Math extra practice problems #1

1. Define the following 2π-periodic functions by their values for $-\pi < x < \pi$:

$$\begin{aligned} f(x) =& x, & -\pi <& x < \pi \\ g(x) =& \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}, & -\pi <& x < \pi \\ h(x) =& \sqrt{\pi^2 - x^2} & -\pi <& x < \pi \end{aligned}$$

For each of the following statements, answer "True" or "False".

(a) By Dirichlet's pointwise convergence theorem, $S_n f(x) \to \frac{1}{2} f(x^+) + \frac{1}{2} f(x^-)$ for all x.

(b) By Dirichlet's pointwise convergence theorem, $S_n g(x) \to \frac{1}{2} g(x^+) + \frac{1}{2} g(x^-)$ for all x.

(c) By Dirichlet's pointwise convergence theorem, $S_n h(x) \to \frac{1}{2} h(x^+) + \frac{1}{2} h(x^-)$ for all x.

(d) The Fourier series for f converges uniformly.

(e) The Fourier series for g converges uniformly.

(f) The Fourier series for h converges uniformly.

(g) The Fourier series for f converges with respect to L^2 norm.

(h) The Fourier series for g converges with respect to L^2 norm.

(i) The Fourier series for h converges with respect to L^2 norm.

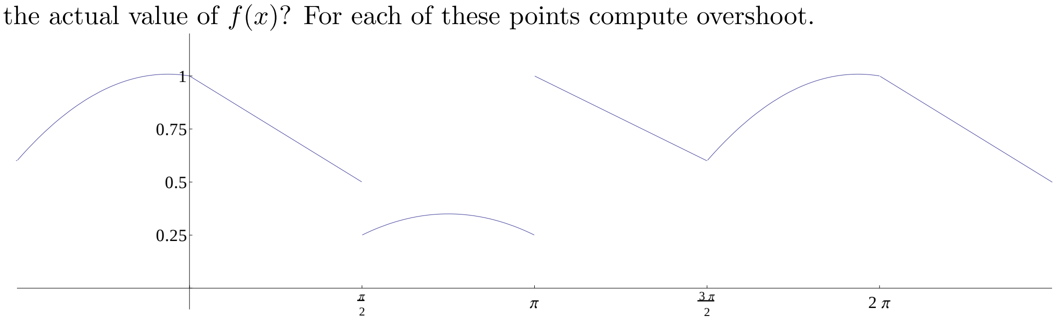
2. The following Fourier series holds for $-\pi \leq x < \pi$

$$x(\pi - |x|) = \sum_{k \geq 0} \frac{8}{\pi(2k+1)^3} \sin((2k+1)x)$$

(a) Evalulate both sides of the above expression at $x = \pi/2$ to get the the value of an infinite sum.

(b) Apply Parseval's identity to the functions to get the value of another infinite sum.

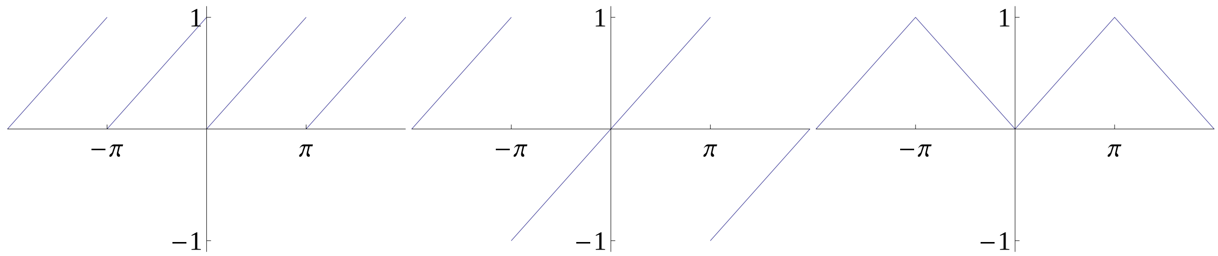
3. The graph below shows a 2π-periodic function $f(x)$. Where does the Fourier series for $f(x)$ overshoot the actual value of $f(x)$? For each of these points compute overshoot.



4. Give a Fourier series to the following 1-d heat problem where $u(x,t)$ is the temperature of a rod at position $0 \le x \le \pi$ and time $t \ge 0$.

$$\begin{aligned} u_t =& 3u_{xx} \\ u(x,0) =& x/\pi \\ u_x(0,t) =& 0 \\ u_x(\pi,t) =& 0 \end{aligned}$$

You can use any of the following Fourier series in your solution (The boundary conditions tell you which one to use).



$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-1}{\pi k}\sin(2kx), \qquad g(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k}\sin(kx), \qquad h(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4\cos((2k+1)x)}{\pi^2(2k+1)^2}$$

5. Suppose that 3 standard 52 card decks of playing cards are shuffled together into a pile
   (a) How many different 5 card poker hands could possibly be dealt from this pile?
   (b) What is the probability of 5-of-a-kind?
   (c) What is the probability of 2 pairs?

6. Suppose that 10% of patients in a hospital have disease X. If a patient has disease X, there is a 90% chance he/she will test positive. If the patient does not have disease X, then there is a 95% chance he/she will test negative. If a patient tests positive, what is the conditional probability that he/she has disease X?

7. Let $f(x) = x^3 - 2x$ and X be a continuous random variable with Uniform$[0,2]$ distribution.
   (a) What is the probability density of X?
   (b) Compute the expected value of $f(X)$.

8. Suppose a student is typing a document. On average, his typing has 0.23 errors per page. Suppose he types a ten page document. Let X be the number of errors he makes in typing this document.
   (a) Explain why it is reasonable to model X with a Poisson random variable.
   (b) Calculate the probability $P\{X \le 1\}$
   (c) Calculate the probability $P\{X > 3\}$