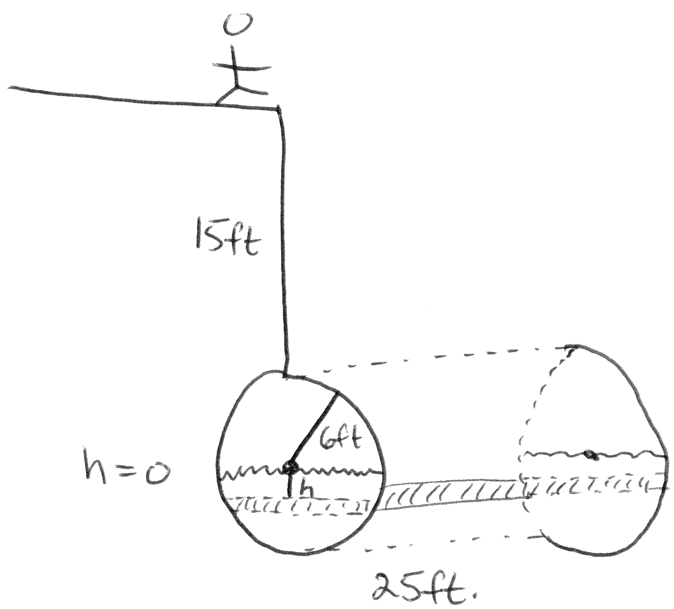
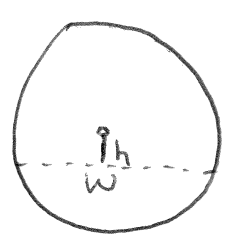
(1) A cylindrical tank half-filled with oil is lying horizontally with its top 15 feet under ground. If the radius of the tank is 6 feet and the length of the tank is 25 feet, find an integral which describes the work done to pump the oil to ground level. The density of the oil is 35 lb/ft^3. **You need not evaluate the integral. To receive any partial credit, write out the corresponding Riemann sum.**





width $= w = \left(\sqrt{36-h^2} - (-\sqrt{36-h^2})\right)$

$= 2\sqrt{36-h^2}$

Total Work Done $= \sum$ work done on each slice

$= \sum$ (Force)(distance) on each slice.

$= \sum \left[(\text{density})\cdot(\text{volume})\right](\text{distanc}$

(length)(width)(height)

$= \sum \left[(35\ \text{lb/ft}^3)(25\text{ft})(2\sqrt{36-h^2})(\Delta h)\right](15+6-h)$

$\rightarrow \int_{-6}^{0} 35\cdot 25\cdot 2\sqrt{36-h^2}\cdot(21-h)$

$= 1750\int_{-6}^{0}\sqrt{36-h^2}\cdot(21-h)d$

also

$= 1750\int_{0}^{6}\sqrt{36-h^2}\cdot(21+h)\,dh$

(2) a) A student invests \$100 at the beginning of each month for 2 years. If the investment yields 0.7% per month, compounded monthly, what is its value after 2 years of investing? **Label your answer carefully.**

Let $A(N)$ be the amount in the account at the end of the N^{th} month.

$A(1) = 100(1.007)$

$A(2) = 100(1.007) + 100(1.007)^2$

$A(3) = 100(1.007) + 100(1.007)^2 + 100(1.007)^3$

$\vdots$

$$A(N) = \sum_{n=1}^{N} 100(1.007)^n$$

$$A(24) = \sum_{n=1}^{24} 100(1.007)^n = 1.007 \cdot 100\left(\frac{1-(1.007)^{24}}{1-1.007}\right) \cong \$2621.71$$

Also accept \$2603.49

b) Calculate

$$\sum_{n=2}^{\infty} \frac{3^{2n} + 6(-4)^n}{5^{3n}}$$

$$\sum_{n=2}^{\infty} \frac{3^{2n}+6(-4)^n}{5^{3n}} = \sum_{n=2}^{\infty} \frac{3^{2n}}{5^{3n}} + \sum_{n=2}^{\infty} \frac{6(-4)^n}{5^{3n}}$$

$$= \sum_{n=2}^{\infty}\left(\frac{3^2}{5^3}\right)^n + 6\sum_{n=2}^{\infty}\left(\frac{-4}{5^3}\right)^n$$

$$= \left(\frac{9}{125}\right)^2 \cdot \frac{1}{1-\frac{9}{125}} + 6\cdot\left(\frac{-4}{125}\right)^2 \cdot \frac{1}{1-\left(\frac{-4}{125}\right)}$$

$$= \frac{81\cdot 129 + 6\cdot 16\cdot 116}{(125)(116)(129)} \cong 0.01154$$

(3) Apply the integral test to the following series. For full credit, give the value of the integral you calculate, and write a sentence describing your conclusion.

