

## HW #2 Answer Key

- 1 a) Since  $f$  is absolutely continuous except for jump discontinuities,

$$c_k(f) = O(k^{-1}) \text{ as } k \rightarrow \infty.$$

Comments: Let  $s(x)$  be the sawtooth

wave 
$$s(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi} & \text{if } -\pi < x < 0 \\ \frac{1}{2} - \frac{x}{2\pi} & \text{if } 0 < x < \pi \end{cases}$$

then  $s(x)$  has a jump of 1 at  $x=0$ , and we know that  $c_k(s) = \begin{cases} 0 & \text{if } k=0 \\ \frac{1}{2\pi i k} & \text{else.} \end{cases}$

$f(x)$  in problem (1a) has jumps of 2 at  $x=0$  and -5 at  $x=\pi$ , so

$$g(x) := f(x) - 2s(x) + 5s(x-\pi)$$

is absolutely continuous. Also  $g'(x)$

is A.C. except for jumps, so

$$c_k(g) = O(k^{-2}).$$

Therefore

$$c_k(f) = 2c_k(s) - 5c_k(s(x-\pi)) + c_k(g)$$

$$= 2\left(\frac{1}{2\pi i k}\right) - 5\left(e^{i\pi k} \frac{1}{2\pi i k}\right) + O(k^{-2})$$

$$= \left(\frac{1}{\pi i} - (-1)^k \frac{5}{2\pi i}\right) k^{-1} + O(k^{-2})$$

b) The overshoot of a function at some jump discontinuity can be calculated by

$$\begin{aligned}\text{Overshoot}(f; x) &= \left( \frac{\text{Si}(\pi)}{\pi} - \frac{1}{2} \right) \times |\text{Jump}(f; x)| \\ &\approx 0.089 \times |\text{Jump}(f; x)|\end{aligned}$$

Since  $\text{Jump}(f; 0) = -2$ , the overshoot of  $f$  at  $x=0$  is

$$\begin{aligned}\text{Overshoot}(f; 0) &\approx 0.089 \times 2 \\ &\approx 0.18\end{aligned}$$

$$\begin{aligned}\text{Since } \text{Jump}(f; \pi) &= 5, \\ \text{Overshoot}(f; 5) &\approx 0.089 \times 5 \\ &\approx 0.45\end{aligned}$$

These are the  $x$ -values near which  $S_n(f)$  fails to converge uniformly.

2 The Parseval identity for sine & cosine series is

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \frac{1}{4} |a_0|^2 + \frac{1}{2} \sum_{k \geq 1} |a_k|^2 + |b_k|^2$$

$$\text{Here } g(x) = \begin{cases} 1 & \text{if } 0 \leq x < \pi \\ -1 & \text{if } \pi \leq x < 2\pi \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} |g(x)|^2 dx = \frac{1}{2\pi} \int_0^{2\pi} 1 dx = 1$$

So Parseval tells us that

$$\begin{aligned} 1 &= \frac{1}{4} |a_0|^2 + \frac{1}{2} \sum_{k \geq 1} |a_k|^2 + |b_k|^2 \\ &= \frac{1}{4} \cdot 0 + \frac{1}{2} \sum_{k \geq 1} \begin{cases} \left| \frac{4}{k\pi} \right|^2 & \text{if } k \text{ is odd} \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$1 = \frac{1}{2} \sum_{n \geq 0} \left( \frac{4}{(2n+1)\pi} \right)^2 \quad \left( \begin{array}{l} \text{Since } k \text{ is odd,} \\ \text{let } k = 2n+1. \end{array} \right)$$

$$= \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n \geq 0} \frac{1}{(2n+1)^2}$$

3

We'll solve

$$u_t = u_{xx}$$

$$u(x,0) = \begin{cases} 100 & \text{if } 0 \leq x < .5 \\ 0 & \text{if } .5 \leq x \leq 1 \end{cases}$$

$$u(0,t) = 100$$

$$u(1,t) = 0$$

Notice that  $v(x,t) := 100 - 100x$ 

solves

$$v_t = v_{xx}$$

$$v(x,0) = 100 - 100x$$

$$v(0,t) = 100$$

$$v(1,t) = 0.$$

Then  $w(x,t) := u(x,t) - v(x,t)$ 

solves

$$w_t = w_{xx}$$

$$w(x,0) = \begin{cases} 100x & \text{if } 0 \leq x < .5 \\ 100x - 100 & \text{if } .5 \leq x \leq 1 \end{cases}$$

$$w(0,t) = w(1,t) = 0$$

We must find a Fourier sine series for  $w(x,0)$ .

$$\begin{aligned} b_k &= 2 \int_0^1 w(x,0) \sin(2\pi kx) dx \\ &= 2 \int_0^{1/2} 100x \sin(2\pi kx) dx \\ &\quad + 2 \int_{1/2}^1 (100x - 100) \sin(2\pi kx) dx \end{aligned}$$

$$= \frac{-100(-1)^k}{k\pi}$$

$$w(x,0) = \sum_{k \geq 1} \frac{100(-1)^{k-1}}{k\pi} \sin(2\pi kx)$$

Notice that each function

$$\sin(2\pi kx) e^{-(2\pi k)^2 t}$$

solves  $u_{xx} = u_t$ . Then so does

$$w(x,t) = \sum_{k \geq 1} \frac{100(-1)^{k-1}}{k\pi} \sin(2\pi kx) e^{-(2\pi k)^2 t}$$

Finally, the solution  $u(x,t)$  to our original problem is

$$u(x,t) = 100 - 100x + \sum_{k \geq 1} \frac{100(-1)^{k-1}}{k\pi} \sin(2\pi kx) e^{-(2\pi k)^2 t}$$