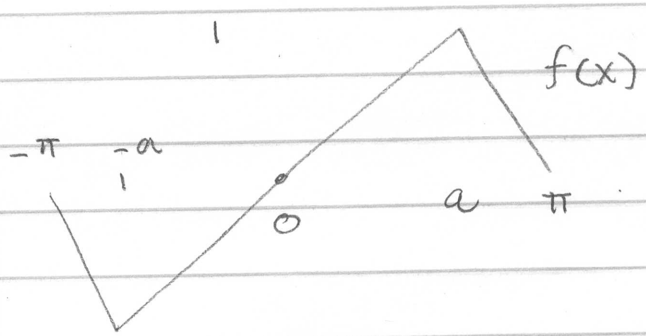
Math 182
Spring 2014 HW#1

1



$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$
$$= \int_{-\pi}^{\pi} \text{Odd function}$$
$$= 0$$

(Since the function is periodic, $\int_{-\pi}^{\pi} = \int_0^{2\pi}$)

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$$

(Since integrand is odd function)

$$= \frac{2}{\pi}\int_0^{a} (x/a)\sin(kx)\,dx$$
$$+ \frac{2}{\pi}\int_a^{\pi}\left(\frac{-1}{\pi-a}(x-a)+1\right)\sin(kx)\,dx$$

This integral can be computed using integration by parts. (Differentiate the f(x) part and integrate the sin(kx))

The answer is

$$b_k = \frac{2}{\pi-a}\,\frac{\sin(ka)}{k^2}$$

Therefore

$$f(x) = \sum_{k=1}^{\infty} \frac{2}{\pi-a}\,\frac{\sin(ka)}{k^2}\,\sin(kx)$$

2 $u_t = \tilde{a} u_{xx}$ heat equation

$$\left.\begin{aligned} u(-\pi, t) &= 0 \\ u(\pi, t) &= 0 \end{aligned}\right\} \text{Boundary conditions}$$

$u(x, 0) = f(x)$ from problem 1

$$= \sum_{k=1}^{\infty} \frac{2}{\pi - a} \frac{\sin(ka)}{k^2} \sin(kx)$$

Comment: $f(x)$ involves parameter a which was not intended to equal the constant a in heat eq. Thus I have used $\tilde{a}$ in heat eq.

Solution: The functions

$$\psi_k(x,t) = \sin(kx)\, e^{-ak^2 t}$$

are solutions to the heat eq.

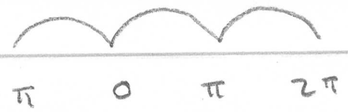
Also
$$\psi_k(\pm\pi, t) = \sin(k(\pm\pi))\, e^{-ak^2 t}$$
$$= 0 \cdot e^{-ak^2 t}$$
$$= 0,$$

so they satisfy the boundary conditions.

Since at $t = 0$ we have $e^{-ak^2 t} = 1$, the following function satisfies the initial data.

$$u(x,t) = \sum_{k=1}^{\infty} \frac{2}{\pi - a} \frac{\sin(ka)}{k^2} \sin(kx)\, e^{-ak^2 t}$$

3 $f(x) = |\sin(x)|$



$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\sin(kx)\,dx$$
$$= \int_{-\pi}^{\pi} \text{Odd function}$$
$$= 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\cos(kx)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(kx)\,dx \qquad \left(\begin{array}{c}\text{Integrand is}\\ \text{even function}\end{array}\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{4}{\pi}$$

If $k \neq 0$, then the integrand can be evaluated either by doing integration by parts twice, or by expressing sin and cos in terms of complex exponentals.

The answer is

$$a_k = \begin{cases} \dfrac{-4}{\pi(k^2-1)} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

Thus
$$f(x) = \tfrac{1}{2}a_0 + \sum_{k\geq 1} a_k\cos(kx)$$
$$= \frac{2}{\pi} + \sum_{\ell \geq 1} \frac{-4}{\pi((2\ell)^2-1)}\cos(2\ell x)$$

(used $k = 2\ell$ to sum over even #'s)

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Evaluate at $x = 0$:

$$0 = f(0)$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{l \geq 1} \frac{\cos(2l \cdot 0)}{(2l)^2 - 1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{l \geq 1} \frac{1}{4l^2 - 1}$$

$$\Rightarrow \sum_{l \geq 1} \frac{1}{4l^2 - 1} = \frac{1}{2}$$

Evaluate at $x = \pi/2$

$$\cos(2l\pi/2) = \cos(l\pi)$$

$$= (-1)^l$$

$$= f(\pi)$$

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{l \geq 1} \frac{\cos(2l\pi/2)}{4l^2 - 1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{l \geq 1} \frac{(-1)^l}{4l^2 - 1}$$

$$\Rightarrow \sum_{l \geq 1} \frac{(-1)^l}{4l^2 - 1} = \frac{-\pi}{4}\left(1 - \frac{2}{\pi}\right)$$

$$= \frac{1}{2} - \frac{\pi}{4}$$

4 2 decks shuffled together

(a) 5 of a kind

$$\text{Prob.} = \frac{\overset{\#\text{ ranks}}{13}\binom{8}{5}}{\binom{104}{5}}$$

$\binom{8}{5}$ ← ways of choosing 5 cards from 8 of that rank

$\binom{104}{5}$ ← Total # of hands

(b) 4 of a kind

$$\text{Prob} = \frac{13\binom{8}{4}\,100}{\binom{104}{5}}$$

100 ← # ways to choose 5th card from remaining 104−4 cards

(c) 3 of a kind

$$\text{Prob.} = \frac{13\binom{8}{3}\binom{101}{2}}{\binom{104}{5}}$$

$\binom{101}{2}$ ← # ways to choose 2 remaining cards after the 3 matching

(d) A pair

$$\text{Prob.} = \frac{13\binom{8}{2}\binom{102}{3}}{\binom{104}{5}}$$

5 D = event dice roll different #'s
S = event at least one dice is 6

$D \cap S = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$P(D \cap S) = 10/36$

$$P(D) = \frac{36-6}{36} = \frac{5}{6}$$

(Since D is everything except $(1,1), (2,2), \ldots, (6,6)$)

$$P(S \mid D) = \frac{P(D \cap S)}{P(D)} = \frac{10/36}{5/6}$$
$$= 1/3$$

6 F = farmer $\quad W$ = wooden teeth
G = gold miner
O = outlaw

Must calculate $P(O \mid W)$.

(Mult. rule) $P(O \mid W) P(W) = P(O \cap W) = P(W \mid O) P(O)$

$$\Rightarrow P(O \mid W) = \frac{P(W \mid O) P(O)}{P(W)} = \frac{0.7 \times 0.2}{P(W)}$$

(Partition theorem) $P(W) = P(W \mid O) P(O) + P(W \mid G) P(G) + P(W \mid F) P(F)$
$= 0.7 \times 0.2 + 0.3 \times 0.45 + 0.15 \times 0.35$
$= .3275$

$P(O \mid W) = .7 \times .2 / .3275 \approx 0.427$