As we build smaller and more powerful devices, understanding and exploiting nonlinearity has become a necessity. The evolution of ocean waves, the propagation of light in optical fibers, and the behavior of plasmas and Bose-Einstein condensates are all described by nonlinear evolution equations. As diverse as these systems are, under very generic modeling assumptions, certain canonical *integrable* nonlinear models emerge. Integrable models are invaluable because we can explore their behavior in far greater detail and with far greater precision than any other class of nonlinear systems. Integrability is not only a feature of dynamical systems; problems in probability, statistical mechanics, combinatorics and approximation theory can also be solved using integrable techniques. My work explores the behavior of these fundamental integrable models, with a particular emphasis on discovering and describing the coherent, and often universal, structures that emerge in various physical limits. Here is a brief summary of my work in the field:

**Soliton resolution and stability of solitons**

- Working with various collaborators [4, 10, 27], we have proved the soliton resolution conjecture and asymptotic stability of (multi-)soliton solutions for the focusing, defocusing and derivative Nonlinear Schrödinger (NLS) equations.
- In [26] we prove, using integrable techniques, that the derivative NLS equation is global well-posedness, removing small norm assumptions required by standard PDE techniques.
- My continuing work seeks extend the resolution conjecture to other, possibly non-integrable, systems, and to study the behavior of solutions for exceptional initial data for which soliton resolution may not occur.

**Small dispersion limits in PDEs**

- I gave the first rigorous analysis of the small dispersion limit of the focusing NLS equation for non-analytic initial data [25]; our continuing work suggests that an infinite cascade of phase transitions occur generating large rogue waves and increasingly turbulent solutions.
- We developed in [6] a WKB scheme for approximating the scattering data for the multi-field system of three wave resonant interaction (TWRI) equations; we are now using this scheme to studying the small dispersion limit of the TWRI evolution.

**Applications of orthogonal polynomials**

- In [2] we showed that the joint statistics of maximal crossings and nestings in a perfect matching–combinatorial objects related to random growth models–are asymptotically independent and computed asymptotic expansions of their joint and marginal distributions.
- With the graduate student I am supervising, we are studying problems connected to the discrete Toda and Ablowitz-Ladik lattice equations using their connection to orthogonal polynomial on the line and circle respectively.

**Approximation theory**

- Using integrable systems techniques, I found *global* asymptotic expansions of the Taylor polynomials of the Riemann $\xi$-function and other $L$-functions which give as a bi-product Riemann-von Mangoldt type estimates for the number and density of non-trivial zeros.

In what follows I first describe briefly some background material which was part of what inspired me as a student to work in integrable systems theory. Following that, I give a detailed description of the past, current and future research plans in each of the four areas highlighted above.

**Background:** Historically, the first example of an integrable system was the Korteweg-de Vries (KdV) equation, $u_t + uu_x - \epsilon^2 u_{xxx} = 0$, introduced to describe the solitary traveling waves fist observed by Russell in the Edinburgh-Glasgow canal. If the dispersion parameter $\epsilon \ll 1$, then it seems reasonable to approximate the solution by Burgers equation, $u_t + uu_x = 0$. However, the nonlinearity $uu_x$ has a steepening effect on waves and once large gradients appear, the $\epsilon^2 u_{xxx}$ term in the equation can no longer be considered perturbative. The KdV evolution, with its infinite number of conservation laws, cannot dissipate energy, and so regularizes the Burgers shock formation by generating an envelope of slowly modulated rapid oscillations, called a dispersive shock wave (DSW), which disperses the shocks energy into higher frequencies, see Figure 1. How can we describe the behavior of the solution inside this oscillatory region? Also, where in the $(x, t)$ spacetime do these oscillations develop? Approximation techniques like Whitham's averaging method [42] successfully capture the modulation of these waves, but it wasn't until the discovery of the inverse scattering transform (IST) in [17] that rigorous analysis of the system was possible. The subsequent work of [12, 21, 30] and others has provided us with a nearly complete understanding of these oscillations, and it has become clear that certain universal features emerge for broad classes of initial data [8, 20].

