Introduction to Hilbert Space Frames

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What is a frame?

A frame is a generalization of a Hilbert space basis that is not necessarily linearly independent. A frame allows us to represent any vector as a set of "frame coefficients," and to reconstruct a vector from its coefficients in a numerically stable way.

Why frames?

Often want to decompose a function in terms of functions that are not linearly independent.
Examples:

- Windowed Fourier transforms
- Wavelet transforms
- Non-uniform sampling

Coefficient Representations of Hilbert Space Vectors

Often useful to represent a vector by a sequence of coefficients.
Familiar in $\mathbb{R}^n$; consider

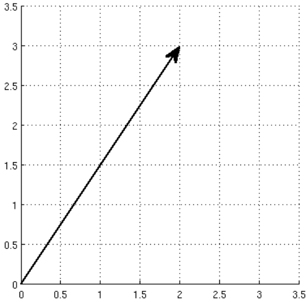

Figure: The vector (2,3) in $\mathbb{R}^2$, equal to $2e_1 + 3e_2$

Another Example: Fourier coefficients

In $L^2(0,1)$, the functions $\phi_n = e^{2\pi in}$ ($n \in \mathbb{Z}$) form an orthonormal basis. We can represent functions in terms of their Fourier coefficients:

$$c_n = \langle \phi_n, f \rangle$$

This is the familiar Fourier series. Coefficients tell us the "frequency content" of a function.
Function can be reconstructed from its coefficients using

$$f = \sum_n c_n e^{2\pi in}$$

We determine coefficient representations by taking inner products with some set of vectors ϕ_n. What sets of vectors will give suitable coefficients?

Frames! Frame theory gives necessary and sufficient condition for ϕ_n to give "suitable" coefficients for every $f \in \mathcal{H}$.

The Frame Operator

For a Hilbert space $\mathcal{H}$ and a set of vectors $\{\phi_n\}$, define a mapping F such that $\forall x \in \mathcal{H}$,

$$(Fx)_n = \langle \phi_n, x \rangle.$$

If the ϕ_n meet the "frame condition," then F is called a frame operator. Note that F is linear.

What Vectors Give Suitable Coefficients?

We want to determine which sets of vectors ϕ_n can be used to represent other vectors in a "suitable" way.

1. Want *unique* coefficients for any $f \in \mathcal{H}$
   - Frame operator should be *injective*
2. Want to be able to go from vector to coefficients, and back again, in a *numerically stable* way
   - Frame operator should be invertible, at least on its range
   - Frame operator and its pseudo-inverse should be *bounded*

For this we need the *frame condition*.

The Frame Condition

A set of vectors $\{\phi_n\} \in \mathcal{H}$ is called a *frame* if there exist $0 < A, B < \infty$ such that

$$A\|x\|^2 \leq \|Fx\|^2 \leq B\|x\|^2$$

where

$$\|Fx\|^2 = \sum |\langle \phi_n, x \rangle|^2$$

for all $x \in \mathcal{H}$.
If $A = B$, ϕ_n is said to be a *tight frame*.

Why this frame condition?

- $\|Fx\|^2 \geq A\|x\|^2$ guarantees *uniqueness* of frame coefficients:

$$\|Fx - Fy\| = 0 \implies \|x - y\| = 0$$

- $\|Fx\|^2 \leq B\|x\|^2$ guarantees *numerical stability* in computing frame coefficients, as long as B is reasonable:

$$\|F(x+\delta) - F(x)\| = \|F\delta\| \leq \sqrt{B}\|\delta\|$$

- Also have stability when reconstructing a vector from its coefficients; we will see that F has a left inverse bounded by $\frac{1}{\sqrt{A}}$.

Bases are frames

Any basis is also a frame. An orthonormal basis[1] is a tight frame with $A = B = 1$. However, a frame does not have to be a basis.

[1] An orthonormal basis is sometimes *defined* as a set of orthonormal vectors satisfying the frame condition with $A = B = 1$

Not every frame is a basis

There are frames which are not bases. Let

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

This is a frame operator, and its rows are the frame vectors. Frame bounds come from the singular value decomposition:

$$\sigma_1 = \sqrt{3}, \sigma_2 = 1$$

so

$$\|x\|^2 \leq \|Fx\|^2 \leq 3\|x\|^2$$

A frame in L^2 which is not a basis

A *windowed Fourier frame* of L^2 has the form

$$\{g_{n,k}(t) = g(t - nu_0)e^{ik\xi_0 t}\}$$

for some window function g(t). This is useful for performing *localized* frequency analysis.