$$\sum_{n=3}^{\infty} \frac{1}{n(1+\ln(n))^2}$$

<u>Use Integral Test:</u>

$$\int_3^\infty \frac{1}{x(1+\ln(x))^2}\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x(1+\ln(x))^2}\,dx \qquad \begin{aligned} u &= 1+\ln(x) \\ du &= \tfrac{1}{x}\,dx \end{aligned}$$

$$= \lim_{b\to\infty} \int_{1+\ln(3)}^{1+\ln(b)} \frac{1}{u^2}\,du$$

$$= \lim_{b\to\infty} -\frac{1}{u}\Bigg|_{1+\ln(3)}^{1+\ln(b)}$$

$$= \frac{1}{1+\ln(3)}$$

Since this integral converges, the above series converges by the integral test.

(4) Apply limit comparison to the following series. For full credit, label carefully the comparison sequence, calculate the relevant limit, and write a sentence describing your conclusion.

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n + 4}{\sqrt[3]{3n^9 - 5n^3 + 6}}$$

Use limit comparison:

$$a_n = \frac{2n^2+3n+4}{\sqrt[3]{3n^9-5n^3+6}} = \frac{n^2\left(2+\frac{3}{n}+\frac{4}{n^2}\right)}{\sqrt[3]{n^9\left(3-\frac{5}{n^6}+\frac{6}{n^9}\right)}} \cong \frac{n^2}{n^3}$$

Set

$b_n = \frac{1}{n}$ It is clear that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p test.

Calculate:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n+4}{\sqrt[3]{3n^9-5n^3+6}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(2+\frac{3}{n}+\frac{4}{n^2}\right)}{\cancel{n^3}\sqrt[3]{3-\frac{5}{n^6}+\frac{6}{n^9}}}\cdot \cancel{n}$$

$$= \frac{2}{\sqrt[3]{3}} > 0$$

Since this limit is positive, we know that the above series diverges by limit comparison.

(5) Determine whether or not the following series converges. For full credit, identify the test you use to reach your conclusion and explain why it is applicable.

$$\sum_{n=3}^{\infty}(-1)^{n-1}\frac{1}{\ln(n)^2}$$

Use Alternating Series test:

Here $a_n = \frac{1}{\ln(n)^2}$

i) $a_n > 0$ since it is a square! ✓

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n)^2} = 0$ ✓

iii) $a_{n+1} < a_n$.

Note: $n < n+1 \implies \ln(n) < \ln(n+1)$ since $y=\ln(x)$ is increasing

$\implies \ln(n)^2 < \ln(n+1)^2$ Since $y=x^2$ is increasing

$\implies \frac{1}{\ln(n+1)^2} < \frac{1}{\ln(n)^2}$ Take reciprocals

$\implies a_{n+1} < a_n$

This series converges because it meets the conditions of the alternating series test.

(6) Consider the following power series.

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-2)^n}{5^n}$$

a) Find the radius of convergence.

$$a_n = (-1)^n \frac{(x-2)^n}{5^n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(-1)^{n+1}\frac{(x-2)^{n+1}}{5^{n+1}}\right|}{\left|(-1)^n \frac{(x-2)^n}{5^n}\right|} = \lim_{n\to\infty} \left|\frac{5^n}{5^{n+1}}\right| \cdot \left|\frac{(x-2)^{n+1}}{(x-2)^n}\right|$$

$$= \frac{1}{5}|x-2|$$

The radius of convergence for this power series is $R = 1/\tfrac{1}{5} = 5$.

b) Find the interval of convergence (you need not consider the end-points).

This power series is centered at $x=2$

$2-5=-3 \qquad 2+5=7$

Thus the interval of convergence is:

$(-3, 7)$