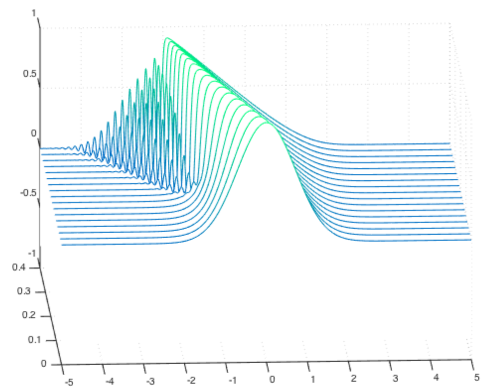
**Fig. 1.** A dispersive shock wave in KdV

## Soliton stability and the resolution conjecture

Nonlinear dispersive equations often possess solitary traveling wave solutions called solitons which do not disperse like typical solutions. The *soliton resolution conjecture*, which loosely states that the solutions of any dispersive PDE which supports solitons will resolve at large times into a sum of independent solitons plus a decaying error, has inspired a flurry of activity in the PDE community [3, 11, 14, 18, 32, 38] with many partial results.

When the evolution is integrable, we are able to make very precise statements. In a series of papers [4, 10, 27] with different collaborators we established soliton resolution for the defocusing NLS equation on a non-zero background, for vanishing solutions of focusing NLS, and most recently, for vanishing solutions of the derivative NLS equations for initial data in dense open subsets of the certain weighted Sobolev spaces. Our results describe both the leading order train of solitons and the leading order dispersive corrections. The technical innovation we made in this series of papers was to extend the DBAR generalization of the Deift-Zhou steepest descent method to incorporate solitons in a uniform and controlled way. Our results also lead to relatively simple proofs of the asymptotic stability of (multi-)soliton initial data compared to traditional PDE methods for proving such results.

Of particular interest is the last of the above works in which we studied the derivative NLS (DNLS) equation. By carefully studying the mapping properties of the forward and inverse spectral maps associated with the IST for DNLS we were able to establish global well-posedness in an open dense subset of the Sobolev space $H^{2,2}(\mathbb{R})$ [26]. Our result removes small data assumptions that had been necessary using traditional PDE techniques.

There are many open problems in this direction that I am trying to understand. Problem 1: What happens to the soliton resolution conjecture for non-generic data? Non-generic data, for which our above results break down, are characterized by having *spectral singularities* and can have infinitely many accumulating solitons and localized solutions with 'fat tails'. Formal scattering theory for such data goes back to [43], but asymptotic methods for extracting information about the evolution have not been developed. We believe that the DBAR steepest descent techniques we've developed are flexible enough to start tackling these problems. As a first step we're studying soliton resolution for the algebraic solitons of DNLS. Problem 2: How stable are integrable systems to non-integrable perturbations? This has been one of the

