Exam #1 Solutions
Math for IPSA exchange 2014

1 (10 points)

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{0} e^{-ikx}\, dx + \frac{1}{2\pi}\int_{0}^{\pi} -e^{-ikx}\, dx$$
$$= \frac{1}{2\pi}\left(\frac{e^{0}}{-ik} - \frac{e^{ik\pi}}{-ik}\right) + \frac{1}{2\pi}\left(\frac{-e^{-ik\pi}}{-ik} - \frac{-e^{0}}{-ik}\right)$$
$$= \frac{(-1)^k - 1}{\pi i k}$$

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k - 1}{\pi i k}\, e^{ikx}$$

2 (12 points, 2 for each true/false)

a True
b True
c True
d False
e False
f True

3 (10 points, 5 for each part a, b)

a $$x^2 = \frac{\pi^2}{3} + \sum_{k\geq 1} \frac{4(-1)^k}{k^2}\cos(kx)$$

$$0 = \frac{\pi^2}{3} + \sum_{k\geq 1} \frac{4(-1)^k}{k^2}\cos(0)$$

$$\Rightarrow \sum_{k\geq 1} \frac{(-1)^k}{k^2} = \frac{-\pi^2}{12}$$

3 b Parseval identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = |a_0|^2 + \frac{1}{2}\sum_{k\geq 1} |a_k|^2 + |b_k|^2$$

Left hand side:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x^2|^2\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x^4\,dx$$

$$= \pi^4/5$$

Right hand side:

$$|a_0|^2 + \frac{1}{2}\sum_{k\geq 1} |a_k|^2 + |b_k|^2$$

$$= \left|\frac{\pi^2}{3}\right|^2 + \frac{1}{2}\sum_{k\geq 1}\left|\frac{4(-1)^k}{k^2}\right|^2 + 0^2$$

$$= \frac{\pi^4}{9} + 8\sum_{k\geq 1}\frac{1}{k^4}$$

Conclusion

$$\sum_{k\geq 1}\frac{1}{k^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}$$

4 (12 points)

x	Jump$(f;x)$	Overshoot$(S_nf;x)$
0	1	$\approx 1 \times 0.0895$
π	$1/4$	$\approx \frac{1}{4} \times 0.0895$
$3\pi/2$	$1/2$	$\approx \frac{1}{2} \times 0.0895$

5 (10 points)

For $u_t = 3u_{xx}$ have fundamental solutions

$$\cos(kx)e^{-3k^2t}$$
$$\sin(kx)e^{-3k^2t}$$

But can only use the "sine" solutions because of boundary conditions.

Of the choices f, g, h, only g uses just sines. Thus

$$u(x,t) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k} \sin(kx) e^{-3k^2t}$$

6 (12 points, 4 for each part)

a Since 32 cards, $\binom{32}{5}$ possible hands

b $\dfrac{8 \cdot 28}{\binom{32}{5}}$ (8 = ranks for matching cards; 28 = choices for 5th card)

c $\dfrac{\binom{8}{2}\binom{4}{2}\binom{4}{2}24}{\binom{32}{5}}$ ($\binom{8}{2}$ = choose two ranks; $\binom{4}{2}$ = choose cards for pair; 24 = choose last card)

If 5th card matches one of 1st four, then you actually have a full house; but if you put 28 here I still gave points

7 (10 points)

Events:

H = hit

M = major league player

Want to find $P(M^c | H)$

$$P(M^c|H) = \frac{P(H|M^c)P(M^c)}{P(H)}$$

$$= \frac{P(H|M^c)P(M^c)}{P(H|M^c)P(M^c) + P(H|M)P(M)}$$

$$= \frac{.22 \times .6}{.22 \times .6 + .29 \times .4}$$

8 (12 points, 4 for each part)

a Unif$[-1,1]$

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } -1<x<1 \\ 0 & \text{else} \end{cases}$$

b $E[e^X] = \int_{-\infty}^{\infty} e^x f_X(x)dx$

$$= \int_{-1}^{1} e^x \tfrac{1}{2}\, dx$$

$$= \tfrac{1}{2}(e + e^{-1})$$

c $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \begin{cases} 0 & \text{if } x<-1 \\ \frac{1}{2}+\frac{1}{2}x & \text{if } -1<x<1 \\ 1 & \text{else} \end{cases}$$

9 (12 points, 4 for each part)

a For each character you have a Bernoulli[p] RV for whether it is typed correctly

The sum of n independent Bernoulli[p] RV's can be approximated as a Poisson RV when n is large, p small.

b $\lambda = 0.15 \times 5 = .75$

$P\{X = k\} = e^{-.75} \dfrac{.75^k}{k!}$

$P\{X \le 2\} = e^{-.75}\left(1 + \dfrac{.75}{1} + \dfrac{.75^2}{2}\right)$

c $E[X] = \lambda$

$= .75$