Math exam #1

1. A 2π-periodic function $f(x)$ is defined by its values for $-\pi \le x < \pi$ by

$$f(x) = \begin{cases} 1 & \text{if } -\pi \le x < 0 \\ -1 & \text{if } 0 \le x < \pi \end{cases}$$

   Calculate a Fourier series representation for this function.
   (Your answer must be of the form $f(x) =$ some infinite series)

2. Define the following 2π-periodic functions by their values for $-\pi < x < \pi$:

$$\begin{aligned} f(x) =& |x|, & -\pi <& x < \pi \\ g(x) =& |x|^{-1/4} & -\pi <& x < \pi \end{aligned}$$

   For each of the following statements, answer "True" or "False".

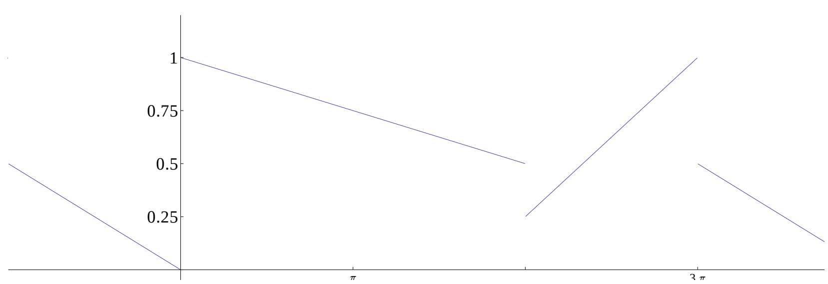
   (a) By Dirichlet's pointwise convergence theorem, $S_n f(x) \to \frac{1}{2} f(x^+) + \frac{1}{2} f(x^-)$ for all x.
   (b) The Fourier series for f converges uniformly.
   (c) The Fourier series for f converges with respect to L^2 norm.
   (d) By Dirichlet's pointwise convergence theorem, $S_n g(x) \to \frac{1}{2} g(x^+) + \frac{1}{2} g(x^-)$ for all x.
   (e) The Fourier series for g converges uniformly.
   (f) The Fourier series for g converges with respect to L^2 norm.

3. The following Fourier series holds for $-\pi \le x < \pi$

$$x^2 = \frac{\pi^2}{3} + \sum_{k \ge 1} \frac{4(-1)^k}{k^2} \cos(kx)$$

   (a) Evalulate both sides of the above expression at $x = 0$ to get the the value of an infinite sum.
   (b) Apply Parseval's identity to find the value of the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^4}$.

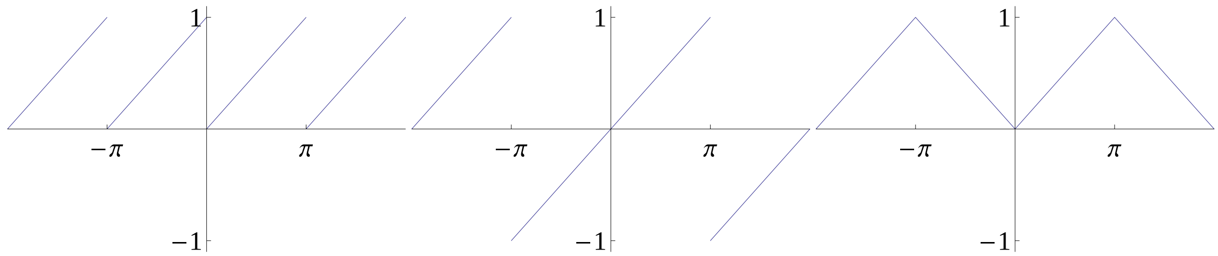
4. The graph below shows a 2π-periodic function $f(x)$. Where does the Fourier series for $f(x)$ overshoot the actual value of $f(x)$? For each of these points compute overshoot.



5. Give a Fourier series to the following 1-d heat problem where $u(x,t)$ is the temperature of a rod at position $0 \le x \le \pi$ and time $t \ge 0$.

$$\begin{aligned} u_t =& 3u_{xx} \\ u(x,0) =& x/\pi \\ u(0,t) =& 0 \\ u(\pi,t) =& 0 \end{aligned}$$

You can use any of the following Fourier series in your solution (The boundary conditions tell you which one to use).



$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-1}{\pi k}\sin(2kx), \qquad g(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k}\sin(kx), \qquad h(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4\cos((2k+1)x)}{\pi^2(2k+1)^2}$$

6. Suppose that all of the cards J,Q,K,A are removed from a standard 52 card deck (so that the the remaining cards are $2, 3, 4, \ldots, 9$ of each suit).
   (a) How many different 5 card poker hands could possibly be dealt from this pile?
   (b) What is the probability of 4-of-a-kind?
   (c) What is the probability of 2 pairs?

7. In Spring training, baseball players from the major leagues and minor leagues play in the same games. Suppose that in a certain league we have the following statistics: minor league players get hits in 22% of their at bats, and major league players get hits in 29% of their at bats. Furthermore, suppose that 40% of the at bats are by major league players. If in a particular at bat a player gets a hit, what is the conditional probability that he is a minor league player?

8. Let $f(x) = e^x$ and X be a continuous random variable with Uniform$[-1, 1]$ distribution.
   (a) What is the probability density of X?
   (b) Compute the expected value of $f(X)$.
   (c) Compute the cumulative distribution function for X

9. Suppose a student is typing a document. On average, his typing has 0.15 errors per page. Suppose he types a five page document. Let X be the number of errors he makes in typing this document.
   (a) Explain why it is reasonable to model X with a Poisson random variable.
   (b) Calculate the probability $P\{X \le 2\}$
   (c) What is the expected value of X?