MATH 129:
SIMS
TEST 4

SPRING 2018

Name	Key
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

(1) Find the Taylor polynomial of degree 4, i.e. $P_4(x)$, for the function $f(x) = \cos(x)$ centered at $a = \pi/2$.

$f(x) = \cos(x)$	$f(\pi/2) = 0$
$f'(x) = -\sin(x)$	$f'(\pi/2) = -1$
$f''(x) = -\cos(x)$	$f''(\pi/2) = 0$
$f'''(x) = \sin(x)$	$f'''(\pi/2) = 1$
$f^{(4)}(x) = \cos(x)$	$f^{(4)}(\pi/2) = 0$

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(\frac{\pi}{2})}{k!}(x - \tfrac{\pi}{2})^k$$

$$= f(\pi/2) + f'(\tfrac{\pi}{2})(x - \tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x - \tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x - \tfrac{\pi}{2})^3 + \frac{f^{(4)}(\frac{\pi}{2})}{4!}(x - \tfrac{\pi}{2})^4$$

$$= 0 - (x - \tfrac{\pi}{2}) + 0 + \frac{1}{3!}(x - \tfrac{\pi}{2})^3 + 0$$

$$= -(x - \tfrac{\pi}{2}) + \frac{1}{3!}(x - \tfrac{\pi}{2})^3$$

(2) Consider the function

$$f(x) = \frac{1}{1-2x}.$$

a) Find the Taylor series centered at $a = 0$ for this function.

Two ways to work this:

1) f is the sum of a geometric series, so

$$f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n \cdot x^n$$

2) Take derivatives:

$$f(x) = (1-2x)^{-1}$$

$$f'(x) = (-1)(1-2x)^{-2}(-2)$$

$$f''(x) = (-1)(-2)(1-2x)^{-3}(-2)^2$$

$$f'''(x) = (-1)(-2)(-3)(1-2x)^{-4}(-2)^3$$

$$\vdots$$

$$f^{(n)}(x) = (-1)(-2)\cdots(-n)(1-2x)^{-(n+1)}(-2)^n$$

$$\Rightarrow f^{(n)}(0) = n! \, 2^n \quad \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{n!}\, 2^n}{\cancel{n!}} x^n$$

$$= \sum_{n=0}^{\infty} 2^n x^n$$

b) Use your result in a) to find the Taylor series for

$$g(x) = \frac{x}{1+2x^3}$$

again centered at $a = 0$.

Note:

$$g(x) = \frac{x}{1+2x^3} = x \cdot \frac{1}{1-2(-x^3)}$$

$$= x \cdot f(-x^3)$$

$$= x \cdot \sum_{n=0}^{\infty} 2^n (-x^3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \, x^{3n+1}$$

(3) Consider the differential equation

$$\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} = 9\,.$$

For what values of a is the function

$$y(x) = \sin(ax)$$

a solution of this differential equation? Show all work to receive any credit.

$y(x) = \sin(ax)$

$y'(x) = a\cos(ax)$

$y''(x) = -a^2\sin(ax)$

$(y')^2 - y \cdot y'' = 9$

$\Rightarrow (a\cos(ax))^2 - (\sin(ax))(-a^2\sin(ax)) = 9$

$\Rightarrow a^2\cos^2(ax) + a^2\sin^2(ax) = 9$

$\Rightarrow a^2\underbrace{(\cos^2(ax) + \sin^2(ax))}_{=1} = 9$

$\Rightarrow a^2 = 9$

$\Rightarrow \boxed{a = \pm 3}$

(4) Find the particular solution of the following differential equation

$$\frac{dy}{dx} = y^2 e^{3x}$$

satisfying $y(0) = 1$.

This D.E. is separable.

STEP 1: $y^2 = 0 \Rightarrow y = 0$.

Thus $y(x) = 0$ is a constant (equilibrium) solution.

We do not want this solution.