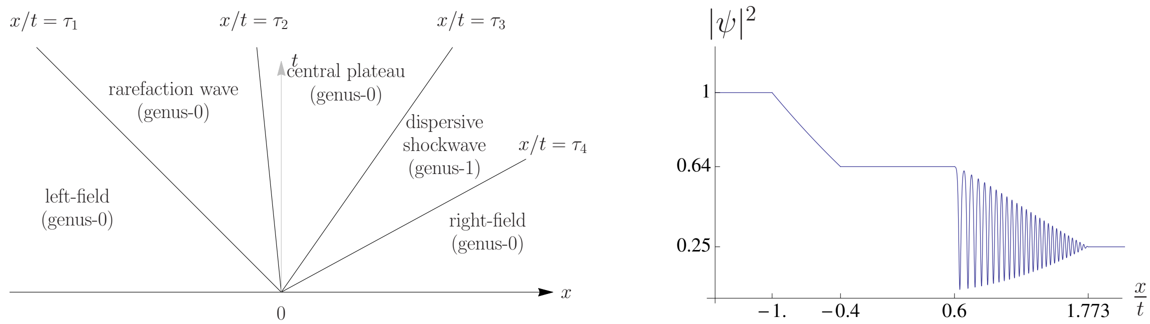


**Figure 2.** The regularization of Riemann (shock) initial data (3) by the defocusing NLS evolution (1). For the chosen parameters ($A = 0.5$, $k = 0.1$) the shock is regularized by a rarefaction on the left and a dispersive shock wave on the right. The dispersion parameter is $\epsilon = 0.001$.

outstanding problems in integrable systems theory for some time. The best result in this direction [13] is for a system without solitons, and essential elements of their proof break down in the presence of solitons. We hope that the techniques we develop studying spectral singularities will shed some light on how to best tackle this problem.

## Small dispersion limits of PDEs

**Dispersive shock waves in hyperbolic systems** like those for KdV in Fig. 1 regularize shock formation by the emergence of slowly modulated rapidly oscillating wavetrains. Whitham averaging theory and matched asymptotic methods have been very successful at describing these waves. However, they fail to capture certain 'slow' properties of the wave which are lost in the averaging process, nor can they estimate the error incurred in the averaging process to establish convergence.

Integrable systems theory can be informed by and improve the results predicted of Whitham theory. In this direction, I studied the Riemann-problem for the defocusing ($\sigma = -1$) nonlinear Schrödinger equation (NLS)

$$i\epsilon\psi_t + \epsilon^2\psi_{xx} + 2\sigma|\psi|^2\psi = 0. \tag{1}$$

The small dispersion limit $\epsilon \downarrow 0$ of (1) is clearer in the hydrodynamic variables $(\rho, u)$ defined by $\psi = \sqrt{\rho}\exp\left(\frac{i}{\epsilon}\int^x u\right)$, where equation (1) takes the form

$$\tfrac{1}{2}\rho_t + (\rho u)_x = 0 \qquad \tfrac{1}{2}u_t + \left(\tfrac{1}{2}u^2 - \sigma\rho\right)_x = \epsilon^2\left(\frac{\rho_x^2}{8\rho^2} - \frac{\rho_{xx}}{4\rho}\right)_x. \tag{2}$$

In this context, the Riemann problem means studying the long time evolution of piecewise plane wave initial data

$$\psi_0(x) = \begin{cases} 1 & x < 0 \\ Ae^{-2ikx/\epsilon} & x > 0 \end{cases} \quad \to \quad (\rho_0, u_0)(x) = \begin{cases} (1, 0) & x < 0 \\ (A^2, -2k) & x < 0. \end{cases} \tag{3}$$

In [23] I derived leading order asymptotic description of the small-dispersion limit of the solution of defocusing (1) for Riemann initial data (3). My result corrects the Whitham theory prediction of [16] by introducing a slowly varying complex phase $\phi = \phi(x/t)$ which, though lost in the Whitham averaging, is explicitly computed by inverse scattering methods. Furthermore, the integrable techniques I employed give explicit error bounds between the approximate and true solutions which rigorously justify the result. Figure 2 shows the long time asymptotic behavior of the regularized shock for a particular choice of initial data (3).