The frame coefficients $(Fx)_n$ can be used to reconstruct any x. But how?

For any frame $\{\phi_n\}$ and any vector x, we have the reconstruction formula

$$x = \sum_n (Fx)_n \tilde{\phi}_n$$

The $\tilde{\phi}_n$ are *not necessarily equal to the* ϕ_n! They make up the *dual frame*.

The Frame Operator's Adjoint

To define the dual frame, we will need the adjoint of F.
For a frame ϕ_n with frame operator F, the adjoint is the function $F^* \; l^2 \to \mathcal{H}$ given by

$$F^* c = \sum_j c_j \phi_j$$

The Dual Frame

For a frame ϕ_n with frame operator F, the dual frame is

$$\tilde{\phi}_n = (F^*F)^{-1}\phi_j$$

This is also a frame. Its frame operator[2] is $\tilde{F} = F(F^*F)^{-1}$. Its frame bounds are given by

$$\frac{1}{B}\|x\|^2 \leq \|\tilde{F}x\|^2 \leq \frac{1}{A}\|x\|^2$$

[2] Note that if the frame is an orthonormal basis, F is unitary and $\tilde{F} = F$!

Reconstruction formula

The frame operator and the dual frame operator satisfy

$$\tilde{F}^* F = F^* \tilde{F} = I$$

which gives the reconstruction formula[3] for any $x \in \mathcal{H}$:

$$x = \sum_j \langle \phi_j, x \rangle \tilde{\phi}_j = \sum_j \langle \tilde{\phi}_j, x \rangle \phi_j$$

[3] If the frame is an orthonormal basis, this reduces to $x = \sum_j \langle \phi_j, x \rangle \phi_j$

Example of a dual frame

Returning to our earlier example, with

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

the dual frame operator is

$$F = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}.$$

The dual frame vectors are the rows of this matrix.

Example of a dual frame (cont.)

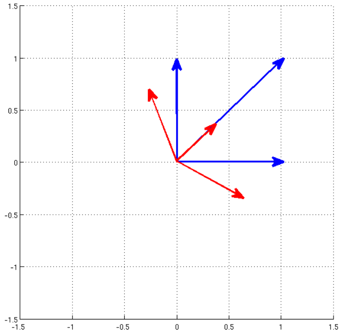

Figure: Frame from example (blue) and dual frame (red)

Say we want to expand $x = (1,2)^T$ in terms of this frame. We have $\tilde{F}x = (0,1,1)^T$, so

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The dual set of coefficients is $Fx = (1,2,3)^T$, so the expansion in terms of the dual frame vectors is

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix} + 2 \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix} + 3 \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}.$$

Classification of Euclidean space frames

A finite set of vectors $\{\phi_n\} \in \mathbb{R}^n$ is a frame iff it spans $\mathbb{R}^n$ and satisfies

$$\sum_n \|\phi_n\|^2 < \infty$$

Thus, any finite spanning set is a frame, and an infinite set can be a frame as long as the magnitudes of the frame vectors decay sufficiently quickly.

A special class of frames called Finite Normalized Tight Frames (FNTFs) exhibit a great deal of symmetry. These are tight frames consisting of finitely many unit vectors. Classification of these is more difficult, but has been done; see paper "Finite Normalized Tight Frames" in bibliography.

Some interesting results about FNTFs in $\mathbb{R}^n$:

- FNTFs in $\mathbb{R}^n$ consisting of k vectors exist for all $k \geq n$
- Frame bound of a FNTF measures "redunancy" of frame vectors; given by $A = \frac{k}{n}$

- The FNTFs with A $= 1$ are precisely the orthonormal bases in $\mathbb{R}^n$
- The vertices of any Platonic solid are a frame in $\mathbb{R}^n$, as well as the vertices of some other highly symmetric shapes like the "soccer ball"
- Problem of classifying FNTFs is related to problem of equidistribution; how to "evenly" distribute k vectors on the unit sphere?

The End

Thanks to Dr. Faris for advising this project.

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- Ten Lectures on Wavelets, Ingrid Daubechies
- "Finite Normalized Tight Frames", J. Benedetto & M. Fickus, Advances in Computational Mathematics 18:357-385, 2003