{\ln\left(\frac{17}{2}\right)}{\ln(17/13)} \cong 7.98$

(6) A bank account earns 3% annual interest compounded continuously. This account is also used to pay bills in a continuous cash flow rate of \$600 per year.

a) Write a differential equation for the balance, $B(t)$, in the bank account at any time t in years.

$$\frac{dB}{dt} = 0.03B - 600$$

b) Find all solutions of the differential equation you found above. For credit, show your work.

This is separable!

$$\frac{dB}{dt} = 0.03\left(B - \frac{600}{0.03}\right) = 0.03(B - 20{,}000)$$

STEP 1

$B(t) = 20{,}000$ is a constant solution!

STEP 2:

$$\frac{1}{B-20{,}000}\frac{dB}{dt} = 0.03$$

STEP 3:

$$\int \frac{1}{B-20{,}000}\,dB = \int 0.03\,dt + C$$

$$\ln|B-20{,}000| = 0.03t + C$$

$$|B-20{,}000| = e^{C} \cdot e^{0.03t}$$

$$B(t) = 20{,}000 + Ae^{0.03t}$$

c) Plot the solutions of this differential equation with the following initial conditions (all on one graph). $B(0) = 10,000$, $B(0) = 20,000$, and $B(0) = 30,000$.

If $10{,}000 = B(0) = 20{,}000 + A$, then $A = -10{,}000$.

If $20{,}000 = B(0) = 20{,}000 + A$, then $A = 0$

If $30{,}000 = B(0) = 20{,}000 + A$, then $A = 10{,}000$