HW #4 Solutions

1 $\mathcal{F}[e^{-2|x|} + \mathbb{1}_{[0,2]}(x)]$

$= \mathcal{F}[e^{-2|x|}] + \mathcal{F}[\mathbb{1}_{[0,2]}(x)]$

use $\mathcal{F}[e^{-|x|}] = \frac{1}{\pi(1+\xi^2)}$

use $\mathcal{F}[f(cx)] = \frac{1}{c}\hat{f}(\xi/c)$ with $c=2$

Use $\mathcal{F}[\mathbb{1}_{[-1,1]}(x)] = \frac{\sin\xi}{\pi\xi}$

Use $\mathcal{F}[f(x-c)] = e^{-ic\xi}\hat{f}(\xi)$ with $c=1$

$$= \frac{1}{2\pi(1+(\xi/2)^2)} + \frac{e^{-i\xi}\sin\xi}{\pi\xi}$$

(Problem #2 next page)

3 $\int_0^x x + \delta_{-2}(x) - 3\delta_5(x)\; dx$

$$= \tfrac{1}{2}x^2 + \left(\int_0^x \delta_{-2}\right) - 3\left(\int_0^x \delta_5\right)$$

$$= \tfrac{1}{2}x^2 + \left(\begin{cases} -1 & \text{if } x<-2 \\ 0 & \text{else}\end{cases}\right) - 3\left(\begin{cases} 0 & \text{if } x<5 \\ 1 & \text{else}\end{cases}\right)$$

$$= \begin{cases} \tfrac{1}{2}x^2 - 1 & \text{if } x<-2 \\ \tfrac{1}{2}x^2 & \text{if } -2<x<5 \\ \tfrac{1}{2}x^2 - 3 & \text{if } 5<x \end{cases}$$

Notice that this function has jumps of 1 at $x=-2$ and -3 at $x=5$, corresponding to δ_{-2} and $-3\delta_5$ in derivative.

2 Step 1

$(1.23), (.45), (-.23), (-.10), (-.79), (-.43), (.25), (.15)$

Step 2 $\theta = e^{2\pi i/2} = -1$

$(1.23 + (-.79), 1.23 + (-.79)\theta), (.45 + (-.43), .45 + (-.43)\theta),$
$(-.23 + (.25), -.23 + (.25)\theta), (-.1 + .15, -.1 + .15\theta)$

$= (.44, 2.02), (.02, .88), (.02, -.48), (.05, -.25)$

Step 3 $\theta = e^{2\pi i/4} = i$

$(.44 + .02, 2.02 + (-.48)\theta, .44 + .02\theta^2, 2.02 + (-.48)\theta^3),$
$(.02 + .05, .88 + (-.25)\theta, .02 + .05\theta^2, .88 + (-.25)\theta^3)$

$= (.46, 2.02 - .48i, .42, 2.02 + .48i), (.07, .88 - .25i, -.03, .88 + .25i)$

Step 4 $\theta = e^{2\pi i/8} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$(.46 + .07, (2.02 - .48i) + (.88 - .25i)\theta, .42 + (-.03)\theta^2,$
$(2.02 + .48i) + (.88 + .25i)\theta^3, .46 + .07\theta^4, (2.02 - .48i) + (.88 - .25i)\theta^5,$
$.42 + (-.03)\theta^6, (2.02 + .48i) + (.88 + .25i)\theta^7)$

$= (.53, 2.82 - .03i, .42 - .03i, 1.22 + .93i, .39, 1.22 - .93i,$
$.42 + .03i, 2.81 + .03i)$

4 $f(x) = |x| + 2\,\mathbb{1}_{[4,7]}(x)$

No jumps

Slope +1 if $x>0$
slope −1 if $x<0$

Jump of +2 at $x=4$
Jump of −2 at $x=7$

slope = 0

$$f'(x) = 2\delta_4(x) - 2\delta_7(x) + \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

5 $f_{XY} = \begin{cases} 2e^{-x-y} & \text{if } 0<x<y \\ 0 & \text{else} \end{cases}$

a Density of Y

$$= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$
$$= \int_0^y 2e^{-x-y}\,dx \qquad (\text{if } y>0)$$
$$= 2e^{-y}\left.\frac{e^{-x}}{-1}\right|_{x=0}^{y}$$
$$= 2e^{-y} - 2e^{-2y}$$

b cond. density for X given $Y=y$

$$= \begin{cases} \dfrac{2e^{-x-y}}{2e^{-y}-2e^{-2y}} & \text{if } 0<x<y \\ 0 & \text{else} \end{cases}$$

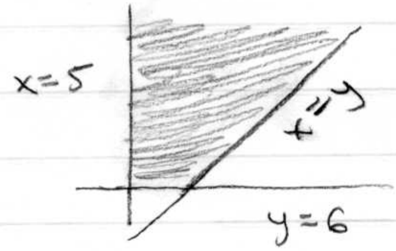
$\left(\text{used } f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} \right)$

5c Integrate conditional density

$$P\{X>5 \mid Y=6\}$$
$$= \int_5^\infty f_{X|Y}(x|6)\,dx$$
$$= \int_5^\infty \left(\begin{cases} \dfrac{2e^{-x-6}}{2e^{-6}-2e^{-12}} & \text{if } 0<x<6 \\ 0 & \text{else} \end{cases} \right) dx$$
$$= \frac{e^{-6}}{e^{-6}-e^{-12}} \int_5^6 e^{-x}\,dx$$
$$= \frac{e^{-6}}{e^{-6}-e^{-12}} \cdot \left(-e^{-6} - (-e^{-5})\right)$$
$$= \frac{e^{-5}-e^{-6}}{1-e^{-6}}$$

5d Use $P\{A \mid B\} = P\{A \cap B\} / P\{B\}$

$$P\{X>5 \mid Y>6\}$$
$$= \int_5^\infty \int_6^\infty \left(\begin{cases} 2e^{-x-y} & \text{if } x<y \\ 0 & \text{else} \end{cases} \right) dy\,dx$$



Region of Integration

$$= \int_5^6 \int_6^\infty 2e^{-x-y}\,dy\,dx + \int_6^\infty \int_x^\infty 2e^{-x-y}\,dy\,dx$$
$$= e^{-12} + 2(e^{-11} - e^{-12})$$