STEP 2

$$\frac{1}{y^2}\frac{dy}{dx} = e^{3x}$$

$$\Rightarrow \int \frac{1}{y^2}dy = \int e^{3x}\,dx + C$$

$$-\frac{1}{y} = \frac{1}{3}e^{3x} + C$$

$$\Rightarrow y = \frac{1}{-\frac{1}{3}e^{3x} - C}$$

we want:

$$1 = y(0) = \frac{1}{-\frac{1}{3}e^{0} - C}$$

$$\Rightarrow -\frac{1}{3} - C = 1$$

$$\Rightarrow C = -\frac{1}{3} - 1 = \frac{-4}{3}$$

$$\boxed{y = \frac{1}{-\frac{1}{3}e^{3x} + \frac{4}{3}}}$$

(5) An egg, initially at $15°C$, is put in a pot of boiling water (at $100°C$).

a) Write a differential equation which describes the temperature $T(t)$ of the egg as a function of time t.

$$\frac{dT}{dt} = -k(T-100)$$

b) Find all solutions of the differential equation you found above. For credit, show your work.

This D.E. is separable.

STEP 1 $T-100=0$

$\Rightarrow T(t)=100$

is a constant solution.

STEP 2:

$$\frac{1}{T-100}\frac{dT}{dt} = -k$$

$$\ln|T-100| = -kt + C$$

$$|T-100| = e^C \cdot e^{-kt}$$

$$T(t) = 100 + Ae^{-kt}$$

$$\boxed{A = \pm e^C}$$

c) Suppose that after 1 minute the egg is $35°C$. At what time will the egg reach $90°C$? Give an exact answer and then approximate this value with two decimal place accuracy.

Note. $15 = T(0) = 100 + A \Rightarrow A = -85$

$\Rightarrow T(t) = 100 - 85e^{-kt}$

Also $35 = T(1) = 100 - 85e^{-k}$

$\Rightarrow e^{-k} = \frac{35-100}{-85}$

$\Rightarrow -k = \ln\left(\frac{65}{85}\right)$

$k = \ln\left(\frac{17}{13}\right)$

$\cong .268264$

Let t^* be the time for which

$90 = T(t^*) = 100 - 85e^{-kt^*}$

$\Rightarrow e^{-kt^*} = \frac{90-100}{-85}$

$\Rightarrow -kt^* = \ln\left(\frac{10}{85}\right)$

$t^* = \frac{\ln\left(\frac{17}{2}\right)}{\ln\left(\frac{17}{13}\right)} \cong 7.98$

(6) A bank account earns 3% annual interest compounded continuously. This account is also used to pay bills in a continuous cash flow rate of \$600 per year.

a) Write a differential equation for the balance, $B(t)$, in the bank account at any time t in years.

$$\frac{dB}{dt} = 0.03B - 600$$

b) Find all solutions of the differential equation you found above. For credit, show your work.

This is separable: $\frac{dB}{dt} = 0.03\left(B - \frac{600}{0.03}\right) = 0.03(B - 20{,}000)$

STEP 1

$B - 20{,}000 = 0$

$\Rightarrow B(t) = 20{,}000$ is a constant solution.

STEP 2

$$\frac{1}{B - 20{,}000}\frac{dB}{dt} = 0.03$$

$\Rightarrow \ln|B - 20{,}000| = 0.03t + C$

$|B - 20{,}000| = e^{C} \cdot e^{0.03t}$

$B(t) = 20{,}000 + A e^{0.03t}$

$\boxed{A = \pm e^{C}}$

c) Plot the solutions of this differential equation with the following initial conditions (all on one graph). $B(0) = 10{,}000$, $B(0) = 20{,}000$, and $B(0) = 30{,}000$.

If $10{,}000 = B(0) = 20{,}000 + A \Rightarrow A = -10{,}000$

If $20{,}000 = B(0) = 20{,}000 + A \Rightarrow A = 0$

If $30{,}000 = B(0) = 20{,}000 + A \Rightarrow A = 10{,}000$