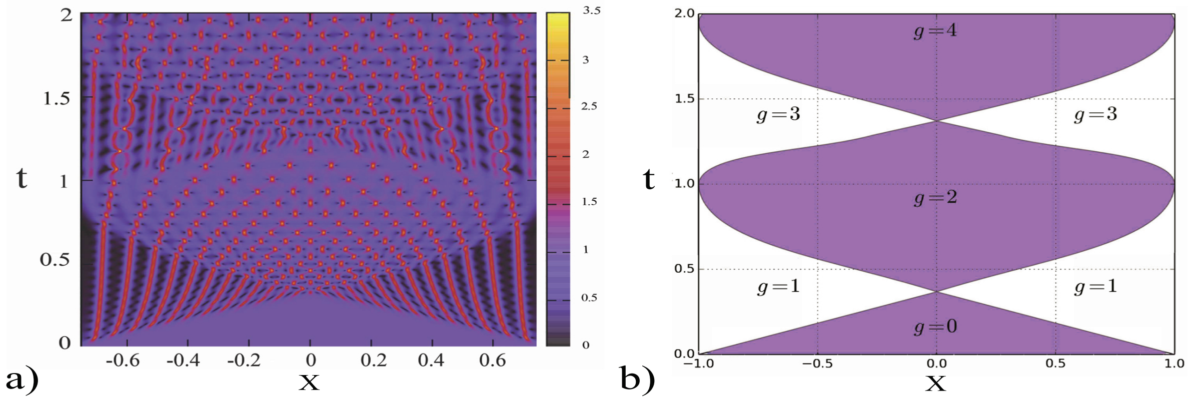


**Figure 3.** The solution of the focusing ($\sigma = 1$) NLS equation (1) for box initial data $\psi_0 = q\chi_{[-L,L]}$ regularizes the initial shock by the formation of two counter propagating one phase waves. The collision of these shock waves is a mechanism for rogue wave generation. *a*) Density plot of $|\psi| = \sqrt{\rho}$ with $q = 1$, $L = 25/33$ and $\epsilon = 1/60$. *b*) The number of wave phases, $g$, and breaking curves suggested by the numerics (normalized to $L = q = 1$).

**Small dispersion limits of non-hyperbolic systems** are more difficult to analyze than problems like KdV and defocusing NLS equation which have hyperbolic zero dispersion limits. I have work extensively on the small dispersion limit of the focusing ($\sigma = 1$) NLS equation (1). The focusing NLS equation is modulationally unstable, which follows from the fact that when $\sigma = 1$, the limiting hydrodynamic equations (2) for NLS are elliptic, and the Cauchy problem is ill-posed. However, numerical simulations [5, 9] suggest that as $\epsilon \downarrow 0$ in (1), the solution possesses a great deal of structure. *Breaking curves*, or nonlinear caustics, emerge which act like phase transitions between regions of space-time in which the solution has disparate wave characteristics. Describing these breaking curves and the modulated wavetrains which describe the solution between breaking curves are the essential questions to be addressed in this setting. The first rigorous studies [28, 31, 39, 40] focused on analytic initial data, avoiding the ill-posedness issue.

In [25], with Ken Mclaughlin, I gave the first rigorous computation of the small dispersion limit of focusing NLS for initial data outside the analytic class. Specifically, I described the evolution of 'box' initial data $\psi_0(x) = q\chi_{[-L,L]}(x)$, showing that the discontinuities are regularized by the instantaneous emergence of single phase (elliptic) modulated waves which propagate into the support of the initial data; the solution remains quiescent for $|x| > L$ for all time. When these counter propagating waves collide there is a increase in the genus, i.e., the number of phases in the wavetrain.

This line of research is far from complete. High genus solutions of focusing NLS possess local waves much larger than the background oscillations and have been proposed as a model of *rogue wave* generation in oceans, In [15], exactly my problem was proposed as a deterministic model for generating rogue waves and it gave numerical evidence to support transitions to higher genus, see Figure 3. Problem: can we analytic prove these transitions occur? I am currently working with Alexander Tovbis, one of the authors of [15], to show that at sufficiently large times, the solution of (1) with box initial data will develop arbitrarily many phases, generating increasingly large rogue waves. If successful this would be the first analytically understood example where 'integrable turbulence', in the sense of Zakharov [1], develops from a non-turbulent initial state.

