HW #3 Solutions

1 $f_X(x) = \begin{cases} cx^{-2} & \text{if } x > 3 \\ 0 & \text{else} \end{cases}$

a Calculate c

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx$$
$$= \int_{3}^{\infty} cx^{-2}\,dx$$
$$= c/3$$
$$c = 3$$

b Cumulative distribution

$$F_X(x) = \int_{-\infty}^{x} f_X(w)\,dw$$
$$= \int_{3}^{x} 3w^{-2}\,dw \qquad (\text{if } x > 3)$$
$$= 3\left(\frac{1}{3} - \frac{1}{x}\right)$$
$$= \begin{cases} 1 - 3/x & \text{if } x > 3 \\ 0 & \text{else} \end{cases}$$

c $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_{3}^{\infty} x \cdot 3x^{-2}\,dx$$
$$= 3\log x \Big|_{x=3}^{\infty}$$
$$= \infty$$

d $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$$= \int_{3}^{\infty} x^2\, 3x^{-2}\,dx$$
$$= \int_{3}^{\infty} 3\,dx$$
$$= \infty,$$

so the variance is undefined.

2 a Let $N = \#$ heads (RV)

then player A's profit is

$P = 2^N - n$

P is a function of N.

$$E[P] = \sum_{k=0}^{\infty} (2^k - n) P\{N=k\}$$
$$= \sum_{k=0}^{\infty} (2^k - n) 2^{-k-1}$$
$$= \sum_{k=0}^{\infty} \frac{1}{2} - \frac{n}{2^{k+1}}$$
$$= \infty \quad \left(\text{since } \tfrac{1}{2} \text{ not summable}\right)$$

b Now $P = \begin{cases} 2^N - n & \text{if } N \leq 40 \\ 2^{40} - n & \text{if } N > 40 \end{cases}$

$$E[P] = \sum_{k=0}^{40} (2^k - n) P\{X=k\} + \sum_{k=41}^{\infty} (2^{40} - n) P\{X=k\}$$
$$= \left(\sum_{k=0}^{40} \frac{1}{2} - \frac{n}{2^{k+1}} \right) + \left(\sum_{k=41}^{\infty} \frac{2^{40} - n}{2^{k+1}} \right)$$
$$= \left(20 - \frac{n}{2} \frac{1 - 2^{-41}}{1 - 2^{-1}} \right) + (2^{40} - n) 2^{-41}$$
$$\approx 20 - n + \frac{1}{2}$$
$$= 20.5 - n$$

c Assuming player A never gets 40 heads in a row (You'd need to play 10^{12} times before this is likely to happen), player B can expect to earn \$19.50 per game.

3 Height = Normal RV with

$\mu = 71, \ \sigma = 2.5$

$\stackrel{d}{=} N[71, 2.5]$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\, 2.5} e^{-(x-71)^2/2\cdot 2.5^2}$$

$P\{74 < X < 77\}$

$$= \int_{74}^{77} \frac{1}{\sqrt{2\pi}\, 2.5} e^{-\frac{(x-71)^2}{2\cdot 2.5^2}} dx$$

≈ 0.06

4 $f_X(x) = \begin{cases} 4e^{-4x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$

$g(X) = \begin{cases} 1 & \text{if } X \leq 5 \\ 0 & \text{else} \end{cases}$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \int_0^5 1 \cdot 4e^{-4x}\, dx$$

$$= 1 - 4e^{-20}$$

(Comment: This is the probability that $X \leq 5$. Also $X \stackrel{d}{=} \text{Exp}[4]$.)

5 $f_X(x) = \frac{1}{2} \quad (1 < x < 3)$

$Y = e^X$

$P\{Y < y\} = P\{e^X < y\} = P\{X < \log y\}$

$= \int_1^{\log y} \frac{1}{2}\, dx \quad (\text{if } 1 < \log y < 3)$

$= \int_e^y \frac{1}{2} w^{-1}\, dw \quad (\text{if } e^1 < y < e^3)$

$f_Y(y) = \begin{cases} (2y)^{-1} & \text{if } e^1 < y < e^3 \\ 0 & \text{else} \end{cases}$

6 $$f_X(x) = \begin{cases} 1 & \text{if} \quad -2<x<-1 \\ -3 & \text{if} \quad 4<x<7 \\ 0 & \text{else} \end{cases}$$

$$= \mathbb{1}_{[-2,-1]}(x) - 3\,\mathbb{1}_{[4,7]}(x)$$

$$\mathcal{F}[\mathbb{1}_{[-a,a]}(x)] = \frac{\sin a\xi}{\pi\xi}$$

Notice that

$$\mathbb{1}_{[-2,-1]}(x) = \mathbb{1}_{[-1/2,1/2]}(x+3/2)$$
$$\mathbb{1}_{[4,7]}(x) = \mathbb{1}_{[-3/2,3/2]}(x-11/2)$$

Now use $\mathcal{F}[f(x-c)] = e^{-ic\xi}\hat{f}(\xi)$.

We get

$$\hat{f}(\xi) = e^{i3\xi/2}\frac{\sin(\xi/2)}{\pi\xi} - 3e^{-i11\xi/2}\frac{\sin(3\xi/2)}{\pi\xi}$$

$$g(x) = e^{ix} \quad (-1<x<1)$$

$$\hat{g}(\xi) = \frac{1}{2\pi}\int_{-1}^{1} e^{ix}e^{-i\xi x}\,dx$$
$$= \frac{1}{2\pi}\left.\frac{e^{i(1-\xi)x}}{i(1-\xi)}\right|_{x=-1}^{1}$$
$$= \frac{1}{2\pi}\frac{e^{i(1-\xi)} - e^{-i(1-\xi)}}{i(1-\xi)}$$

7 $\hat{a}(\xi) = e^{-10\xi^2}$ Find $a(x)$.

$$\mathcal{F}^{-1}[e^{-10\xi^2}] = 2\pi\,\mathcal{F}[e^{-10\xi^2}](-x)$$
$$= 2\pi\,\frac{1}{20}\frac{1}{\sqrt{2\pi}}\left. e^{-(x/20)^2/2}\right|_{x\to -x}$$
$$= \frac{\sqrt{2\pi}}{20}e^{-x^2/800}$$

(Used $\mathcal{F}[e^{-x^2/2}] = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}$ and $\mathcal{F}[f(cx)] = \frac{1}{c}\hat{f}\left(\frac{\xi}{c}\right)$)