**Extending the IST to coupled systems** is a challenging subject. With Peter Miller and Robert Buckingham, I am studying the small dispersion limit of the three wave resonant interaction (TWRI) equations:

$$\epsilon\psi_j^*(\partial_t + c_j\partial_x)\psi_j = \gamma_j\psi_1^*\psi_2^*\psi_3^* \qquad j = 1, 2, 3, \tag{4}$$

where $(\gamma_j)^2 = \pm 1$ and the $c_j$'s are distinct wave speeds. The TWRI have a wide variety of physical applications [29], following from the fact that resonant wave coupling is a such basic nonlinear phenomenon, and that the quadratic nonlinearity is the first to be excited.

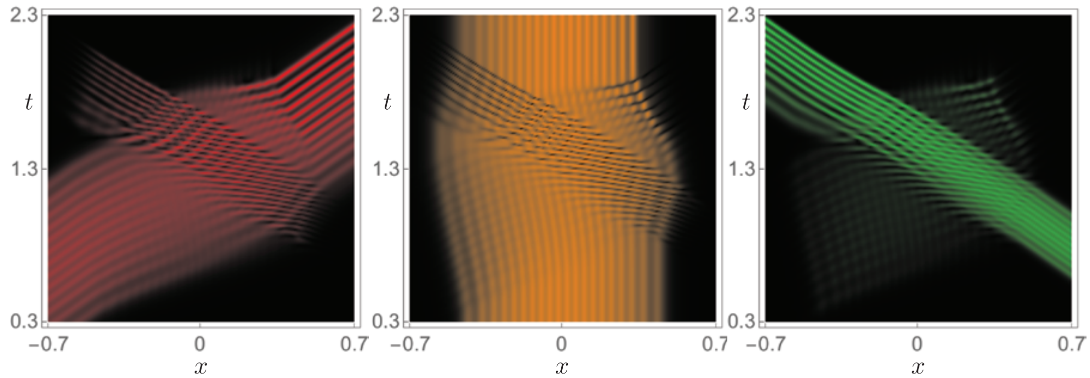


**Figure 4.** Spacetime plots of $|\psi_j(x,t)|$ for an exact soliton ensemble solution of the TWRI equations (4) approximating the collision of three initial separated semicircular fields $\psi_j$, $j = 1,2,3$ (pictured left-to-right). Here $\epsilon = 1/80$.

Though the formal IST has been worked out for several systems of higher order [23, 36], effective asymptotic methods for extracting useful information from the formal solutions has thus far been restricted to system that are either scalar or $2 \times 2$ Lax-Pair systems. We have recently [6] described an effective WKB strategy for computing approximate scattering data in the small dispersion limit of the TWRI for a special class of initial data. This approximate scattering data corresponds to a *soliton ensemble*, an exact multi-soliton solution of the TWRI equations in which the number of solitons grows like $\mathcal{O}\left(\epsilon^{-1}\right)$, as $\epsilon \downarrow 0$.

There are many open questions to address here. Problem 1: Can we prove analytically that the soliton ensembles we've constructed converge to the original initial data as $\epsilon \downarrow 0$? How do the ensembles evolve in time? This will involve modifying the so called $g$-function mechanism akin to what has been done for the study of multiply orthogonal polynomials [41]. Interestingly, our numerical simulations of these soliton ensembles, see Figure 4, show regions of quiescent, slowly varying, and rapidly oscillatory oscillations reminiscent of the dispersive shock waves seen in KdV and NLS, although the physical mechanism for their generation must be different since the TWRI is dispersionless. Problem 2: Can we develop WKB methods to address the TWRI equations or other coupled systems directly? A limitation of our current work is that it relies on reductions to lower order systems to make effective computations which places a strong restriction on the class of initial data we can consider. It's far from clear how this would be done, but there exist abstract formal methods for computing scattering data which have yet to be studied in the small-dispersion limit. Even marginal progress in this direction would open avenues for studying many open problems.

## Orthogonal Polynomials and Random Matrix Theory

Integrable systems theory is not only concerned with PDEs, the same methods that have grown up to describe the IST for integrable PDE can be used to analyze orthogonal polynomial problems intimately related with random matrix theory, statistical mechanics, determinantal point processes and growth models like TASEP and its connection to KPZ.

**Random complete matchings** of $[2n]$ are pairings $M = \{(i_1, j_1), \ldots, (i_n, j_n)\}$. Given a matching $M$, a subset $\{(i_{s_1}, j_{s_1}), \ldots, (i_{s_r}, j_{s_r})\}$ is called an *r-crossing* if $i_{s_1} < \cdots < i_{s_r} < j_{s_1} < \cdots < j_{s_r}$ and an *r-nesting* if $i_{s_1} < \cdots < i_{s_r} < j_{s_r} < \cdots < j_{s_1}$. Let $\mathrm{cr}_n(M)$ be the largest number $k$ such that $M$ has

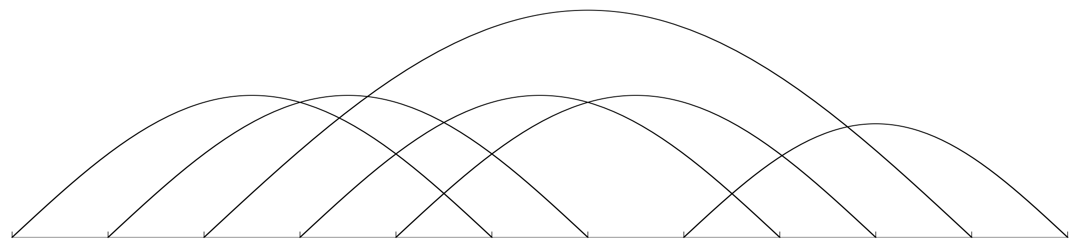


**Figure 5.** A complete matching $M$ of [12]. In this sample $\mathrm{cr}_6(M) = 4$, achieved by $\{(1,6),(2,7),(4,9),(5,10)\}$, and $\mathrm{ne}_6(M) = 2$, achieved by $\{(3,11),(4,9)\}$.

a $k$-crossing (maximal crossing) and $\mathrm{ne}_n(M)$ denote the largest number $j$ such that $M$ has a $j$-nesting (maximal nesting). See Figure 5 for an example. Various combinatorial properties of $\mathrm{cr}_n$ and $\mathrm{ne}_n$ were studied in [7]. There are many connections between these numbers and other combinatorial questions of interest, among them the longest decreasing subsequence of a fixed point free involution and the height of the polynuclear growth (PNG) model with a flat initial condition.

Equipping the space $\mathcal{M}_n$ of all such matchings with uniform measure, and allowing the size to be $2\mathcal{N}$ where $\mathcal{N}$ is a Poisson random variable with rate $t^2/2$, the maximal crossing and nesting numbers $\mathrm{CR}_t = \mathrm{cr}_{\mathcal{N}}$ and $\mathrm{NE}_t = \mathrm{ne}_{\mathcal{N}}$ become random variables. With Jinho Baik I analyzed the asymptotic distributions of joint and marginal statistics of these randomized crossing and nesting numbers as the Poisson rate parmeter $t \to \infty$. The principal results of my work were to establish the asymptotic independence of the joint distribution of the maximal crossing and nesting numbers and to give an explicit first correction to the limiting Tracy-Widom GOE behavior of the marginal distributions of each of the crossing and nesting numbers. For instance, letting $\widehat{\mathrm{CR}}_t = 2^{-1}t^{-1/3}(\mathrm{CR}_t - t)$ we show that the marginal distribution satisfies $\mathbb{P}\left(\widehat{\mathrm{CR}}_t < x_t\right) = F_1(x_t) + t^{-2/3}E(x_t) + \mathcal{O}\left(t^{-1}\right)$ where $F_1$ is the Tracy-Widom GOE distribution, $E(x)$ is an explicit function related to Painlevé II, and $x_t$ is a certain fine translation of $x$ by a term of size $t^{-1/3}$.

**Schur flows and the Ablowitz-Ladik equation** are intimately related to orthogonal polynomials on the circle [19]. An essential ingredient in the analysis I carried out in [2], was the careful analysis of the monic orthogonal polynomials $\pi_n(z,t)$ on the unit circle associated with the discrete measure $d\mu(t) = \frac{1}{2m}\sum_{r=0}^{2m-1} e^{t(z+z^{-1})}\delta_{\omega^r}(z)$ where $\omega$ is a primitive $2m^{\text{th}}$ root of unity. We observed at the time, but did not pursue the connection to Schur flows. To wit, the Verblunsky coefficients $\alpha_n(t) := -\pi_n(0;t)^*$ satisfy the discrete system

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = (1 - |\alpha_n|^2)(\alpha_{n+1} - \alpha_{n-1}), \tag{5}$$

for initial boundary value data:

$$\alpha_{-1}(t) = -1, \qquad \alpha_{2m-1}(t) = 1, \qquad \alpha_k(0) = 0, \text{ for } 1 < k < 2m-2. \tag{6}$$

Of course, both the Schur flow and Ablowitz-Ladik systems are integrable in their own right [34]. In the context of Ablowitz-Ladik the system (5)-(6) describes in the large $m$ limit, the interaction of two counter propagating wave fronts. In work in preparation [22] I am in the process of completing the asymptotic description of the discrete orthogonal polynomials $\pi_n(z;t)$ began in [2], which will give a complete description of these colliding shock fronts.

**Singular orbits of the finite Toda lattice** As part of a separate project, I am also currently overseeing the doctoral thesis project of Kyle Pounder, who is studying the discrete Toda lattice of $N$ particles:

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \exp(x_{n+1} - x_n) - \exp(x_n - x_{n-1}), \quad n = 1, \ldots, N, \tag{7}$$

where for simplicity we set $x_0(t) = -x_{N+1}(t) \equiv \infty$. The Toda lattice is a special case of the Fermi-Pasta-Ulam lattice which supports soliton solutions, and is a simple model in statistical mechanics of one dimensional crystals. The Toda lattice was shown to be integrable by Flaschka, and the inverse scattering transformation can be encoded into a family of orthogonal polynomials on the real line with respect to a discrete point measure with exponentially evolving weights. Pounder's thesis results provide a sharper description of the classical scattering results of Moser [33], and go on to give a very detailed description of the sorting process of the Toda evolution. In the future we plan to study the continuum limit as the number of particles $N \to \infty$, and separately to study the effects of randomness on the initial data. This in turn should have interesting connections to the recent work of Deift and Menon [35] on the time needed to compute the eigenvalues of a symmetric matrix using the Toda algorithm.

**Uniform asymptotic descriptions of Taylor polynomials.** The classical convergence results on Taylor polynomials work on fixed size compact sets. Szegő [37] instead considered the *global* asymptotic behavior of the Taylor polynomials of $e^z$; his work was later extended by others to describing the Taylor series of $\cos(z)$ and $\sin(z)$. Using modern Riemann-Hilbert techniques from integrable systems theory, I derived a uniform asymptotic description of the Taylor polynomials of the Riemann $\xi$-function valid everywhere in the complex plane [24]. This description gives interesting estimates on the nontrivial zeros of the Riemann $\zeta$-function and other $L$-functions, allowing us to prove results similar to the Riemann-von Mangoldt theorem.

**In summary,** the ability of integrable methods to give detailed and rigorous descriptions of complex, nonlinear systems and its versatility to answer questions in diverse areas of mathematics is what drew me to the topic and continues to inspire my interest. Integrable systems research bridges the artificial gap between pure and applied mathematics. My favorite problems blur these lines entirely; problems that come from canonical physical models but present difficult and complex mathematical challenges. I try in my research to apply integrable methods in new ways and to extend techniques beyond their usual reach. Nonlinear physics govern the most interesting and surprising phenomena in the natural world and integrable methods provide me with a way to explore their behavior.

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